

Viscous shock profile for 2×2 systems of hyperbolic conservation laws with an umbilic point

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1 Introduction

Let us consider a 2×2 system of conservation laws in one space dimension:

$$U_t + F(U)_x = 0, \quad (x, t) \in \mathbf{R} \times \mathbf{R}_+ \quad (1)$$

where $U = {}^t(u, v) \in \Omega$ for a domain $\Omega \subseteq \mathbf{R}^2$ and $F = {}^t(F_1, F_2) : \Omega \rightarrow \mathbf{R}^2$ is a smooth map. We suppose that this system of equations (1) is *hyperbolic*, i.e. the Jacobian matrix $F'(U)$ has *real* eigenvalues $\lambda_1(U), \lambda_2(U)$ for any $U \in \Omega$. If, in particular, these eigenvalues are *distinct*: $\lambda_1(U) < \lambda_2(U)$, the system is called *strictly hyperbolic* at U . A state $U^* \in \Omega$ is called an *umbilic* point, if $\lambda_1(U) = \lambda_2(U)$ and $F'(U)$ is diagonal at $U = U^*$. We suppose that the system of equations (1) is strictly hyperbolic at any $U \in \Omega \setminus \{U^*\}$ and that U^* is a single umbilic point in Ω . Since $U = U^*$ is an isolated umbilic point, we have the Taylor expansion of $F(U)$ near $U = U^*$:

$$F(U) = F(U^*) + \lambda^*(U - U^*) + Q(U - U^*) + O(1)|U - U^*|^3$$

where $\lambda^* = \lambda_1(U^*) = \lambda_2(U^*)$ and $Q : \mathbf{R}^2 \rightarrow \mathbf{R}^2$ is a homogeneous quadratic mapping. After the Galilean change of variables: $x \rightarrow x - \lambda^*t$ and $U \rightarrow U + U^*$, we observe that the system of equations (1) is reduced to

$$U_t + Q(U)_x = 0, \quad (x, t) \in \mathbf{R} \times \mathbf{R}_+ \quad (2)$$

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modulo higher order terms. Now by a change of unknown functions $V = S^{-1}U$ with a regular constant matrix S , we have a new system of equations $V_t + P(V)_x = 0$ where $P(V) = S^{-1}Q(SV)$. Thus we come to

Definition 1.1 *Two quadratic mappings $Q_1(U)$ and $Q_2(U)$ are said to be equivalent, if there is a constant matrix $S \in GL_2(\mathbf{R})$ such that*

$$Q_2(U) = S^{-1}Q_1(SU) \quad \text{for all } U \in \mathbf{R}^2. \quad (3)$$

A general quadratic mapping $Q(U)$ has six coefficients and $GL_2(\mathbf{R})$ is a four dimensional group. Thus by the above equivalence transformations, we can eliminate four parameters. These procedures are successfully carried out by Schaeffer-Shearer [25] and they obtained the following *normal forms*.

Let $Q(U)$ be a hyperbolic quadratic mapping with an isolated umbilic point $U = 0$, then there exist two real parameters a and b with $a \neq 1 + b^2$ such that $Q(U)$ is equivalent to $\frac{1}{2}\nabla C$ where $\nabla = (\partial_u, \partial_v)$ and

$$C(U) = \frac{1}{3}au^3 + bu^2v + uv^2. \quad (4)$$

Moreover, if $(a, b) \neq (a', b')$, then the corresponding quadratic mappings: $\frac{1}{2}\nabla C$ and $\frac{1}{2}\nabla C'$ are not equivalent.

In the following argument, we shall confine ourselves to the quadratic mapping:

$$F(U) = Q(U) = \frac{1}{2}\nabla C(U) = \frac{1}{2} \begin{pmatrix} au^2 + 2buv + v^2 \\ bu^2 + 2uv \end{pmatrix} \quad (a \neq 1 + b^2). \quad (5)$$

Mathematical properties of the systems of equations (1) depends on (a, b) . Schaeffer-Shearer classify in [25] ab -plane into four cases: Case I is $a < \frac{3}{4}b^2$; Case II is $\frac{3}{4}b^2 < a < 1 + b^2$; for $a > 1 + b^2$, the boundary between Case III and Case IV is $4\{4b^2 - 3(a - 2)\}^3 - \{16b^3 + 9(1 - 2a)b\}^2 = 0$. We notice that these 2×2 system of hyperbolic conservation laws with an isolated umbilic point is a generalization of a three phase Buckley-Leverett model for oil reservoir flow where the flux functions are represented by a quotient of polynomials of degree two. In Appendix of [25]: in collaboration with Marchesin and Paes-Leme, they show that the quadratic approximation of the flux functions is either Case I or Case II.

The Riemann problem for (1) is the Cauchy problem with initial data of the form

$$U(x, 0) = \begin{cases} U_L & \text{for } x < 0, \\ U_R & \text{for } x > 0 \end{cases} \quad (6)$$

where U_L, U_R are constant states in Ω . A jump discontinuity defined by

$$U(x, t) = \begin{cases} U_L & \text{for } x < st, \\ U_R & \text{for } x > st \end{cases} \quad (7)$$

is a piecewise constant weak solution to the Riemann problem, provided these quantities satisfy the *Rankine-Hugoniot condition*:

$$s(U_R - U_L) = F(U_R) - F(U_L). \quad (8)$$

We say that the above discontinuity is a *j-compressive shock wave* ($j = 1, 2$) if it satisfies the *Lax entropy conditions* :

$$\lambda_j(U_R) < s < \lambda_j(U_L), \quad \lambda_{j-1}(U_L) < s < \lambda_{j+1}(U_R) \quad (9)$$

(Lax [16], [17]). Here we adopt the convention $\lambda_0 = -\infty$ and $\lambda_3 = \infty$. The presence of an umbilic point bring us to face with non-classical: overcompressive shocks and crossing shocks. We say that a piecewise constant weak solution (7) is a *overcompressive shock* if it satisfies

$$\lambda_1(U_R) < s < \lambda_1(U_L), \quad \lambda_2(U_R) < s < \lambda_2(U_L). \quad (10)$$

We say also that a piecewise constant weak solution (7) is a *crossing shock* if it satisfies

$$\lambda_1(U_R) < s < \lambda_2(U_R), \quad \lambda_1(U_L) < s < \lambda_2(U_L). \quad (11)$$

In this note, we shall confine ourselves to Case II of the representative quadratic mapping $F(U) = Q(U)$ defined by (5). Our aim is to show that there is no crossing shock with viscous profile on the complement of medians $M_1 \cup M_3$ hence the associated vector field $X_s(U_L, U)$ is structurally stable on the complement of $M_1 \cup M_3$ in Case II. In Section 2, we introduce the vector field $X_s(U, U_L)$ which allows us to determine the existence of a viscous profile to the shock wave solutions. Then we classify the character of critical points for the vector field $X_s(U_L, U)$. In Section 3, we show that there is no crossing shock with viscous profile on the complement of $M_1 \cup M_3$. In Section 4, as conclusion, we show that the vector field $X_s(U_L, U)$ is structurally stable on the complement of $M_1 \cup M_3$ in Case II.

2 Viscous Shock Profiles

One admissibility condition for shock wave solutions (7) to the Riemann problem (6) for a hyperbolic system of conservation laws (1) is to obtain these

solutions as limits of travelling wave solutions to an associated parabolic equation:

$$U_t + F(U)_x = \epsilon(B(U)U_x)_x, \epsilon > 0 \quad (12)$$

with an admissible matrix $B(U)$ in [4, 8, 9, 21, 28, 31]. More precisely, let U_L and U_R be two constant states to Riemann problem (1), (6). If there exists a shock $U(x, t)$ (7) with speed s to this Riemann problem and the two constant states U_L and U_R are connected through a travelling wave solution $U_\epsilon(x, t) = U\left(\frac{x - st}{\epsilon}\right)$ to (12) with shock speed s which converges to the shock wave $U(x, t)$ (7) as ϵ tends to 0, then we say that this shock (7) satisfies the *viscosity admissibility criterion* and that it has a *viscous shock profile* $U_\epsilon(x, t) = U\left(\frac{x - st}{\epsilon}\right)$. The travelling wave $U_\epsilon(x, t) = U\left(\frac{x - st}{\epsilon}\right)$ should satisfy, by integrating (12), the 2×2 system of nonlinear ordinary equations:

$$B(U)U_\xi = -s(U - U_L) + f(U) - f(U_L) \quad (13)$$

with $\xi = \frac{x - st}{\epsilon}$ and the boundary conditions at the infinity

$$\lim_{\xi \rightarrow -\infty} U(\xi) = U_L, \lim_{\xi \rightarrow \infty} U(\xi) = U_R. \quad (14)$$

The conditions (13), (14) required for the travelling wave solution imply automatically the Rankine-Hugoniot condition (8) for the Riemann problem. The existence of shock with a viscous profile is equivalent to the system of (13) with the boundary condition (14).

Let $X_s(U, U_L)$ be the vector field

$$X_s(U, U_L) = -s(U - U_L) + F(U) - F(U_L). \quad (15)$$

The shock wave solution (7) has a viscous shock profile if and only if there exists an orbit along the vector-field $X_s(U, U_L)$ from the critical point U_L to the critical point U_R of this vector-field.

Let p be a critical point of a vector field X . We say that p is hyperbolic if dX has two eigenvalues with non-zero real part at p . Clearly the eigenvalues of $dX_s(U, U_L)$ are $-s + \lambda_j(U)$. In particular, $dX_s(U, U_L)$ has real eigenvalues.

The critical point U of X_s is not hyperbolic if and only if $s = \lambda_j(U)$ ($j = 1$ or 2).

Proposition 2.1 *The shock wave (7) is*

- 1-compressive shock if and only if U_L is repeller and U_R is saddle.
- 2-compressive shock if and only if U_L is saddle and U_R is attractor.
- overcompressive shock if and only if U_L is repeller and U_R is attractor.
- crossing shock if and only if U_L and U_R are saddles.

For all above shocks, both critical point U_L and U_R are hyperbolic. Moreover there exists a shock wave (7) with a viscous profile if and only if there exists an orbit connecting two critical points of the vector field X_s .

We say, for example, *repeller-saddle connection* or simply *R-S connection* an orbit from a repeller point to a saddle point.

In Case II, we investigate the critical points of the vector-field $X_s(U, U_L)$ in the finite part of the U -plane and at the infinity. The Poincaré transformation [2, 9] enables us to make a one-to-one correspondence from U -plane including the infinity to the sphere S^2 by identifying two antipodal points. The line joining two antipodal points of $S^2 = \{(x_1, x_2, x_3) \in \mathbf{R}^3; x_1^2 + x_2^2 + x_3^2 = 1\}$ intercepts the plane $P_1 = \{(u, v, -1); (u, v) \in \mathbf{R}^2\} \simeq U$ -plane at one point. This mapping induces the vector field $X_s(U, U_L)$ on U -plane to the vector field $X_s^{S^2}(U, U_L)$ on the sphere S^2 minus the equator $\{x_3 = 0\}$. The equator $\{x_3 = 0\}$ corresponds to $\infty \times S^1$ of U -plane. Similarly the line joining the origin and a point on $P_2 = \{(1, w, -z); (w, z) \in \mathbf{R}^2\}$ intercepts S^2 at two antipodal points. By this mapping, a vector field on P_2 is induced to a vector field on the sphere S^2 minus the equator $\{x_1 = 0\}$. Therefore the composition of two mappings above transforms a point $(1, w, -z) \in P_2$ to a point $(u, v, 1) \in P_1$:

$$u = 1/z, v = w/z \text{ if } z \neq 0,$$

or equivalently

$$w = v/u, z = 1/u \text{ if } u \neq 0.$$

For $u = 0$, we take instead of the plane P_2 the plane $P'_2 = \{(w, 1, -z); (w, z) \in \mathbf{R}^2\}$. Similarly a point $(w, 1, -z) \in P'_2$ corresponds to a point $(u, v, 1) \in P_1$:

$$w = u/v, z = 1/v \text{ if } v \neq 0.$$

By the mapping from P_2 to P_1 , the differential equation $\frac{dv}{du} = \frac{-sv + F_2(U)}{-su + F_1(U)}$ of the vector field $X_s(U, U_L)$ is induced to the differential equation

$$\frac{dz}{dw} = \frac{\Psi}{\Xi} \quad (16)$$

where

$$\begin{aligned}\Psi &= -z\{-sz(1 - zu_L) + F_1(1, w) - z^2F_1(U_L)\}, \\ \Xi &= -w\{-sz(1 - zu_L) + F_1(1, w) - z^2F_1(U_L)\} + F_2(1, w) \\ &\quad - z^2F_2(U_L) - sz(w - zv_L).\end{aligned}$$

The right-hand side of the differential equation (16) is well-defined also for $\{z = 0\}$ which corresponds to the equator $\{x_3 = 0\}$ of S^2 then to the infinity of U -plane.

We consider the critical points of $X_s(U, U_L)$ at the infinity. They satisfy $z = 0$ then

$$-wF_1(1, w) + F_2(1, w) = -\Phi(w) = -(w^3 + 2bw^2 + (a - 2)w - b) = 0$$

which has three distinct real roots μ_1, μ_2, μ_3 for $a < 1 + b^2$. The corresponding vector field of (16) is $\dot{w} = \Xi$, $\dot{z} = \Psi$ and its Jacobian matrix at $z = 0$ is

$$\begin{pmatrix} -F_1(1, w) - wF_1'(1, w) + F_2'(1, w) & 0 \\ 0 & -F_1(1, w) \end{pmatrix}. \quad (17)$$

We have already known [3] the configuration of the roots μ_i of $\Phi(w) = 0$. For $b > 0$,

$$\text{in Case II, } \mu_1 < -b < \mu_2 < -b/2 < 0 < \mu_3. \quad (18)$$

Then we have

$$-F_1(1, w) - wF_1'(1, w) + F_2'(1, w) = -\Phi'(w) \begin{cases} < 0 & \text{for } w = \mu_1, \mu_3, \\ > 0 & \text{for } w = \mu_2 \end{cases} \quad (19)$$

and

$$-F_1(1, w) = -\frac{1}{w}(\Phi(w) + 2w + b) \begin{cases} < 0 & \text{for } \mu_1, \mu_2, \\ > 0 & \text{for } \mu_3. \end{cases} \quad (20)$$

Therefore in Case II, μ_1 is a attractor, μ_2 is a saddle and μ_3 is a repeller. On account of the fact that, at the antipodal point, the character of a critical point is the inverse, we have

Theorem 2.1 *The vector field $X_s(U, U_L)$ has six singularities at infinity. In Case II, two are repellers, two are attractors and two are saddles.*

We investigate critical points of $X_s(U, U_L)$ in the bounded region of U -plane. Owing to the Poincaré-Hopf theorem, we can show

Theorem 2.2 *The vector field $X_s(U, U_L)$ has two, three or four critical points in the bounded region of U -plane. In Case II,*

(i) if the vector field $X_s(U, U_L)$ has four critical points in the bounded region of U -plane, then the critical points are two nodes and two saddles.

(ii) if the vector field $X_s(U, U_L)$ has three critical points in the bounded region of U -plane, then the critical points are one node, one saddle and one saddle-node.

(iii) if the vector field $X_s(U, U_L)$ has two critical points in the bounded region of U -plane, then the critical points are one node and one saddle or two saddle-nodes.

Let us recall the notion of structurally stable vector fields. Let $\chi(M^2)$ be the space of all vector fields of C^1 class on a 2-dimensional compact manifold M^2 with the C^1 -topology.

Definition 2.1 *A vector field $X \in \chi(M^2)$ is said to be structurally stable if there exists a neighborhood N of X in $\chi(M^2)$ such that for any $Y \in N$, there exists a homeomorphism $\rho : M^2 \rightarrow M^2$ which maps any orbit of X to an orbit Y .*

The following theorem due to Peixoto [24] gives a characterization of structurally stable vector fields.

Theorem 2.3 *A vector field $X \in \chi(M^2)$ is structurally stable if and only if it satisfies the following conditions:*

- *there are only a finite number of critical points and all are hyperbolic,*
- *there are only a finite number of closed orbits and all are hyperbolic,*
- *the ω -limit sets and α -limit sets of any orbit consist only of critical points or closed orbits,*
- *there are no saddle-saddle connections.*

Since both eigenvalues of $X_s(U_L, U)$ are real, we have

Proposition 2.2 *The vector field $X_s(U_L, U)$ has no closed orbits, nor singular closed orbit, nor ω -limit sets, nor α -limit sets.*

The most unstable connection is clearly saddle-saddle connection. We will show in the next section that there are no saddle-saddle connections on the complement of $M_1 \cup M_3$ in Case II.

3 Saddle-Saddle Connections

The aim of this section is to show that there is no crossing shock on the complement of $M_1 \cup M_3$ in the Case II.

Theorem 3.1 *A crossing shock has a viscous profile if and only if this profile comes from a saddle-saddle connection which is a straight line on the median $M_j = \{U = {}^t(u, v); v = \mu_j u\}$ ($j = 1, 2, 3$).*

Proof. Suppose that there is a crossing shock. It is obvious, from Proposition 2.1 and its following remark, that the existence of a crossing shock is equivalent to the existence of a S-S connection. The next lemma is due to Chicone [6].

Lemma 3.1 *Let $X = {}^t(\Psi, \Xi)$ be a quadratic vector field on the plane where Ψ and Ξ are relatively prime polynomials. Then every saddle-saddle connection lies on a straight line.*

To accomplish the proof of the theorem, we make use of a strategy of Gomes [9]. Let U_L and U_R be two saddle points connected by an straight orbit $L: U = {}^t(1, k)t + U_L$. Owing to the fact that the segment $\tilde{L}$ from U_L to U_R is invariant under the vector field X_s , we have $(X_s|_{\tilde{L}}, {}^t(-k, 1)) = 0$.

Denoting $U = {}^t(u, v)$ and $U_L = {}^t(u_L, v_L)$, we have, from the above equation,

$$F_2(U) - F_2(U_L) = k(F_1(U) - F_1(U_L)), \quad (21)$$

i.e. $(kF_1(1, k) - F_2(1, k))u^2 = 0$ modulo polynomial of u of degree ≤ 1 . It implies that

$$kF_1(1, k) - F_2(1, k) (= \Phi(k)) = 0, \quad (22)$$

then $k = \mu_j$ ($j = 1, 2$ or 3). Substituting $k = \mu_j$ into (21), we obtain

$$k^2(bu_L + v_L) + k((a-1)u_L + bv_L) - (bu_L + v_L) = 0. \quad (23)$$

(22) $\times u_L$ - (23) gives us $(k^2 + bk - 1)(ku_L - v_L) = 0$. Because clearly $k^2 + bk - 1 \neq 0$, we have $ku_L = v_L$. Then L is on a median.

Therefore the straight orbit lies on the medians and every median is invariant of the vector field X_s , which proves the assertion. The converse is quite clear.

In the context of the above proof, we showed

Corollary 3.1 *i) Every median M_j is invariant under the vector field X_s and every straight line orbit lies on a median. ii) The orbit of any saddle-saddle connection lies on a median.*

Let us investigate the structure of orbits on the medians. Let $U_L = {}^t(u_L, v_L)$ be a point on a median $M = \{U = {}^t(u, v); v = \mu u\}$ where $\mu = \mu_j$ ($1 \leq j \leq 3$). Owing to Corollary 3.1, the orbit through U_L lies on the median M . Then we have

$$X_s(U, U_L) = \{(a + 2b\mu + \mu^2)(u^2 - u_L^2) - s(u - u_L)\} \begin{pmatrix} 1 \\ \mu \end{pmatrix}. \quad (24)$$

Let $U_1 = {}^t(u_1, v_1)$ be a point $X_s(U_1, U_L) = 0$ ($U_1 \neq U_L$). Then we have $v_1 = \mu u_1$ and

$$u_1 = -u_L + \frac{\mu}{b + 2\mu}s. \quad (25)$$

If $u_1 < u_L$ i.e. $u_L > \frac{\mu}{2(b + 2\mu)}s$, then both components of $X_s(U, U_L)$ are negative for $u_1 < u < u_L$ and positive for $u < u_1$ and for $u > u_L$. Hence there is an orbit from U_L to U_1 .

If $u_1 > u_L$ i.e. $u_L < \frac{\mu}{2(b + 2\mu)}s$, then both components of $X_s(U, U_L)$ are negative for $u_L < u < u_1$ and positive for $u < u_L$ and for $u > u_1$. Hence there is an orbit from U_1 to U_L .

In any case, there is an orbit between U_L and U_1 . Therefore we have

Theorem 3.2 *Any point U_L on a median M_j ($1 \leq j \leq 3$) can be connected via one shock to a point U_1 on the common median M_j and this shock has a viscous profile.*

Furthermore the character of shock waves on the median M_j ($1 \leq j \leq 3$) can be determined in Case II by the following two propositions

Proposition 3.1 *Let $b \geq 0$. On the median M_2 , there is no crossing shock in Case II.*

Proof. On the median $M_2 = \{{}^t(u, v); v = \mu_2 u\}$, the system (1) is reduced to the equation

$$v_t + \left(\frac{b}{\mu_2^2} + \frac{2}{\mu_2} \right) \left(\frac{v^2}{2} \right)_x = 0. \quad (26)$$

Then the speed of shock wave joining $U_+ = {}^t(u_+, v_+)$ and $U_- = {}^t(u_-, v_-)$ is $s(U_+, U_-) = \frac{b + 2\mu_2}{2\mu_2^2}(v_+ + v_-)$. The Jacobian matrix $F'(U)$ on the median M_2 is

$$F'(U) = \begin{pmatrix} au + bv & bu + v \\ bu + v & u \end{pmatrix} = \frac{1}{\mu_2} \begin{pmatrix} a + b\mu_2 & b + \mu_2 \\ b + \mu_2 & 1 \end{pmatrix} v.$$

As we have already seen in Proposition 5.1 [3], the eigenvalues of $F'(U)$ are

$$\lambda(U) = \left(\frac{a}{\mu_2} + 2b + \mu_2 \right) v = \frac{b + 2\mu_2}{\mu_2^2} v \text{ and } \lambda^\perp(U) = \left(\frac{1}{\mu_2} - b - \mu_2 \right) v$$

and its eigenvectors are ${}^t(v, \mu_2 v)$ and ${}^t(-\mu_2 v, v)$ respectively. We can determine $\lambda_1(U)$ and $\lambda_2(U)$ according to the sign of v (or u). In fact, we have

$$\lambda(U) - \lambda^\perp(U) = \frac{v}{\mu_2^2}(1 + \mu_2^2)(\mu_2 + b). \quad (27)$$

On the median M_2 , taking into account of (18), for $v > 0$, $\lambda_1(U) = \lambda^\perp(U)$, $\lambda_2(U) = \lambda(U)$ and, for $v < 0$, $\lambda_1(U) = \lambda(U)$, $\lambda_2(U) = \lambda^\perp(U)$.

Suppose that there is a crossing shock on the median M_2 . We have four cases: *i*) $v_+ \geq 0, v_- > 0$, *ii*) $v_+ > 0, v_- \leq 0$, *iii*) $v_+ < 0, v_- \geq 0$, *iv*) $v_+ \leq 0, v_- < 0$. In case *i*), we would have

$$\begin{aligned} s(U_+, U_-) - \lambda_2(U_+) &= \frac{2\mu_j + b}{\mu_j^2}(v_- - v_+) < 0, \\ s(U_+, U_-) - \lambda_2(U_-) &= \frac{2\mu_j + b}{\mu_j^2}(v_+ - v_-) < 0 \end{aligned}$$

which is not possible to realize. In case *ii*), we would have

$$s(U_+, U_-) - \lambda_1(U_-) = \frac{2\mu_j + b}{2\mu_j^2}(v_+ - v_-) > 0 \text{ then } v_+ < v_-$$

which is not possible to realize. In case *iii*), we would have

$$s(U_+, U_-) - \lambda_1(U_+) = \frac{2\mu_j + b}{2\mu_j^2}(v_- - v_+) > 0 \text{ then } v_- < v_+$$

which is not possible to realize. In case *iv*), we would have

$$\begin{aligned} s(U_+, U_-) - \lambda_1(U_+) &= \frac{2\mu_j + b}{\mu_j^2}(v_- - v_+) < 0, \\ s(U_+, U_-) - \lambda_1(U_-) &= \frac{2\mu_j + b}{\mu_j^2}(v_+ - v_-) < 0 \end{aligned}$$

which is not possible to realize.

Therefore there is no crossing shock on the median M_2 .

Proposition 3.2 *Let $b \geq 0$. Suppose that (a, b) belongs to Case II. On the median M_1 , there is a saddle-saddle connection from U_- to U_+ if and only if $v_- < 0 < v_+$. On the median M_3 , there is a saddle-saddle connection from U_- to U_+ if and only if $v_+ < 0 < v_-$.*

We can prove this proposition using a similar strategy as Proposition 3.1. Combining Corollary 3.1, Proposition 3.1 and Proposition 3.2, we have

Theorem 3.3 *There is no saddle-saddle connection nor crossing shock with viscous profile on the complement of $M_1 \cup M_3$ in Case II.*

The relation $X_s(U, U_L) = 0$ is the intersection of two quadratic equations $F_1(U) - F_1(U_L) - s(u - u_L) = 0$ and $F_2(U) - F_2(U_L) - s(v - v_L) = 0$. Then it consists of at most four points including U_L and U_1 . In fact, the others are two saddle points. More precisely

Proposition 3.3 *Let U_L be a point on a median M_j ($1 \leq j \leq 3$). The set $X_s(U, U_L) = 0$ consists of at most four points. The others critical points than U_L and U_1 consist only of saddle points.*

Proof. Let U_L be a point on a median $M_j : v_L = \mu_j u_L$. The equation $X_s(U, U_L) = 0$ implies that

$$F_1(U) - F_1(U_L) - s(u - u_L) = 0, \quad (28)$$

$$F_2(U) - F_2(U_L) - s(v - v_L) = 0. \quad (29)$$

(29) - (28) $\times \mu_j$ implies that

$$(a\mu_j - b)u^2 + 2(b\mu_j - 1)uv + \mu_j v^2 - s\mu_j u + sv + \{F_2(U_L) - \mu_j F_1(U_L)\} = 0.$$

Here

$$\begin{aligned} F_2(U_L) - \mu_j F_1(U_L) &= (b - a\mu_j)u_L^2 + 2(1 - b\mu_j)u_L v_L - \mu_j v_L^2 \\ &= u_L^2 \{(b - a\mu_j) + 2\mu_j(1 - b\mu_j) - \mu_j^3\} \\ &= -u_L^2 \{\mu_j^3 + 2b\mu_j^2 + (a - 2)\mu_j - b\} \\ &= 0. \end{aligned}$$

Hence we have

$$\begin{aligned} 0 &= (a\mu_j - b)u^2 + 2(b\mu_j - 1)uv + \mu_j v^2 - s\mu_j u + sv \\ &= (v - \mu_j u) \left\{ \mu_j v - \frac{1}{\mu_j}(a\mu_j - b)u + s \right\} \\ &= (v - \mu_j u) \{ \mu_j v + (\mu_j^2 + 2b\mu_j - 2)u + s \}. \end{aligned}$$

Therefore we have $v = \mu_j u$ and

$$v = \frac{1}{\mu_j^2}(a\mu_j - b)u - \frac{s}{\mu_j} \quad (30)$$

$$\text{or equivalently } v = \left(-\mu_j - 2b + \frac{2}{\mu_j}\right)u - \frac{s}{\mu_j}. \quad (31)$$

Substituting $v = \mu_j u$ into $X_s(U, U_L) = 0$, we obtain as above $U = U_L, U_1$.

Similarly substituting $v = \left(-\mu_j - 2b + \frac{2}{\mu_j}\right)u - \frac{s}{\mu_j}$ into $X_s(U, U_L)$, we obtain

$$X_s(U, U_L) = x_s^1(U, U_L) \begin{pmatrix} 1 \\ \mu_j \end{pmatrix} \quad (32)$$

$$\text{where } x_s^1(U, U_L) = \left(-3b - 2\mu_j + \frac{4}{\mu_j}\right)u^2 + s \left(2b + \mu_j - \frac{4}{\mu_j}\right)u \quad (33)$$

$$+ \frac{s^2}{\mu_j} - (b + 2\mu_j)u_L^2 + s\mu_j u_L. \quad (34)$$

Therefore on the line $v = \left(-\mu_j - 2b + \frac{2}{\mu_j}\right)u - \frac{s}{\mu_j}$, the vector field $X_s(U, U_L)$ has the constant direction $\pm^t(1, \mu_j)$ and passing through the critical point, $X_s(U, U_L)$ changes the sign. It occurs only in the case of saddle points, which proves the proposition.

4 Structural Stability

Applying Theorem 3.3 and Proposition 2.2 to Theorem 2.3, a vector field $X_s(U_L, U)$ is structurally stable on the complement of $M_1 \cup M_3$ if and only if there are only a finite number of singularities and all are hyperbolic. Even if there are many variations of critical points as stated in Theorem 2.2, in any case, a vector field $X_s(U_L, U)$ has at most four critical points in bounded region and six critical points at infinity of U -plane and all of these are hyperbolic. Therefore we have

Theorem 4.1 *A vector field $X_s(U_L, U)$ is structurally stable on the complement of $M_1 \cup M_3$ in Case II.*

References

- [1] A. V. Azevedo-D. Marchesin-B. Plohr-K. Zumbrum, Bifurcation of Non-classical Viscous Shock Profiles from the Constant State, *Comm. Math. Phys.*, 202 (1999), 267–290.
- [2] A. A. Andronov-E. A. Leontovich-I. I. Gordon-A. G. Maier, *Qualitative Theory of Second-Order Dynamics Systems*, Wiley New York, 1966.
- [3] F. Asakura-M. Yamazaki, Geometry of Hugoniot curves in 2×2 systems of hyperbolic conservation laws with quadratic flux functions. preprint.
- [4] S. Čanić-B. J. Plohr, Shock Wave Admissibility for Quadratic Conservation Laws, *J. Differential Equations*, 118 (1995), 293–335.
- [5] G.-Q. Chen-P. T. Kan, Hyperbolic Conservation Laws with Umbilic Degeneracy, I, *Arch. Rational Mech. Anal.*, 130 (1995), 231–276.
- [6] C. Chicone, Quadratic gradients on the plane are generically Morse-Smale, *J. Differential Equations*, 33(1979), 159–166.
- [7] G. Darboux, Sur la forme des lignes de courbure dans le voisinage d'un omblic, *Leçon sur la Théorie Générale des Surfaces*, quatrième partie, Note VII, 448–465, Gauthier-Villars, Paris, 1896 (Chelsea, New York, 1972).
- [8] I. M. Gelfand, Some problems in the theory of quasilinear equations, *Uspekhi mat. Nauk.* 14 (1959) 87–158. English transl: *Amer. Math. Soc. Trans. Ser. 2* No 29 (1963).
- [9] M. E. S. Gomes, Riemann problems requiring a viscous profile entropy condition, *Adv. Appl. Math.* 10 (1989), 285–323.
- [10] E. Isaacson-D. Marchesin-B. Plohr-B. Temple, The Riemann Problem near a hyperbolic singularity: The classification of solutions of quadratic Riemann problems I, *SIAM J. Appl. Math.* 48 (1988), 1009–1032.
- [11] E. Isaacson-D. Marchesin-B. Plohr, Transitional waves for conservation laws, *SIAM J. Math. Anal.* 21 (1990), 837–866.
- [12] E. Isaacson-D. Marchesin-C. Palmeira-B. Plohr, A global formalism for nonlinear waves in conservation laws, *Comm. Math. Phys.* 146 (1992), 505–552.

- [13] E. Isaacson-B. Temple, The Riemann Problem near the hyperbolic singularity II, SIAM J, Appl. Math. 48 (1988), 1287–1301.
- [14] E. Isaacson-B. Temple, The Riemann Problem near the hyperbolic singularity III, SIAM J, Appl. Math. 48 (1988), 1302–1318.
- [15] B. Keyfitz-H. Kranzer, Existence and uniqueness of entropy solutions to the Riemann problem for hyperbolic systems of two conservation laws, J. Differential Equations, 27 (1978), 444–476.
- [16] P. D. Lax, Hyperbolic systems of conservation laws II, Comm. Pure Appl. Math., 10 (1957), 537–566.
- [17] P. D. Lax, Shock waves and entropy. *E. Zarantonello (ed.), Contributions to nonlinear Functional Analysis*, Academic Press, New York, 1971, 603–634.
- [18] T.-P. Liu, The Riemann problem for general 2×2 conservation laws, Trans. Amer. Math. Soc. 199 (1974), 89–112.
- [19] T.-P. Liu, Existence and uniqueness theorems for Riemann problems, Trans. Amer. Math. Soc. 212 (1975), 375–382.
- [20] T.-P. Liu, The Riemann problem for general systems of conservation laws, J. Differential Equations, 18 (1975), 218–234.
- [21] A. Majda-R. Pego, Stable Viscosity Matrices fo Systems of Conservation Laws, J. Diff. Eq. 56 (1985), 229–262.
- [22] J. W. Milnor, Topology from the Differentiable View Point, The University Press of Virginia, Charlottesville, 1965.
- [23] O. Oleĭnik, Discontinuous solutions of nonlinear differential equations, Usp. Mat. Nauk, 12, 1957, 3–73 (Amer. Math. Soc. Transl. Ser. 2, 26, 1957, 95–172).
- [24] M.-M. Peixoto, Structural stability on two-dimensional manifolds, Topology 1 (1962), 101–120.
- [25] D. Schaeffer-M. Shearer, The classification of 2×2 systems of non-strictly hyperbolic conservation laws, with applications to oil recovery, Comm. Pure Appl. Math., 40 (1987), 141–178.
- [26] D. Schaeffer-M. Shearer, The Riemann problem for nonstrictly hyperbolic 2×2 systems of conservation laws, Trans. Amer. Math. Soc., 304 (1987), 267–306.

- [27] S. Schechter-M. Shearer, Undercompressive shocks for nonstrictly hyperbolic conservation laws, *J. Dyn. Diff. Equations* 3 (1991), 199–271.
- [28] D. Serre, *Systèmes de lois de conservation I et II*, Diderot Editeur, Arts et Sciences, Paris, 1996 (*Systemes of conservation laws I & II*, Cambridge University Press, Cambridge, 1999).
- [29] M. Shearer, The Riemann problem for 2×2 systems of hyperbolic conservation laws with case I quadratic nonlinearities, *J. Diff. Equations*, 80 (1989), 343–363.
- [30] M. Shearer–D. Schaeffer–D. Marchesin–P. L. Paes-Leme, Solution of the Riemann problem for a prototype 2×2 system of nonstrictly hyperbolic conservation laws, *Arch. Rational Mech. Anal.*, 97 (1987), 299–320.
- [31] J. Smoller, *Shock Waves and Reaction-Diffusion Equations (2nd ed.)*, Springer Verlag, New York, 1994.
- [32] B. Wendroff, The Riemann problem for materials with nonconvex equations of state II: General flow, *J. Math. Anal. Appl.*, 38 (1972), 640–658.