

THEORETICAL RESEARCH ON
PHASE DYNAMICS AND INFORMATION PROCESSING OF
NEURONAL RHYTHMICAL NETWORKS

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Abstract

We theoretically probe neuronal rhythmical models based on the phase reduction theory. In the first two parts, plausible neuronal properties of asymmetry and multifrequencies in their natural frequencies are considered in phase oscillator models. Further, we analyze the dynamics using the Ott-Antonsen ansatz. We find that the asymmetry in natural frequency distributions leads to nonstandard transition in the Kuramoto model, and that different frequency distributions across populations along with the coupling function derived from phase reduction induces the clustered chimera states. In addition, we study information processing in coupled phase oscillators by introducing the transfer entropy in the last part. This research reveals that a type of coupling that does not always evoke synchronization has the maximum value in the transfer entropy rate.

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Chapter 1

Introduction

1.1 Rhythmic Phenomena and Synchronization

We can observe many various rhythms in the fields of physics, chemistry, biology, engineering and so on [104, 90, 117, 15]. For example, human has circadian clocks, which give us periodic time [117, 86]. Owing to the circadian clocks we have the daily and seasonal periodism. As another example, a frog croaks periodically and multiple frogs interact each other [5].

Some rhythmic systems consist of a large number of elements and exhibit various behaviors. Heart cell contracts periodically and, their population cooperate to show synchronized patterns. The synchronization induced by interactions among rhythmic elements can be observed in so many systems [104].

1.2 Neural Oscillation and Network Structure in Brain

The brain consists of a huge number of neurons. The neurons have connections between them and transfer information by firing [41, 22]. Many studies have reported that the neurons have specific properties to realize many functions [41, 16, 102]. Among them we focus on the rhythmic and network properties.

A single neuron with an external input generates subsequent action potentials and shows periodic motion. As a population of neurons the brain exhibits rhythmic behaviors, which can be observed by the EEG and fMRI [15]. The neural oscillation seems to be related to the cognitive functions of the brain and therefore have been investigated [16, 24].

Another point we focus on is the network structure in the brain. Neurons connect with each other through synapses and transmit information. Complex network structure in the brain have been observed at various scales [102]. Network structure plays an important role in many systems [8], and the similar situation is suggested in the brain to realize many functions. Therefore, we should pay attention to the connectivity networks among neurons.

1.3 Mathematical Modeling by Dynamical Systems

To study the rhythmic systems, they can be effectively described mathematically by the dynamical systems. The dynamical systems are dealt with ordinary differential equations and their time evolutions are studied.

Like any other system, a neuron has been described by dynamical systems and so many mathematical models have succeeded in extracting the dynamical properties of a neuronal electrical potential [39]. Concretely, the Hodgkin-Huxley model describes well neuronal dynamics [31] (Fig. 1.1).

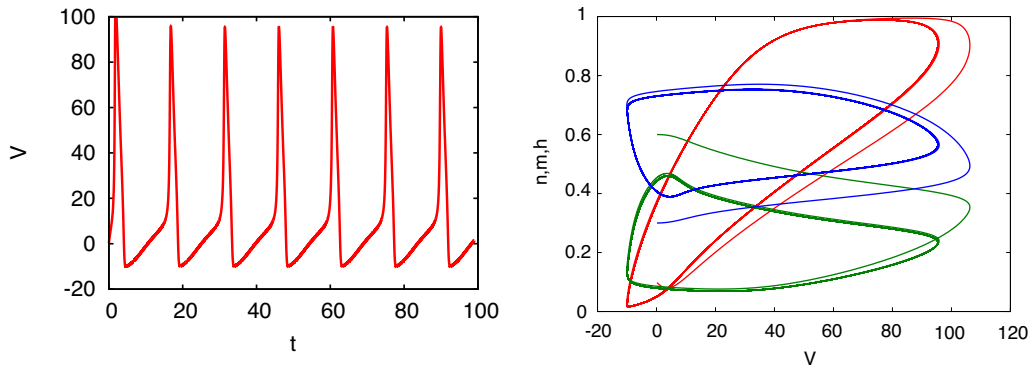


Figure 1.1: The Hodgkin-Huxley model with an external input. Left: The good mathematical model of a neuron exhibits a periodic behavior as seen in V - t diagram. Right: In the phase space we can see it attracting to a stable limit cycle. The red, blue and green lines represent the motions of channel variables n, m, h , respectively.

We extend our attention from a single element to a population consisting of many elements. Neurons in the brain, cells in the heart and fireflies generate rhythms spontaneously, and cooperate to adjust the individual timing, which leads often the synchronization. These systems are generally described by dynamical systems with a huge number of degrees of freedom, which induces difficulties on theoretical analyses or computations.

To overcome the difficulties we will employ the phase reduction theory [50] in our study. The phase reduction reduces rhythmic dynamical systems with a large number of degrees

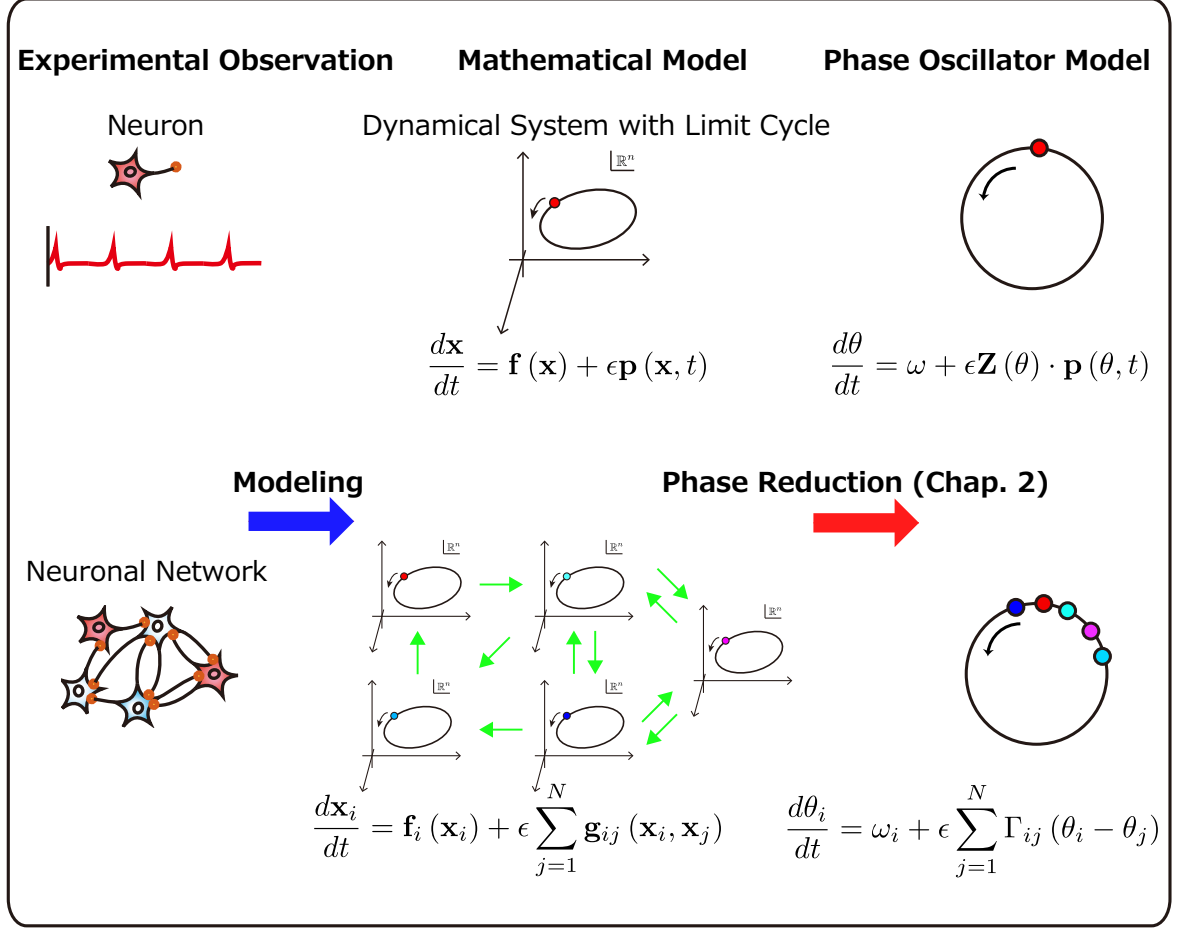


Figure 1.2: Schematic representation on our study.

of freedom to phase oscillator systems with phase variables (Fig. 1.2). This reduction is a powerful tool, which helps us analyze rhythmic systems. Our strategy is to make use of the phase reduction theory in order to understand neuronal rhythmic networks.

1.4 Structure of Thesis

From consideration above, we employ coupled phase oscillator models to study dynamics and information processing in rhythmic systems. The Kuramoto model [49] is a standard phase oscillator model, which has been extensively studied [103]. Owing to its simplicity, this model can be solved exactly. However, some simplifications cannot include realistic properties. We will study the phase oscillator systems by introducing two plausible properties.

One is the asymmetry in natural frequency distributions, which has often been assumed to be symmetric. In chapter 3, We will show that the asymmetry brings nonstandard transition in the Kuramoto model with the help of bimodality. Next, we will consider multifrequency systems. Many studies have assumed that the oscillators have similar natural frequencies. The breaking of this assumption will be considered in chapter 4.

Finally, we try to connect dynamics and information in coupled phase oscillator systems. To quantify causality we will introduce the transfer entropy [95], which is a information-theoretic measure. We will calculate the transfer entropy rate in coupled oscillators in chapter 6. Then, we will discuss on the relation between the coupling functions and information transfer.

The thesis is based on two papers published in [109, 111] and one paper in preparation [110].

Chapter 2

Review of Phase Oscillator Model

2.1 Phase Reduction

In this section we give a short review of the phase reduction theory [50]. The phase reduction theory reduces a limit-cycle oscillator with high dimensions to a simple phase oscillator, which permit us to analyze its dynamics easily. In fact, the phase reduction has been applied to various systems and succeeded in reducing their dynamics with some extensions [112, 51, 66, 97].

We consider nonlinear dynamics described by the equation

$$\frac{d\mathbf{x}(t)}{dt} = \mathbf{f}(\mathbf{x}(t)), \quad (2.1)$$

where $\mathbf{x}(t)$ and $\mathbf{f}(t)$ are n -dimensional vectors. This equation is assumed to have a stable limit-cycle solution χ , which we focus on in this section. Then, we define the phase θ on the limit cycle so that θ increases constantly with frequency $\omega = 2\pi/T$, where T is the period of the limit-cycle solution. In this process we regard a certain point A on χ as the origin of the phase $\theta = 0$ although this choice does not matter. By the definition above, we obtain the phase θ on χ , where $\theta = \omega t \pmod{2\pi}$ holds.

Next, we would extend the phase variable to the state $\mathbf{x}$ in the basin of χ not only on the limit cycle. As a natural choice, we define the phase not on the limit cycle as

$$\lim_{t \rightarrow \infty} |\mathbf{x}(t) - \mathbf{x}_0(\theta(t))| = 0, \quad (2.2)$$

where $\mathbf{x}_0(\theta(t))$ denotes the state on the limit cycle with the phase $\theta(t)$. This induces the map Θ which maps from a state $\mathbf{x}$ in the basin of χ to the phase value $\theta \in [0, 2\pi)$ as

$$\theta = \Theta(\mathbf{x}). \quad (2.3)$$

From this definition a state $\mathbf{x}(t)$ in the basin of χ has the corresponding phase $\theta(t)$, for which

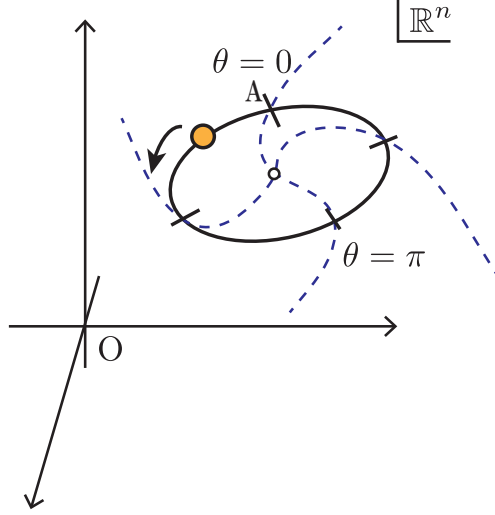


Figure 2.1: We can extend the phase definition to states not on the limit cycle.

$$\frac{d\theta(t)}{dt} = \omega \quad (2.4)$$

holds. From the fact by the chain rule that

$$\begin{aligned} \frac{d\theta(t)}{dt} &= \left. \frac{\partial \Theta(\mathbf{x})}{\partial \mathbf{x}} \right|_{\mathbf{x}=\mathbf{x}(t)} \cdot \frac{d\mathbf{x}(t)}{dt} \\ &= \left. \frac{\partial \Theta(\mathbf{x})}{\partial \mathbf{x}} \right|_{\mathbf{x}=\mathbf{x}(t)} \cdot \mathbf{f}(\mathbf{x}(t)), \end{aligned} \quad (2.5)$$

the function $\Theta(\mathbf{x})$ satisfies the equation

$$\left. \frac{\partial \Theta(\mathbf{x})}{\partial \mathbf{x}} \right|_{\mathbf{x}=\mathbf{x}(t)} \cdot \mathbf{f}(\mathbf{x}(t)) = \omega. \quad (2.6)$$

Further, we can consider similarly the limit cycle with perturbations such as external forces and interactions. We perturbate the limit-cycle oscillator (2.1) with a weak perturba-

tion $\epsilon \mathbf{p}(\mathbf{x}, t)$ as

$$\frac{d\mathbf{x}}{dt} = \mathbf{f}(\mathbf{x}) + \epsilon \mathbf{p}(\mathbf{x}, t), \quad (2.7)$$

where we suppose $|\epsilon| \ll 1$. Then,

$$\begin{aligned} \frac{d\theta(t)}{dt} &= \left. \frac{\partial \Theta(\mathbf{x})}{\partial \mathbf{x}} \right|_{\mathbf{x}=\mathbf{x}(t)} \cdot (\mathbf{f}(\mathbf{x}) + \epsilon \mathbf{p}(\mathbf{x}, t)) \\ &= \omega + \epsilon \left. \frac{\partial \Theta(\mathbf{x})}{\partial \mathbf{x}} \right|_{\mathbf{x}=\mathbf{x}(t)} \cdot \mathbf{p}(\mathbf{x}, t). \end{aligned} \quad (2.8)$$

To obtain a closed equation for the phase variable, we approximate Eq. (2.8) to

$$\frac{d\theta(t)}{dt} = \omega + \epsilon \left. \frac{\partial \Theta(\mathbf{x})}{\partial \mathbf{x}} \right|_{\mathbf{x}=\mathbf{x}_0(\theta(t))} \cdot \mathbf{p}(\mathbf{x}_0(t), t), \quad (2.9)$$

where $\mathbf{x}_0(\theta(t))$ denotes the state on the limit cycle with the same phase θ as the state $\mathbf{x}(t)$ from (2.2). This approximation holds for $\mathcal{O}(\epsilon)$. We denote

$$\mathbf{Z}(\theta) = \left. \frac{\partial \Theta(\mathbf{x})}{\partial \mathbf{x}} \right|_{\mathbf{x}=\mathbf{x}_0(\theta)} \quad (2.10)$$

and call $\mathbf{Z}(\theta)$ the phase sensitivity function. Then, we obtain a simple equation for the phase variable:

$$\frac{d\theta}{dt} = \omega + \epsilon \mathbf{Z}(\theta) \cdot \mathbf{p}(\theta, t). \quad (2.11)$$

As a simple example, we consider a pair of oscillators described by a pair of equations:

$$\frac{d\mathbf{x}_1}{dt} = \mathbf{f}_1(\mathbf{x}_1) + \epsilon \mathbf{g}_{12}(\mathbf{x}_1, \mathbf{x}_2), \quad (2.12)$$

$$\frac{d\mathbf{x}_2}{dt} = \mathbf{f}_2(\mathbf{x}_2) + \epsilon \mathbf{g}_{21}(\mathbf{x}_2, \mathbf{x}_1), \quad (2.13)$$

where $\mathbf{f}_1$ and $\mathbf{f}_2$ are the dynamics of the oscillators and $\mathbf{g}_{12}$ and $\mathbf{g}_{21}$ determine the interactions between them. We apply the phase reduction and the averaging method [93] to this system to obtain simple equations for phase variables. Here, for simplicity we assume that the natural frequencies of the two oscillators ω_1 and ω_2 , which are induced by the intrinsic dynamics $\mathbf{f}_1$

and $\mathbf{f}_2$, are similar, that is, we can set $\omega = \omega_1 = \omega_2 + \epsilon$. The other cases would be considered in the later chapter. By using (2.11) we apply the phase description to (2.12) and (2.13) and obtain the phase equations:

$$\frac{d\theta_1}{dt} = \omega_1 + \epsilon \mathbf{Z}_1(\theta_1) \cdot \mathbf{g}_{12}(\theta_1, \theta_2), \quad (2.14)$$

$$\frac{d\theta_2}{dt} = \omega_2 + \epsilon \mathbf{Z}_2(\theta_2) \cdot \mathbf{g}_{21}(\theta_2, \theta_1), \quad (2.15)$$

where $\mathbf{Z}_1(\theta_1)$ and $\mathbf{Z}_2(\theta_2)$ are the phase sensitivity functions for the oscillators 1 and 2, respectively. To separate the slow dynamics from (2.14) and (2.15), we define the slow phases ψ_1 and ψ_2 so that they satisfy $\theta_1 = \omega_1 t + \psi_1$ and $\theta_2 = \omega_2 t + \psi_2$. Then, the equation for the phases can be written as

$$\frac{d\psi_1}{dt} = \epsilon \mathbf{Z}_1(\psi_1 + \omega_1 t) \cdot \mathbf{g}_{12}(\psi_1 + \omega_1 t, \psi_2 + \omega_2 t), \quad (2.16)$$

$$\frac{d\psi_2}{dt} = \epsilon \mathbf{Z}_2(\psi_2 + \omega_2 t) \cdot \mathbf{g}_{21}(\psi_2 + \omega_2 t, \psi_1 + \omega_1 t). \quad (2.17)$$

We denote ω_1 as ω and average the right hand sides of (2.16) and (2.17) over the period $2\pi/\omega$. As a result we obtain a pair of the averaged phase equations as

$$\frac{d\psi_1}{dt} = \epsilon \Gamma_{12}(\psi_1 - \psi_2), \quad (2.18)$$

$$\frac{d\psi_2}{dt} = \epsilon \Gamma_{21}(\psi_2 - \psi_1), \quad (2.19)$$

where

$$\Gamma_{12}(\psi_1 - \psi_2) = \frac{\omega}{2\pi} \int_0^{2\pi/\omega} dt \mathbf{Z}_1(\psi_1 - \psi_2 + \omega t) \cdot \mathbf{g}_{12}(\psi_1 - \psi_2 + \omega t, \omega t),$$

$$\Gamma_{21}(\psi_2 - \psi_1) = \frac{\omega}{2\pi} \int_0^{2\pi/\omega} dt \mathbf{Z}_2(\psi_2 - \psi_1 + \omega t) \cdot \mathbf{g}_{21}(\psi_2 - \psi_1 + \omega t, \omega t).$$

This averaging approximation is valid to order ϵ . We return to the original phases θ_1 and θ_2 and get the averaged equations

$$\frac{d\theta_1}{dt} = \omega_1 + \epsilon \Gamma_{12}(\theta_1 - \theta_2), \quad (2.20)$$

$$\frac{d\theta_2}{dt} = \omega_2 + \epsilon \Gamma_{21}(\theta_2 - \theta_1), \quad (2.21)$$

where $\Gamma_{ij}(\theta)$ are 2π -periodic functions. What we should pay attention to is that the coupling functions $\Gamma_{ij}(\theta)$ depend only on the phase difference. We will study a concrete example in the next section using this result.

2.2 Kuramoto Model and Synchronization Transition

In this section we consider N -body problem in coupled phase oscillator systems. We analyze a simple paradigmatic model, the Kuramoto model, and by the self-consistent equation show that the system undergoes the synchronization transition. The self-consistent approach was originally done by Kuramoto and we review the analysis following Ref. [50].

From the previous section a wide class of the systems with N oscillators is reduced to a system described by

$$\frac{d\theta_i}{dt} = \omega_i + \sum_{j=1}^N \Gamma_{ij}(\theta_i - \theta_j), \quad (2.22)$$

where $\Gamma_{ij}(\theta)$ represents the coupling function between the oscillators i and j . The natural frequency ω_i is decided by the original nonlinear dynamics. As the representative model, the coupling function $\Gamma_{ij}(\theta)$ is chosen as $(K/N) \sin \theta$, and the equation (2.22) becomes

$$\frac{d\theta_i}{dt} = \omega_i + \frac{K}{N} \sum_{j=1}^N \sin(\theta_j - \theta_i), \quad (2.23)$$

where the oscillators couple to each other (the interaction is all-to-all) and the coupling strength K is homogeneous. The natural frequencies $\{\omega_i\}$ are supposed to obey a distribution $g(\omega)$, and we call $g(\omega)$ the natural frequency distribution. Without loss of generality we set the mode at $\omega = 0$. We also remark the important assumption on $g(\omega)$. Following the analysis by Kuramoto [50], we also assume that $g(\omega)$ is unimodal and symmetric. In other words, $g'(\omega) < 0$ for positive ω , and $g(\omega) = g(-\omega)$ holds. To investigate the extent of the synchrony in the system, we use the complex order parameter z , which is, as an analogy in the spin systems, defined as

$$z = r e^{i\psi} = \frac{1}{N} \sum_{j=1}^N e^{i\theta_j}. \quad (2.24)$$

The radius $r(\geq 0)$ represents the extent of the synchronization and the declination $\psi \in [0, 2\pi)$ does the mean direction.

To see the macroscopic behavior, by considering the limit $N \rightarrow \infty$ we derive the self-consistent equation from (2.23), which decides the solutions with constant r . From the definition (2.24), the equation (2.23) is written as

$$\frac{d\theta_i}{dt} = \omega_i + Kr \sin(\psi - \theta_i). \quad (2.25)$$

Then, we think of separating the oscillators into two groups, where the oscillators in one group have the fixed point for (2.25) but in the other they do not. In the former group with $|\omega_i| \leq Kr$, the oscillators become entrained because their natural frequencies do not deviate from the center $\omega = 0$. On the other hand, the oscillators in the latter fail to be entrained and drift. In the following we consider the density function of the oscillators $n(\theta, t)$, where $n(\theta, t) d\theta$ denotes the fraction of the oscillators with the phase θ at time t and $n(\theta, t)$ satisfies the normalized condition

$$\int_0^{2\pi} d\theta n(\theta, t) = 1. \quad (2.26)$$

We represent the density of the former group as $n_s(\theta, t)$ and the latter as $n_d(\theta, t)$, respectively.

We separate $n(\theta, t)$ as

$$n(\theta, t) = n_s(\theta, t) + n_d(\theta, t) \quad (2.27)$$

and calculate them separately. For the synchronized group, the equation (2.25) has a fixed point, which satisfies $\theta_i = \psi + \text{Sin}^{-1}(\omega_i/(Kr))$, where $\text{Sin}^{-1}(\cdot)$ represent the principal value between $-\pi/2$ and $\pi/2$. Therefore,

$$n_s(\theta, t) = g(\omega(\theta)) \left| \frac{d\omega}{d\theta} \right| \quad (2.28)$$

$$= \begin{cases} Kr \cos(\theta - \psi) g(Kr \sin(\theta - \psi)), & \text{if } |\theta - \psi| \leq \frac{\pi}{2} \\ 0. & \text{if } |\theta - \psi| > \frac{\pi}{2} \end{cases} \quad (2.29)$$

On the other hand, for the desynchronized oscillators the equation (2.25) does not have any fixed point. The probability $f(\theta|\omega_i)d\theta$ that the oscillator i with ω_i has the phase value between θ and $\theta + d\theta$ at time t is proportional to $|d\theta_i/dt|$, and

$$f(\theta|\omega_i) = \left| \frac{d\theta_i}{dt} \right|^{-1} \left| \int_{\theta_i=\theta}^{2\pi} d\theta_i \left| \frac{d\theta_i}{dt} \right|^{-1} \right)^{-1} \quad (2.30)$$

$$= \frac{\sqrt{\omega_i^2 - (Kr)^2}}{2\pi |\omega_i - Kr \sin(\theta - \psi)|}. \quad (2.31)$$

From this we can obtain the expression for $n_d(\theta, t)$ as

$$n_d(\theta, t) = \int_{|\omega| > Kr} d\omega g(\omega) f(\theta|\omega) \quad (2.32)$$

$$= \frac{1}{2\pi} \int_{|\omega| > Kr} d\omega g(\omega) |\omega + Kr \sin(\theta - \psi)| \frac{\sqrt{\omega^2 - (Kr)^2}}{\omega^2 - (Kr \sin(\theta - \psi))^2}. \quad (2.33)$$

Substituting Eqs. (2.27), (2.29) and (2.33) into Eq. (2.24), considering the even property of $g(\omega)$ and taking no contribution of the desynchronized oscillators to the order parameter into account, we obtain the self-consistent equation for r :

$$r = Kr \int_{-\pi/2}^{\pi/2} d\theta g(Kr \sin \theta) \cos \theta e^{i\theta}, \quad (2.34)$$

which has the trivial solution $r = 0$. We, therefore, are investigating the nontrivial solution $r \neq 0$, which satisfies

$$1 = K \int_{-\pi/2}^{\pi/2} d\theta g(Kr \sin \theta) \cos^2 \theta. \quad (2.35)$$

Focusing on the neighbor of the criticality, we expand the right hand side of (2.35) for $r \ll 1$ as

$$1 = \frac{\pi}{2} K g(0) - \frac{\pi}{16} K^3 g''(0) r^2 + \mathcal{O}(r^3). \quad (2.36)$$

From this form and $g''(0) < 0$, we find that at $K = K_c = 2/(\pi g(0))$ the nontrivial solution emerges continuously. Another method to find K_c will be introduced in the next chapter.

From the consideration above, we can suppose that the Kuramoto model shows the continuous transition as in figure 2.2. Especially, for a Lorentzian $g(\omega)$ the integral in Eq. (2.35)

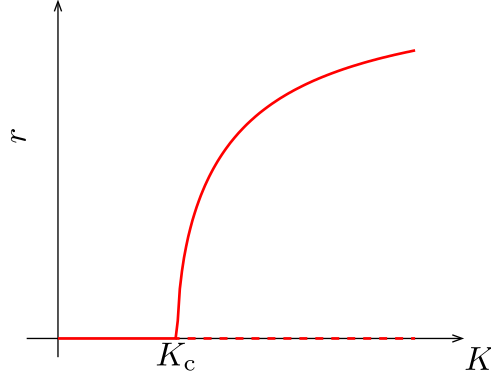


Figure 2.2: The Kuramoto model with symmetric unimodal distributions exhibits the continuous synchronization transition.

can be conducted and the exact solution is explicitly obtained. However, we remark that the result on the synchronization transition holds for the Kuramoto model with symmetric unimodal natural frequency distributions, and we would consider other cases in the next chapter.

2.3 Ott-Antonsen Ansatz

We showed in the previous section that the Kuramoto model with symmetric unimodal natural frequency distribution exhibits the continuous synchronization transition. On the other hand, recent work by Ott and Antonsen [76] reported a prominent theory on the coupled phase oscillators, which reduces the oscillator systems to low-dimensional systems. We review the Ott-Antonsen ansatz here by revisiting the Kuramoto model and derive the exact diagram.

We use the Kuramoto model described by Eq. (2.23) and assume that natural frequency distribution has the form of the Lorentzian distribution:

$$g(\omega) = \frac{\gamma}{\pi((\omega - \Omega)^2 + \gamma^2)}, \quad (2.37)$$

where the function $g(\omega)$ has the poles in the complex plane and this property permits us the use of the Ott-Antonsen ansatz as seen later. Considering the limit $N \rightarrow \infty$, we use

the probability density function $f(\theta, \omega, t)$ of oscillators, where $f(\theta, \omega, t)d\theta d\omega$ represents the fraction with phase between θ and $\theta + d\theta$ and natural frequency between ω and $\omega + d\omega$ at time t . Then, since the number of the oscillators conserves the equation of continuity

$$\frac{\partial f}{\partial t} + \frac{\partial}{\partial \theta}(fv) = 0 \quad (2.38)$$

holds. The velocity field v is defined from Eq. (2.23) as

$$v(\theta, \omega, t) = \omega + \frac{Kr}{2i} \left(e^{-i(\theta-\psi)} - e^{i(\theta-\psi)} \right). \quad (2.39)$$

Ott and Antonsen assume that the probability density function f can be expanded as

$$f(\theta, \omega, t) = \frac{g(\omega)}{2\pi} \left[1 + \sum_{n=1}^N \left(a(\omega, t)^n e^{in\theta} + \bar{a}(\omega, t)^n e^{-in\theta} \right) \right], \quad (2.40)$$

where $|a(\omega, t)| < 1$ is assumed for the convergence of f [76]. This form determines the conditional probability density function $f(\theta|\omega, t)$ as figure 2.3. Substituting Eq. (2.40) to

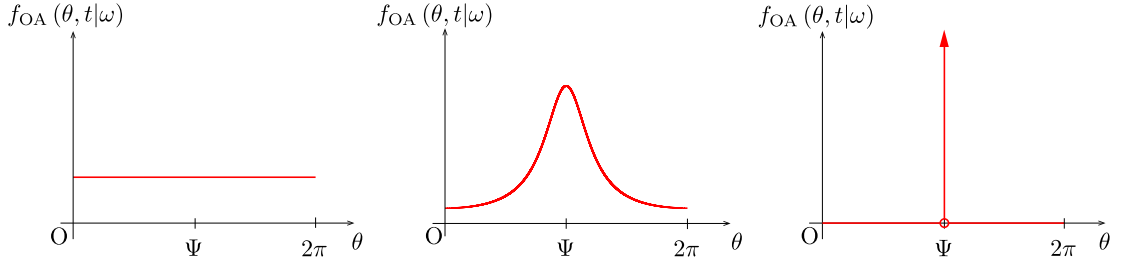


Figure 2.3: Equation (2.40) means that the conditional probability density function $f(\theta, t|\omega)$ has the form which is called the Poisson kernel. the left, center and right forms represent the cases for $|a(\omega, t)| = 0, 1/2, 1 - 0$, respectively. The center phase Ψ is the declination of $a(\omega, t)$ in the polar coordinates as $a(\omega, t) = |a(\omega, t)| e^{-i\Psi}$.

the equation of continuity, we obtain

$$\frac{\partial a}{\partial t} + i\omega a + \frac{K}{2} (a^2 z - \bar{z}) = 0. \quad (2.41)$$

Now, the order parameter z is expressed as

$$z(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega \int_0^{2\pi} d\theta f(\omega, \theta, t) e^{i\theta} \quad (2.42)$$

$$= \int_{-\infty}^{\infty} d\omega g(\omega) \bar{a}(\omega, t). \quad (2.43)$$

We remark that $g(\omega)$ is assumed to be Lorentzian (2.37) and therefore have the poles at $\omega = \Omega \pm i\gamma$. Using this fact and the residue theorem,

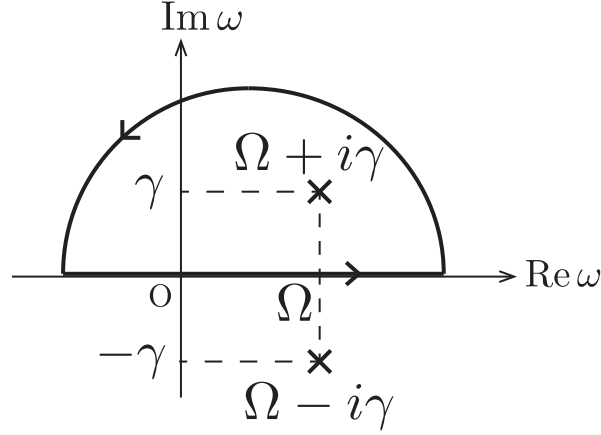


Figure 2.4: We consider the contour integral in complex plain to conduct the integral in Eq. (2.43).

$$z(t) = \bar{a}(\Omega + i\gamma) \quad (2.44)$$

is obtained. From this relation we obtain

$$\frac{dz}{dt} = \left(\frac{K}{2} - \gamma + i\Omega \right) z - \frac{K}{2} |z|^2 z, \quad (2.45)$$

which describes the order parameter dynamics with two degree of freedom. The polar form $z = re^{-i\psi}$ is used and the equation (2.45) is expressed as

$$\frac{dr}{dt} = \left(\frac{K}{2} - \gamma \right) r - \frac{K}{2} r^3, \quad (2.46)$$

$$\frac{d\psi}{dt} = \Omega t. \quad (2.47)$$

Since the equation (2.46) is closed for r we can straightforwardly obtain the same result on the synchronization transition to that by the self-consistent equation in the previous section.

The Ott-Antonsen ansatz is applicable for a wide class of coupled-phase-oscillator systems [59, 76, 57, 99, 85, 108, 83, 87] and we will use this ansatz in the following chapter.

Chapter 3

The Kuramoto Model with Asymmetric Distributions of Natural Frequencies

This chapter is constructed based on the published paper [111].

3.1 Introduction

Rhythmic phenomena are ubiquitous in various fields, such as biology, chemistry, engineering, physics and so on [50, 104, 117, 90]. Rhythmic phenomena are classified into the individual and collective levels, and individual rhythmic units cooperate to synchronize and organize the collective rhythmic motion. The individual rhythmic unit is mathematically described by a dynamical system having a limit cycle, and a limit-cycle oscillator is reduced to a phase oscillator by using the phase reduction technique [50], which extracts the rhythmic part from the limit-cycle oscillator. The phase reduction technique is applicable even under interactions with other oscillators as perturbations, and we can consider coupled phase-oscillators, which describes the synchronization.

The Kuramoto model is a paradigmatic phase-oscillator system, which has the all-to-all, mean-field interaction [49, 103, 4, 28]. In this model, each oscillator has the so-called natural frequency, which is constant in time and obeys a probability distribution. The coupling function between a pair of oscillators is represented by the sine function, where the argument

is just the phase difference between the coupled oscillators. This simple model describes the transition from the incoherent state to the partially synchronized state by strengthening the couplings even when the natural frequencies are not identical. This transition is called the synchronization transition, and is continuous when natural frequency distributions are symmetric and unimodal as originally considered [49] as seen in Sec. 2.2. The steady states (the incoherent and the partially synchronized states for instance) and the transitions are represented on bifurcation diagrams depending on parameters like the coupling strength. The bifurcation diagrams depend on the natural frequency distributions as well as the coupling function, and drawing bifurcation diagrams is one of the central issues in the coupled phase-oscillator systems as the phase diagrams in condensed matters.

The bifurcation diagram crucially depends on symmetry of the system as illustrated by the simple dynamical system $\dot{x} = ax - x^3 - h$ [105, 26]. In the case $h = 0$, where the equation is invariant with respect to the transformation $x \rightarrow -x$, the system undergoes the pitchfork bifurcation. On the other hand, when $h \neq 0$, the equation is asymmetric and the system exhibits not the pitchfork but the saddle-node bifurcation. Similarly, the Kuramoto model shows different bifurcation diagrams from the originally obtained one by breaking symmetry of the coupling function or of the natural frequency distributions.

We first turn our attention to the coupling function. In the Sakaguchi-Kuramoto model [92], which introduces the phase lag parameter in the sine coupling function of the Kuramoto model, the non-zero lag breaks the odd symmetry of the coupling function and the discontinuous transition can appear [74, 75]. The phase lag induces a further drastic change that one bifurcation diagram includes both a continuous and a subsequent discontinuous transitions, where the latter emerges not from the incoherent state but from the partially synchronized state [74].

Coming back to the original Kuramoto model, which is the target of this study, we focus on the natural frequency distributions, which have been assumed to be symmetric or unimodal so far. By breaking the symmetry of distributions, the synchronization transition can change from continuous to discontinuous ones [10, 11]. However, asymmetric unimodal distributions have not generated a bifurcation diagram including a continuous and a discontinuous transitions, which is reported in the Sakaguchi-Kuramoto model.

As discussed above, there are two ways to introduce asymmetry in the Kuramoto model, but asymmetric distributions have not been widely investigated except for the limited cases [10, 11]. Nevertheless, the asymmetric case should be considered, since the symmetry of natural frequency distributions is not trivial and slight asymmetry may induce qualitative changes in bifurcation diagrams. Therefore, the first question considered in this chapter is: Can we reproduce the nonstandard bifurcation diagram which includes the two transitions and is observed in the Sakaguchi-Kuramoto model by introducing asymmetry of the natural frequency distribution in the Kuramoto model?

Asymmetric unimodal distributions could not realize it yet. Our idea is, therefore, to break the other assumption of unimodality, and to consider asymmetric bimodal distributions. Before going to the second question, we review bifurcation diagrams in the symmetric bimodal case.

As well as the asymmetric unimodal distributions, symmetric bimodal ones give discontinuous synchronization transitions [12, 19, 59, 82, 88] but the prior continuous transition does not appear. Apart from the stationary states and extending objects to nonstationary states, we have the collective oscillation with noise [12, 19] and later without noise [59, 82, 88]. This oscillatory state directly bifurcates from the incoherent state as the partially synchronized state going through the continuous synchronization transition. Thus, the second question arises: Is it possible for the oscillatory state to bifurcate from the partially coherent state as the subsequent discontinuous transition observed in the Sakaguchi-Kuramoto model?

The purposes of this chapter are to answer the above two questions by introducing a family of smooth asymmetric bimodal distributions, and to demonstrate that the asymmetry with the bimodality in distributions plays a similar role to the asymmetry in the coupling function.

A simple way to draw bifurcation diagrams is to perform N -body simulations, but the finite-size fluctuation is an obstacle. The finite-size fluctuation is eliminated by taking the limit $N \rightarrow \infty$, and the Kuramoto model is described by the equation of continuity, but the infinite dimensionality is another difficulty. To overcome this difficulty, we introduce the Ott-Antonsen ansatz [76, 77, 78], which reduces the infinite-dimensional equation of continuity to a finite-dimensional dynamical system and succeeds in various oscillator systems

[59, 76, 57, 99, 108, 83, 109]. The theoretical results obtained by the ansatz are verified by comparing with direct N -body simulations.

This chapter is organized as follows. We introduce the Kuramoto model of coupled phase-oscillators in Sec. 3.2. The reduced system is derived by the Ott-Antonsen ansatz in Sec. 3.3. The theoretical predictions in the reduced system are compared with the N -body simulations in Sec. 3.4. We summarize this chapter in the final section 3.5.

3.2 The Kuramoto Model

The Kuramoto model is, as seen in the previous chapter, described by the equation (2.23), where $\theta_i \in [0, 2\pi)$ represents the phase of the i th oscillator and real ω_i the natural frequency. The natural frequencies are obeyed by the probability density function $g(\omega)$, where $g(\omega) d\omega$ is a fraction of oscillators having natural frequency between ω and $\omega + d\omega$. The coupling function is chosen as to be odd and 2π -periodic, and sine is one of the simplest choices. We assumed that the interaction is all-to-all, and that the coupling strength $K > 0$ does not depend on coupling pairs. The order parameter (2.24) measures the extent of synchronization by $r \geq 0$ and the collective phase by ψ for the N oscillators.

In the limit of $N \rightarrow \infty$, we consider the probability density function $f(\theta, \omega, t)$, where $f(\theta, \omega, t) d\theta d\omega$ denotes the fraction of oscillators with phase between θ and $\theta + d\theta$ and natural frequency between ω and $\omega + d\omega$ at time t under the normalization condition

$$\int_{-\infty}^{\infty} d\omega \int_0^{2\pi} d\theta f(\theta, \omega, t) = 1 \quad (3.1)$$

which induces

$$\int_0^{2\pi} d\theta f(\theta, \omega, t) = g(\omega). \quad (3.2)$$

Since the number of the oscillators conserves, the probability density function satisfies the equation of continuity (2.38). By defining the velocity field v as

$$v(\theta, \omega, t) = \omega + K \int_{-\infty}^{\infty} d\omega' \int_0^{2\pi} d\theta' f(\theta', \omega', t) \sin(\theta' - \theta), \quad (3.3)$$

the equation of continuity corresponds to the evolution equations of the phase-oscillators (2.23) [55].

In the continuous limit, the order parameter is

$$z = r e^{i\psi} = \int_{-\infty}^{\infty} d\omega \int_0^{2\pi} d\theta f(\theta, \omega, t) e^{i\theta}. \quad (3.4)$$

For a symmetric and unimodal $g(\omega)$, after some calculations, Kuramoto derived the self-consistent equation for r which must be satisfied in stationary states [50]. The self-consistent equation (2.34) has the unique trivial solution $r = 0$ for $K \leq K_c$, but has one more positive solution for $K > K_c$ as discussed in the previous chapter. The system exhibits a continuous spontaneous synchronization above K_c , where $K_c = 2/[\pi g(0)]$ [49, 103, 28, 94]. However, the breaking of symmetry or unimodality may change the continuity of transitions. Indeed, asymmetric unimodal [10, 11] and symmetric bimodal [59, 82] cases produce discontinuous transitions.

In this chapter, to investigate the asymmetric bimodal case, we consider the following family of smooth natural frequency distributions:

$$g(\omega) = \frac{c}{\left[(\omega - \Omega)^2 + \gamma_1^2\right] \left[(\omega + \Omega)^2 + \gamma_2^2\right]}, \quad (\Omega, \gamma_1, \gamma_2 > 0) \quad (3.5)$$

where

$$c = \frac{\gamma_1 \gamma_2 \left[(\gamma_1 + \gamma_2)^2 + 4\Omega^2\right]}{\pi (\gamma_1 + \gamma_2)} \quad (3.6)$$

is the normalization constant. This family, whose form is the product of two Lorentzian distributions, can be bimodal and/or asymmetric depending on parameters Ω , γ_1 , and γ_2 . We note that the Kuramoto model (2.23) is invariant under appropriate scalings of $t, \omega_i, K, \Omega, \gamma_1$ and γ_2 . Thus, we impose the arbitrariness of γ_2 to the other parameters and set $\gamma_2 = 1$ without loss of generality. Figure 3.1 represents the diagram of the distribution forms. We explore only the left-half plane from the symmetry line without loss of generality, since bifurcations in the other side are replicated by putting $\theta \rightarrow -\theta$.

3.3 The Ott-Antonsen Reduction

The Ott-Antonsen ansatz [76, 77, 78] is a powerful tool to reduce the infinite-dimensional equations of continuity to finite-dimensional ordinary differential equations. In practice,

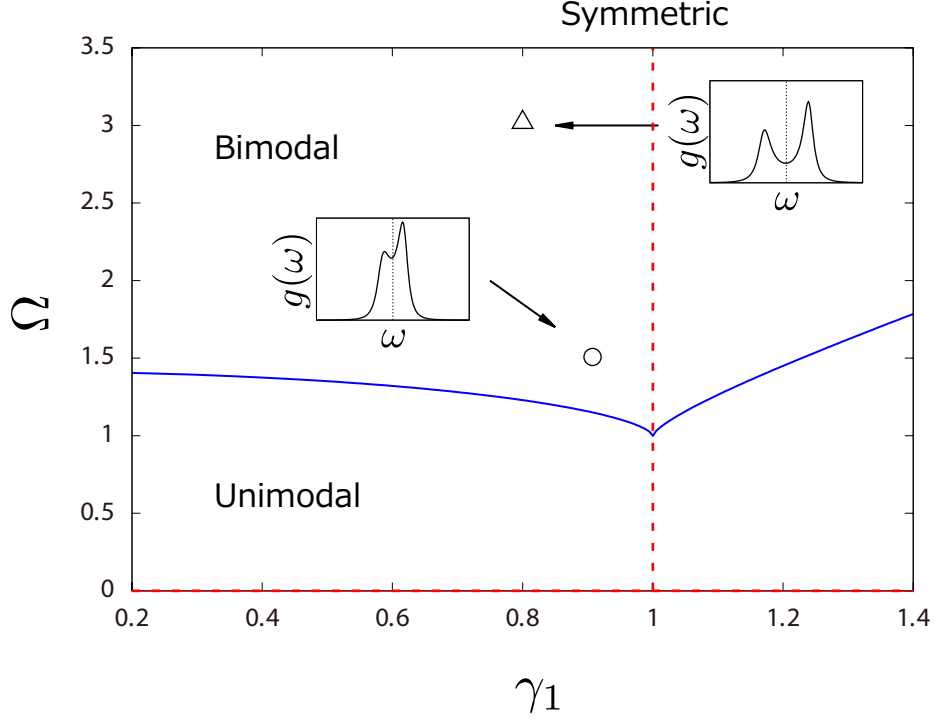


Figure 3.1: Classification of the family of distributions (3.5) with $\gamma_2 = 1$. The red dashed lines denote the symmetric distributions. The blue solid lines are the borders between the unimodality and the bimodality. The circle and the triangle represent the parameter sets studied in Sec. 3.4. (insets) Forms of the two distributions, where the dashed vertical lines denote $\omega = 0$.

many works succeed in reducing the dynamics of the coupled phase-oscillators [59, 76, 57, 99, 85, 108, 83, 109]. In this section, we employ the ansatz and derive the reduced equation to obtain precise bifurcation diagrams.

The Ott-Antonsen ansatz [76] introduces the Fourier expansion of $f(\theta, \omega, t)$ as

$$f(\theta, \omega, t) = \frac{g(\omega)}{2\pi} \left[1 + \sum_{n=1}^{\infty} \left(a(\omega, t)^n e^{in\theta} + \bar{a}(\omega, t)^n e^{-in\theta} \right) \right], \quad (3.7)$$

where we assume $|a(\omega, t)| < 1$ to ensure convergence of the sum. From the assumption (3.7) and the continuous equation (2.38), the Kuramoto model with the distribution (3.5) is reduced to the system of the two complex variables A and B (see appendix 3.A for the derivation):

$$\frac{dA}{dt} = (i\Omega - \gamma_1) A - \frac{K}{2} [(\bar{k}_1 \bar{A} + \bar{k}_2 \bar{B}) A^2 - (k_1 A + k_2 B)], \quad (3.8)$$

$$\frac{dB}{dt} = -(i\Omega + \gamma_2) B - \frac{K}{2} [(\bar{k}_1 \bar{A} + \bar{k}_2 \bar{B}) B^2 - (k_1 A + k_2 B)], \quad (3.9)$$

where we set the constant parameters k_1 and k_2 as

$$k_1 = \frac{\gamma_2 [2\Omega - i(\gamma_1 + \gamma_2)]}{(\gamma_1 + \gamma_2) [2\Omega + i(\gamma_1 - \gamma_2)]}, \quad k_2 = \frac{\gamma_1 [2\Omega + i(\gamma_1 + \gamma_2)]}{(\gamma_1 + \gamma_2) [2\Omega + i(\gamma_1 - \gamma_2)]}. \quad (3.10)$$

We have fixed as $\gamma_2 = 1$, but the symbol γ_2 remains to clarify how it appears in the above equations. We note that the relation $k_2 = \bar{k}_1$ holds if the distribution is symmetric, namely $\gamma_1 = \gamma_2$. The complex order parameter z is represented by A and B as

$$z = k_1 A + k_2 B, \quad (3.11)$$

and we concentrate on the amplitude $r = |z|$, which expresses the extent of the synchronization.

We construct bifurcation diagrams by searching steady states in which the amplitude $r(t)$ is constant or oscillatory. The former and the latter states in the reduced system correspond to the stationary states and the oscillatory states, respectively in the original system. The formers are further classified into the trivial fixed-point solution of $A = B = 0$ corresponding to the incoherent state with $r = 0$, and nontrivial solutions with constant $r > 0$ corresponding to the partially synchronized states. Stability of the trivial solution is analyzed by considering the 4×4 Jacobian matrix derived from Eqs. (3.8) and (3.9). For detecting the nontrivial solutions, we perform one more reduction since r depends on r_A, r_B and ϕ only, where we introduced the polar forms $A = r_A e^{-i\phi_A}, B = r_B e^{-i\phi_B}$ and the phase difference variable $\phi = \phi_A - \phi_B$. The three variables are obeyed by the equations

$$\frac{dr_A}{dt} = -\gamma_1 r_A + \frac{K}{2} (1 - r_A^2) (\text{Re} k_1 r_A + \text{Re} k_2 r_B \cos \phi + \text{Im} k_2 r_B \sin \phi), \quad (3.12)$$

$$\frac{dr_B}{dt} = -\gamma_2 r_B + \frac{K}{2} (1 - r_B^2) (\text{Re} k_1 r_A \cos \phi - \text{Im} k_1 r_A \sin \phi + \text{Re} k_2 r_B), \quad (3.13)$$

$$\begin{aligned} \frac{d\phi}{dt} = & 2\Omega + \frac{K}{2} \left[(1 + r_A^2) \left(\text{Im} k_1 + \text{Im} k_2 \frac{r_B}{r_A} \cos \phi - \text{Re} k_2 \frac{r_B}{r_A} \sin \phi \right) \right. \\ & \left. - (1 + r_B^2) \left(\text{Im} k_2 + \text{Im} k_1 \frac{r_A}{r_B} \cos \phi + \text{Re} k_1 \frac{r_A}{r_B} \sin \phi \right) \right], \end{aligned} \quad (3.14)$$

where $\text{Re } k_i$ and $\text{Im } k_i$ represent the real and the imaginary parts of k_i ($i = 1, 2$), respectively. The nontrivial solutions are found as the fixed points of the 3-dimensional system, and their stabilities are sufficiently captured by the associated 3×3 Jacobian matrix. Based on this

discussion, we search the oscillatory states by performing numerical simulations of the 3-dimensional system.

3.4 Bifurcation Diagrams

In this section we obtain two new types of bifurcation diagrams by computing fixed points of the reduced equations (3.12)-(3.14) theoretically, and oscillating solutions numerically. The bifurcation diagrams are validated by comparing with direct N -body simulations of Eq. (2.23). All the simulations of the reduced system and the N -body system are performed by the fourth-order Runge-Kutta method with the time step $dt = 0.01$. The synchronization and the collective oscillation are investigated by calculating the averages and the standard deviations of r in the time interval $t \in [4500, 5000]$, in which we confirmed that the system reaches a steady state. In section 3.4.1, we show the first bifurcation diagram consisting of stationary states corresponding to the fixed points of the reduced equations (3.12)-(3.14). In section 3.4.2, the second bifurcation diagram is reported which includes the oscillatory states corresponding to the limit-cycle solutions of the reduced system.

3.4.1 Bifurcations of Stationary States

We here concentrate on exploring nontrivial fixed-point solutions by setting $dr_A/dt = dr_B/dt = d\phi/dt = 0$ in Eqs. (3.12)-(3.14). Analytical solutions for the parameter set $(\Omega, \gamma_1) = (1.5, 0.9)$, which makes $g(\omega)$ asymmetric and bimodal, are plotted as lines in Fig. 3.2 with the trivial solution $r = 0$. We marked K_c derived theoretically from the equation of continuity (see appendix 3.B) at which the incoherent state changes the stability. The derived K_c is in good agreement with the analysis of the reduced system. At K_c , the stable incoherent state continuously bifurcates into the unstable incoherent state and the partially synchronized state as the symmetric unimodal case. Further increasing K , we find the discontinuous transition and coexistence of the two partially synchronized stable branches. The subsequent discontinuous transition is observed in the Sakaguchi-Kuramoto model [74], but according to our best knowledge, this type of bifurcation diagram has not been discovered in the Kuramoto model.

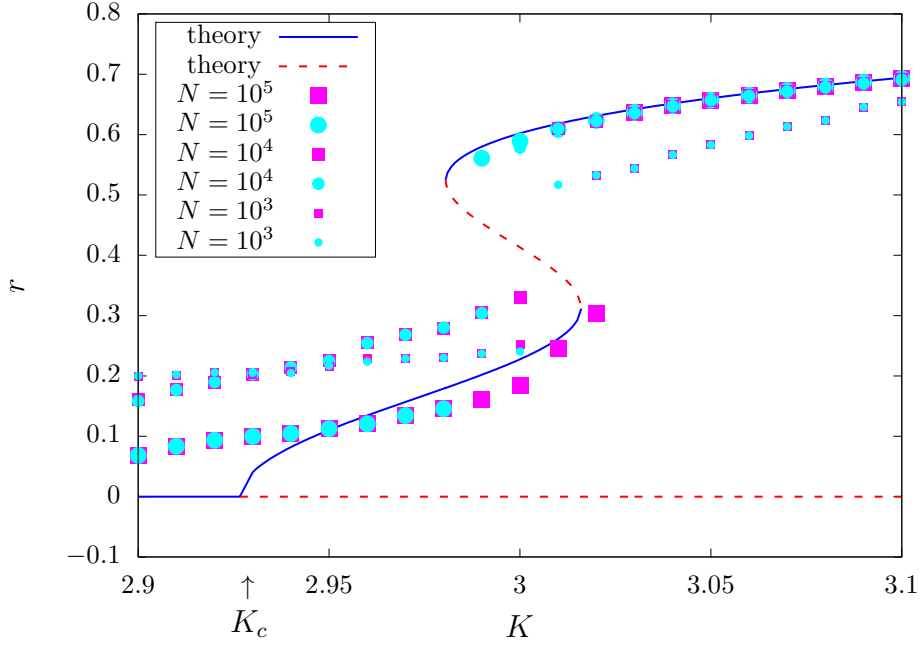


Figure 3.2: Bifurcations of stationary states in the Kuramoto model for $(\Omega, \gamma_1) = (1.5, 0.9)$. The lines are theoretically obtained fixed-point solutions to the reduced equations (3.12)-(3.14). The blue solid and the red dashed lines represent the stable and the unstable branches, respectively. The points are the time averages in the N -body simulations. The triangles and the inverse triangles indicate the forward and the backward processes, respectively, where the forward process increases K with $dK = 0.01$ whereas the backward process decreases K after the forward simulation. The sizes of the points indicate the system sizes.

To verify the bifurcation diagram obtained under the Ott-Antonsen ansatz, we numerically simulate the N -body system. Here, we set $N = 10^3, 10^4, 10^5$ and integrate Eq. (2.23) with natural frequencies randomly generated from the distribution Eq. (3.5). To see the hysteresis in the discontinuous transition the simulations are performed with firstly increasing and subsequently decreasing the coupling strength K , where we call the former process with the increase of K the forward process and the latter the backward process, respectively. The forward simulation denoted as the triangles in Fig. 3.2 increases K with the step $dK = 0.01$ from $K = 2.9$ to $K = 3.1$ and the initial state is set to the previous final state at each K except for $K = 2.9$ in which the initial conditions of the phase variables are chosen randomly from the uniform distribution on the circle. The backward process is represented as the inverse triangles in Fig. 3.2, which is realized by decreasing in the same interval $[2.9, 3.1]$ of K and the step $dK = 0.01$. All the numerically computed points, and the discontinuous

transition in particular, are qualitatively in good agreements with the Ott-Antonsen analysis. Consequently, the theoretically obtained new bifurcation diagram has been verified.

3.4.2 Bifurcations with Oscillatory States

Previous works on the Kuramoto model with symmetric bimodal distributions [59, 82] reported that the reduced system has not only the fixed-point solutions but also limit-cycle solutions, which correspond to the collective oscillations. We search oscillatory states in our asymmetric bimodal case by performing numerical simulations of the reduced system. The main strategy is to increase the parameter Ω representing the distance between the two peaks of the distribution $g(\omega)$ from the value examined in Sec. 3.4.1.

For the parameter set $(\Omega, \gamma_1) = (3.0, 0.8)$, the averages and the standard deviations of $r(t)$ are plotted in Fig.3.3 for the steady states realized after transient durations. Large standard deviations indicate the oscillation of r , and the oscillatory states are observed in our asymmetric bimodal case around the interval $[5, 6]$ of K .

We give two remarks on the bifurcation diagram. First, the discontinuous transition emerges from the oscillatory states, and the hysteresis is confirmed by performing the forward and the backward processes as done in the previous section 3.4.1 but for the reduced system with the step $dK = 0.05$, where we set $r_A, r_B \sim 0$ and ϕ to the arbitrary constant as the initial state of the forward process and give the perturbation to r_A, r_B at each K to see the stability. Accordingly, the coexistence of an oscillatory and a stationary states is observed as in the symmetric bimodal case [59, 82, 88]. Second, the oscillatory state appear after the continuous transition, and arise from the partially synchronized states around $K = 5$. This emergence manner of the collective oscillation shows a sharp contrast with the symmetric bimodal case, in which it arises from the incoherent state.

We examine the bifurcation diagram obtained in the reduced system by performing N -body simulations at the points A ($K = 4.0$), B ($K = 5.5$), C ($K = 6.0$), D ($K = 6.5$) in the forward process and E ($K = 6.0$) in the backward process marked in Fig. 3.3. At each point, temporal evolutions of $r(t)$ are compared between the reduced system and the N -body system in Fig. 3.4. For the comparisons, we shifted the N -body lines properly, since the initial phase of the collective oscillation do not matter. The two lines show good agreements, even under the finite-size fluctuation.

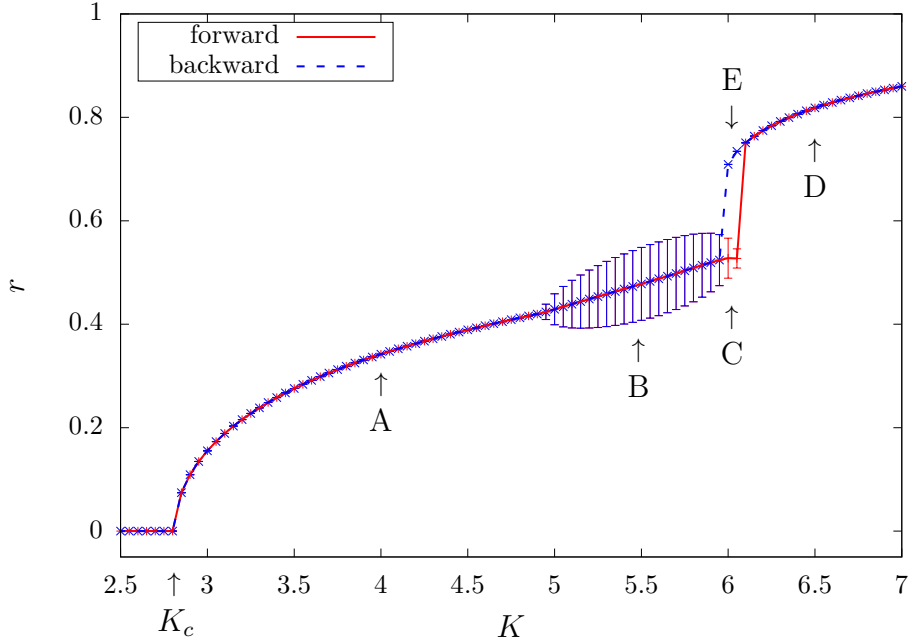


Figure 3.3: Bifurcation diagram for $(\Omega, \gamma_1) = (3.0, 0.8)$ obtained by simulating the reduced 3-dimensional system. The averages of the order parameter are plotted by points, which are connected with lines to emphasize the continuous and the discontinuous transitions at $K = K_c$ and around $K = 6$, respectively. The standard deviations are represented by vertical bars associated with the points, where large deviations indicate oscillation of the order parameter. Red (blue) points and lines are for the forward (backward) process, where the forward process increases K with $dK = 0.05$ whereas the backward process decreases K after the forward simulation. For the points A, B, C, D and E, temporal evolutions of the order parameter are plotted in Fig. 3.4, and synchronizations on the (θ, ω) plane in Fig. 3.5.

To look into the oscillatory states observed here, in Fig. 3.5 we represent the snapshots of $\{(\theta_i, \omega_i)\}$ in the steady states of the N -body simulations at the five points of A, B, C, D and E. The population of the oscillators forms only one synchronized cluster in Figs. 3.5 (a), (d) and (e), but the two clusters exist in Figs. 3.5 (b) and (c). The two-cluster states are called the traveling wave states, which were observed in systems with asymmetric distributions [84] and with random coupling [84, 34]. We observe that existence of the second smaller cluster induces the oscillating states, and coalescing of the two cluster triggers the discontinuous transition. If we had a symmetric bimodal natural frequency distribution, the two groups of the synchronized oscillators would emerge simultaneously as seen in [59, 82, 88], and the oscillatory states would emerge at K_c from the incoherent state. However, the asymmetry permits the two groups to synchronize separately, and provides the continuous synchronization transition at K_c . Another interesting observation is that the natural frequencies in the

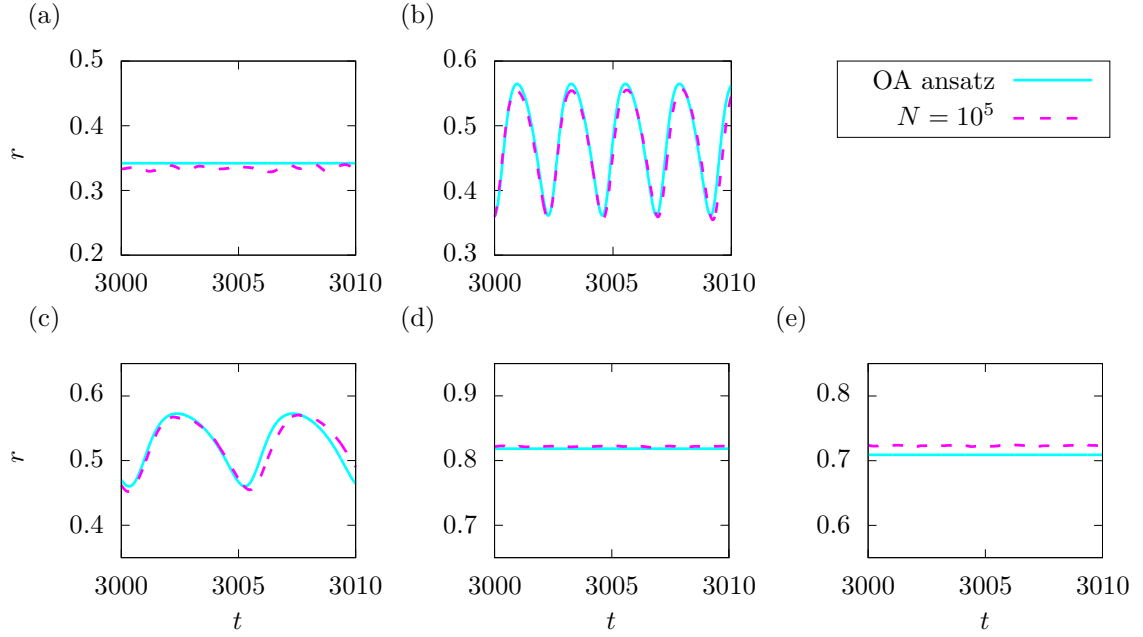


Figure 3.4: Comparison of time evolutions of the order parameters between the reduced system (solid light blue lines) and N -body system (dashed magenta lines) for $(\Omega, \gamma_1) = (3.0, 0.8)$. The examined points are marked in Fig. 3.3: (a) A, (b) B, (c) C, (d) D from the forward process and (e) E from the backward process.

second cluster are not located at the second peak of the natural frequency distribution as shown in Fig. 3.5 (f).

3.5 Chapter Summary

In this chapter we studied bifurcation diagrams in the Kuramoto model having asymmetric bimodal distributions of natural frequencies. With the aid of the Ott-Antonsen ansatz [76], we theoretically found the two new bifurcation diagrams: One includes continuous and subsequent discontinuous transitions and the other shows the oscillatory state bifurcating from the partially coherent states. These theoretical results were validated by comparing with direct N -body simulations.

The first diagram we found has been reported in the Sakaguchi-Kuramoto model with the symmetric unimodal distribution. However, according to our best knowledge, such a bifurcation diagram has not been observed in the Kuramoto model, and this is the first observation by introducing asymmetric bimodal distributions.

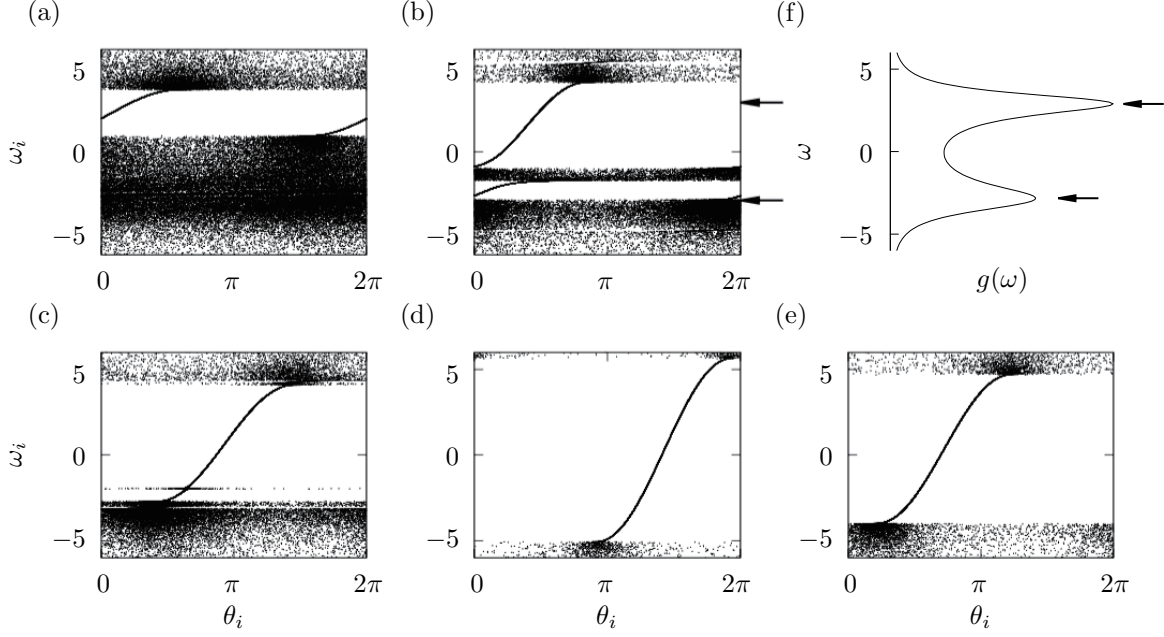


Figure 3.5: Snapshots of $\{(\theta_i, \omega_i)\}$ in the N -body simulations for $(\Omega, \gamma_1) = (3.0, 0.8)$ with $N = 10^5$. The examined points are marked in Fig. 3.3: (a) A, (b) B, (c) C, (d) D from the forward process and (e) E from the backward process. The natural frequency distribution is displayed in the panel (f) for comparing with the panel (b) in particular. The two peaks of the distribution are pointed to by arrows in the panels (b) and (f).

The second diagram has the oscillatory states, in which the magnitude of the order parameter oscillates. With the symmetric bimodal distributions the oscillatory states have been reported to bifurcate from the incoherent state [59, 82, 88]. However, the asymmetry lets the oscillatory state emerge from partially synchronized state in the bifurcation diagram. This emergence happens because the synchrony begins around the higher peak, which forms the first cluster, and the second synchrony subsequently occurs, which forms the relatively traveling second cluster as discussed in the section 3.4.2. It is worth noting that the second synchrony is not located at the center of the lower peak.

From the perspective obtained by the second diagram, we reconsider the first diagram. Contrastingly with the second bifurcation diagram, we numerically confirmed by the N -body simulations that the first diagram has no second cluster. We suppose that, as the separation of the two peaks become smaller, the oscillatory states tend to disappear and the bifurcation diagram approaches from the second bifurcation diagram to the first.

We give the two remarks to emphasize the worth of this study. First, Kuramoto sketched the successive synchronies observed in the second bifurcation diagram for the asymmetric

bimodal case in [50], where the first and the second synchronies were expected to appear around the higher and the lower peaks of $g(\omega)$, respectively, and to be isolated. Beyond the sketch, this article has revealed that the second cluster may not be located around the center of the lower peak as shown in Fig. 3.5, and may not appear as in the first diagram. Second, we demonstrated that the asymmetry of natural frequency distributions discloses the bifurcation diagrams hidden in the symmetric case, as the Sakaguchi-Kuramoto model can do by introducing asymmetry in the coupling function. Indeed, keeping the peak-separation parameter Ω and setting the distribution to be symmetric, we could not find the reported new bifurcation diagrams. We, therefore, conclude that the asymmetry in the natural frequency distributions plays a similar role to the asymmetry in the coupling function, introduced in the Sakaguchi-Kuramoto model.

We note that similar bifurcation diagrams to the reported diagrams in this paper were observed in the noisy Kuramoto model with delta-type trimodal natural frequency distributions [3]. It is not straightforward to construct connection between the Kuramoto model with the noisy Kuramoto model having the delta-type distributions, but we speculate an analogy between the symmetric trimodal case and the asymmetric bimodal case in the noiseless Kuramoto model. For simplicity we consider a smooth symmetric trimodal distribution whose highest peak is at the origin. Then, we conjecture that the first synchronization happens at the highest center peak without oscillation, and that the second occurs around the other two peaks simultaneously. Appearance of oscillation in the second synchronization may depend on the distance of the lower two peaks from the origin. This two-step synchronization is similar to the asymmetric bimodal case which has been revealed in this article. In other words, the symmetry reduces the number of synchronization steps from the number of peaks, while the asymmetry may keep it. Confirmation of this conjecture has to be done.

We further discuss on the asymmetry by using analogies between the Kuramoto model and Hamiltonian systems. We start from referring to two analogies. The Kuramoto model shares macroscopic aspects with a Hamiltonian system in the critical exponents concerning with the applied external force (see [32, 67, 33, 21] for the Kuramoto model and [81, 69, 70, 71, 119] for the Hamiltonian system). Another analogy is used in the analysis on the Landau damping [56], which is originally studied in plasma physics, and is applied to the Kuramoto model [106]. Taking account of the above analogies, the analyticity of natural

frequency distributions is expected to change macroscopic dynamics in the Kuramoto model, since non-analytic momentum distributions in a Hamiltonian system, which corresponds to the natural frequency distributions in the Kuramoto model, make the exponential Landau damping algebraic [115]. This perspective tells us that the study on the asymmetric unimodal case has not completed, because the previously studied distributions are not differentiable [10, 11]. Our preliminary researches did not find nonstandard bifurcation diagrams in the analytic asymmetric unimodal case, but further investigations are necessary. It is still an open question what is the whole role of asymmetry. For instance, an asymmetric unimodal momentum distribution makes the susceptibility tensor non-diagonal even in a simple Hamiltonian system [118]. Thus, it might be interesting to study the response to an external force in the Kuramoto model with asymmetric natural frequency distributions.

3.A Derivation of the Reduced Equations

We derive the reduced equations (3.8) and (3.9) using the Ott-Antonsen ansatz from the original equation (2.23) and the natural frequency distribution (3.5).

We consider the Kuramoto model in the continuous limit $N \rightarrow \infty$, and from Eq. (2.23) the velocity of the phase can be written as

$$v(\theta, \omega, t) = \omega + \frac{K}{2i} \left(z e^{-i\theta} - \bar{z} e^{i\theta} \right), \quad (3.15)$$

where we denote the complex order parameter (3.4) as $z(t) = r e^{i\psi}$. The Ott-Antonsen ansatz [76] assumes that the probability distribution function $f(\theta, \omega, t)$ is expanded into the Fourier series as Eq. (3.7). Substituting the assumed $f(\theta, \omega, t)$ (3.7) and the velocity (3.15) into the continuous equation (2.38), we obtain the equation for $a(\omega, t)$ and $z(t)$ as

$$\frac{\partial a}{\partial t} + i\omega a + \frac{K}{2} (a^2 z - \bar{z}) = 0. \quad (3.16)$$

From Eqs. (3.4) and (3.7), we also find

$$z(t) = \int_{-\infty}^{\infty} d\omega g(\omega) \bar{a}(\omega, t). \quad (3.17)$$

From the complex conjugate of Eq. (3.16), $\partial \bar{a} / \partial t \sim -(\text{Im } \omega) \bar{a}$ as $\text{Im } \omega \rightarrow \infty$. We then find that $\bar{a}(\omega, t) \rightarrow 0$ as $\text{Im } \omega \rightarrow \infty$. The vanishing $\bar{a}(\omega, t)$ permits us to add the upper semicircular contour to the integration of Eq. (3.17) in the complex ω plane for using the residue theorem. Thus, assuming analyticity of $\bar{a}(\omega, t)$ in $\text{Im } \omega > 0$ [76], we have

$$z(t) = k_1 \bar{a}(\Omega + i\gamma_1, t) + k_2 \bar{a}(-\Omega + i\gamma_2, t), \quad (3.18)$$

because the distribution (3.5) has two poles at $\Omega + i\gamma_1$ and $-\Omega + i\gamma_2$ in the upper half plane. Temporal evolutions of the complex variables

$$A(t) = \bar{a}(\Omega + i\gamma_1, t), \quad B(t) = \bar{a}(-\Omega + i\gamma_2, t), \quad (3.19)$$

are obtained from the complex conjugate of Eq. (3.16) by setting ω as $\omega = \Omega + i\gamma_1$ and $\omega = -\Omega + i\gamma_2$, respectively, and are expressed by Eqs. (3.8) and (3.9).

3.B Calculation of K_c from the Linear Stability Analysis in the Continuum Equation

We here derive the equation determining the critical coupling strength K_c , at which the incoherent solution to the equation of continuity (2.38)

$$f_0(\theta, \omega) = \frac{g(\omega)}{2\pi} \quad (3.20)$$

is destabilized, where $g(\omega)$ is the natural frequency distribution. The critical strength K_c is obtained by the linear stability analysis. We consider a perturbed solution $f(\theta, \omega, t)$ with a perturbation $\epsilon f_1(\theta, \omega, t)$ as

$$f(\theta, \omega, t) = f_0(\theta, \omega) + \epsilon f_1(\theta, \omega, t), \quad (3.21)$$

where $|\epsilon| \ll 1$, and investigate when the perturbation grows.

We explicitly write the velocity functional $v[f]$ (3.3) as

$$v[f] = \omega + V[f], \quad V[f] \equiv K \int_{-\infty}^{\infty} d\omega \int_0^{2\pi} d\theta' f(\theta', \omega, t) \sin(\theta' - \theta). \quad (3.22)$$

Substituting the expansions of f (3.21) and v (3.22) into the equation of continuity, and picking up the terms of $O(\epsilon)$, we have

$$\frac{\partial}{\partial t} f_1 + \frac{\partial}{\partial \theta} (v_0 f_1 + V_1 f_0) = 0, \quad (3.23)$$

where we put $v_0 = v[f_0]$ and $V_1 = V[f_1]$. Expanding each function for the Fourier series as

$$f_1(\theta, \omega, t) = \sum_{n=-\infty}^{\infty} \tilde{f}_1(n, \omega, t) e^{in\theta}, \quad (3.24)$$

we obtain the equation

$$\frac{\partial \tilde{f}_1}{\partial t}(n, \omega, t) = -in \left(\tilde{f}_1(n, \omega, t) \tilde{v}_0(0, \omega) + \tilde{f}_0(0, \omega) \tilde{V}_1(n, \omega, t) \right), \quad (3.25)$$

where we utilized the fact $\tilde{v}_0(n, \omega) = \tilde{f}_0(n, \omega) = 0$ for nonzero integer n . Performing the Laplace transforms

$$\hat{f}_1(n, \omega, s) = \int_0^{\infty} dt \tilde{f}_1(n, \omega, t) e^{-st}, \quad (3.26)$$

where $\text{Re } s > 0$ to ensure the convergence, we have

$$(s + in\tilde{v}_0(0, \omega)) \hat{f}_1(n, \omega, s) = \tilde{f}_1(n, \omega, 0) - in\tilde{f}_0(0, \omega) \hat{V}_1(n, \omega, s). \quad (3.27)$$

We consider the cases $n \neq \pm 1$ and $n = \pm 1$ separately. For the case $n \neq \pm 1$ it is easily shown that $\tilde{V}(n, \omega, t) = 0$ and from Eq. (3.27) we obtain

$$\hat{f}_1(n, \omega, s) = \frac{\tilde{f}_1(n, \omega, 0)}{s + in\omega}. \quad (3.28)$$

Since we see $\tilde{f}_1(n, \omega, t) = \tilde{f}_1(n, \omega, 0) e^{-in\omega t}$ from the inverse Laplace transform of $\hat{f}_1(n, \omega, s)$ with Eq. (3.28), it is shown that the magnitude of $\tilde{f}_1(n, \omega, t)$ does not change with respect to time t as

$$|\tilde{f}_1(n, \omega, t)| = |\tilde{f}_1(n, \omega, 0)|. \quad (3.29)$$

This computation implies that the modes for $n \neq \pm 1$ do not contribute to the destabilization of the incoherent solution, and we then focus on the case $n = \pm 1$. In the light of the fact that $|\tilde{f}_1(1, \omega, t)| = |\tilde{f}_1(-1, \omega, t)|$, it is enough to consider only the case $n = 1$. For $n = 1$ the

equation (3.27) becomes

$$(s + i\omega) \hat{f}_1(1, \omega, s) = \tilde{f}_1(1, \omega, 0) + \pi K \tilde{f}_0(0, \omega) \int_{-\infty}^{\infty} d\omega \hat{f}_1(1, \omega, s). \quad (3.30)$$

We define the function

$$\hat{\rho}_1(n, s) = \int_{-\infty}^{\infty} d\omega \hat{f}_1(n, \omega, s), \quad (3.31)$$

and integrate Eq. (3.30) over ω from $-\infty$ to ∞ . Then, we formally obtain

$$\hat{\rho}_1(1, s) = \frac{1}{D_K(s)} \int_{-\infty}^{\infty} d\omega \frac{\tilde{f}_1(1, \omega, 0)}{s + i\omega}, \quad (3.32)$$

where the function $D_K(s)$ is defined by

$$D_K(s) = 1 + i \frac{K}{2} \int_{-\infty}^{\infty} d\omega \frac{g(\omega)}{\omega - is} \quad (3.33)$$

in the domain $\text{Re } s > 0$ by recalling the domain of the Laplace transform and $\tilde{f}_0(0, \omega) = g(\omega)/(2\pi)$. A root of $D_K(s)$, denoted by s_0 , gives a pole of $\hat{\rho}_1(1, s)$ at $s = s_0$, and this pole gives the estimation of $\tilde{\rho}_1(1, t) \sim e^{s_0 t}$ in the linear regime by the inverse Laplace transform of $\hat{\rho}_1(1, s)$. Therefore, roots of $D_K(s)$ in the domain $\text{Re } s > 0$ imply instability of the incoherent state $f_0(\theta, \omega)$ (3.20). The strongest instability is given by the root of $D_K(s)$ whose real part is the maximum, and we denote the root by $s^*(K)$ depending on K . The critical strength K_c is determined by the condition $\lim_{K \rightarrow K_c} \text{Re } s^*(K) = 0$. A benefit of the Lorentzian type $g(\omega)$ (3.5) is that we can conduct explicitly the integral in Eq. (3.33) with the aid of the residue theorem and obtain the critical coupling strength K_c exactly.

Chapter 4

Dynamics of Multifrequency Systems

This chapter includes mainly the content of the paper [109].

4.1 Introduction

Rhythm plays a crucial role in many aspects of physics, biology and chemistry [104, 117]. To study the rhythmic phenomena quantitatively, the phase oscillator model has been widely used since the prominent studies had been done [116, 49, 23]. The Kuramoto model undergoes a transition from nonsynchronous to synchronous as the coupling strength increases as seen in the previous chapters. This model can also be solved exactly with the Lorentzian natural frequency distribution. Although this discovery has triggered many subsequent works, many unsolved, related models are considered important in real world situations [4]. One of such important models is a system composed of multiple heterogeneous populations of phase oscillators. In fact, some real systems appear to have a hierarchical structure of multiple populations. For example, multiple neuronal modules in the brain seem to organize into structural networks [14, 15]. To understand the dynamical properties of such systems, we need to theoretically investigate a model with multiple populations of oscillators. As a first step, we here investigate a two-population system of phase oscillators.

It is plausible that the characteristics of two populations are generally different and in several situations this property seems to play an important functional role. For example,

the synchrony of different neuronal populations in the brain is positively correlated with the success of human tasking [15]. In real systems such as electroencephalograms, the average frequencies often largely differ across populations. Considering such a coupled system of two populations with different average frequency distributions, we can theoretically derive the correct form of the coupling function by the averaging method [93, 35] in the phase reduction method. This derivation is essential if average frequencies between two populations are related by an integer ratio such as $k : 1$ (where k is an integer). This condition is called the resonant condition. [58, 45, 47]. However, resonance has not been considered in most previous studies. Therefore, we will examine multifrequency oscillator systems by applying a specific coupling function.

Moreover, multiple populations of phase oscillators exhibit interesting properties [72, 9, 1, 89, 44, 45, 47]. One of the most remarkable phenomena is the formation of chimera states in which synchronous and asynchronous states coexist [63, 1, 52, 54, 60, 61, 83, 53]. This phenomenon was theoretically discovered by Kuramoto *et al.* [48], the solvable model was found later by Abrams *et al.* [2]. Later, the properties of chimera states were experimentally investigated [113, 29, 68]. Recent theoretical works have explored chimera states in more general situations [65, 27, 80, 98, 73]. However, in most studies on chimera states in multiple phase oscillator populations, the natural frequencies of the oscillators are assumed to be evenly distributed across the populations. As the next stage, we should theoretically examine heterogeneous frequency distributions across the populations. To this end, we treat a general situation in which the oscillator populations have different average frequencies. We focus on the resonant case with an integer frequency ratio. Here, we must derive the appropriate type of phase coupling function under the corresponding resonance condition. We show that such a system develops chimera states under some conditions and investigate the properties of these states.

Recently Ott and Antonsen proposed a remarkable ansatz that reduces an infinite system of coupled phase oscillators in the continuum limit to a low-dimensional system [76, 77, 78]. This ansatz has been applied to a wide range of applications and has yielded many fruitful results [59, 57, 64, 100, 43, 108, 62]. Komarov and Pikovsky considered the resonant interactions among more than two oscillator communities and applied the Ott-Antonsen ansatz to a simple resonant case [45]. In this study, we consider two populations of phase oscillators in

the more general resonant case $k : 1$, where k is not ± 1 . However, because the populations interact through the resonant type coupling function, we cannot straightforwardly reduce the system to low-dimensional equations using the Ott-Antonsen ansatz. To proceed with the analysis, we augment the Ott-Antonsen ansatz with the additional assumptions of Skardal *et al.* [99]. Consequently, our system reduces to a three-dimensional system of ordinary differential equations.

The remainder of this chapter is structured as follows. Section 4.2 describes a coupled system of two populations of phase oscillators. In Sec. 4.3, we reduce this system to a low-dimensional system. The emergent dynamical states such as the clustered chimera states are investigated in Sec. 4.4. In Sec. 4.5, we numerically confirm that our results hold without the additional solvability assumption. Our method is extended to more general resonant conditions in Sec. 4.6. The chapter concludes with a summary in Sec. 4.7.

4.2 Phase Reduction and Model Equations

As a preliminary, we discuss two interacting oscillators. The oscillators evolve by the following equations:

$$\frac{d\mathbf{x}_1}{dt} = \mathbf{f}_1(\mathbf{x}_1) + \epsilon \mathbf{g}_1(\mathbf{x}_1, \mathbf{x}_2), \quad (4.1)$$

$$\frac{d\mathbf{x}_2}{dt} = \mathbf{f}_2(\mathbf{x}_2) + \epsilon \mathbf{g}_2(\mathbf{x}_2, \mathbf{x}_1), \quad (4.2)$$

where $\mathbf{x}_1$ and $\mathbf{x}_2$ are n -dimensional state vectors, $\mathbf{f}_1$ and $\mathbf{f}_2$ represent the intrinsic dynamics of the oscillators, and $\mathbf{g}_1$ and $\mathbf{g}_2$ are the interaction terms between the oscillators. We further suppose $|\epsilon| \ll 1$, and that the oscillators have non-perturbed limit cycles (when $\epsilon = 0$). The periods of oscillators 1 and 2 are $2\pi/\omega_1$ and $2\pi/\omega_2$ respectively, where ω_1 and ω_2 are the respective natural frequencies of the oscillators. The frequencies almost satisfy the resonant relation $k : 1$, where the natural frequency of the fast oscillator ω_1 is approximately k times the frequency of the slow oscillator ω_2 :

$$\omega_1 \simeq k\omega_2. \quad (4.3)$$

If the frequencies satisfy Eq. (4.3), the resonant coupling function can be derived by phase reduction. For simplicity ω_2 is denoted as ω in the following.

We first introduce the original phase variables θ_1 and θ_2 such that $d\theta_1/dt = k\omega$, $d\theta_2/dt = \omega$ near the limit cycle orbits $\mathbf{x}_{1,0}(t)$ and $\mathbf{x}_{2,0}(t)$ in the absence of the perturbations. Applying phase reduction, the dynamics of the phase variables θ_1 and θ_2 are determined from Eqs. (4.1) and (4.2) as

$$\frac{d\theta_1}{dt} = k\omega + \epsilon \mathbf{Z}_1(\theta_1) \cdot \mathbf{g}_{12}(\theta_1, \theta_2), \quad (4.4)$$

$$\frac{d\theta_2}{dt} = \omega + \epsilon \mathbf{Z}_2(\theta_2) \cdot \mathbf{g}_{21}(\theta_2, \theta_1), \quad (4.5)$$

where

$$\mathbf{Z}_1(\theta_1) = \nabla_{\mathbf{x}_1} \theta_1(\mathbf{x}_1) |_{\mathbf{x}_1 = \mathbf{x}_{1,0}(\theta_1)},$$

$$\mathbf{Z}_2(\theta_2) = \nabla_{\mathbf{x}_2} \theta_2(\mathbf{x}_2) |_{\mathbf{x}_2 = \mathbf{x}_{2,0}(\theta_2)}.$$

To separate the slow dynamics from Eqs.(4.4) and (4.5), we define slow phase variables ψ_1 and ψ_2 as $\theta_1 = k\omega t + \psi_1$ and $\theta_2 = \omega t + \psi_2$ respectively. The dynamics of the slow phase variables are described by

$$\frac{d\psi_1}{dt} = \epsilon \mathbf{Z}_1(\psi_1 + k\omega t) \cdot \mathbf{g}_{12}(\psi_1 + k\omega t, \psi_2 + \omega t), \quad (4.6)$$

$$\frac{d\psi_2}{dt} = \epsilon \mathbf{Z}_2(\psi_2 + \omega t) \cdot \mathbf{g}_{21}(\psi_2 + \omega t, \psi_1 + k\omega t). \quad (4.7)$$

Averaging the RHS in Eq. (4.6) over the period of the slow oscillator $2\pi/\omega$, we obtain

$$\begin{aligned} \frac{d\psi_1}{dt} &= \epsilon \frac{\omega}{2\pi} \int_0^{2\pi/\omega} dt \mathbf{Z}_1(\psi_1 + k\omega t) \cdot \mathbf{g}_{12}(\psi_1 + k\omega t, \psi_2 + \omega t) \\ &= \frac{\epsilon}{2\pi} \int_0^{2\pi} d\Theta \mathbf{Z}_1(\psi_1 + k\Theta) \cdot \mathbf{g}_{12}(\psi_1 + k\Theta, \psi_2 + \Theta) \\ &= \frac{\epsilon}{2\pi} \int_0^{2\pi} d\Theta \mathbf{Z}_1(\psi_1 - k\psi_2 + k\Theta) \cdot \mathbf{g}_{12}(\psi_1 - k\psi_2 + k\Theta, \Theta), \end{aligned} \quad (4.8)$$

where ψ_1 and ψ_2 are constants independent of t during one period. When evaluating the integral, we express the RHS of Eq. (4.8) as a function of $\psi_1 - k\psi_2$; namely $\Gamma_{12}(\psi_1 - k\psi_2)$.

Performing similar operations the RHS of Eq. (4.7), we obtain the pair of averaged equations:

$$\frac{d\psi_1}{dt} = \epsilon \Gamma_{12}(\psi_1 - k\psi_2), \quad (4.9)$$

$$\frac{d\psi_2}{dt} = \epsilon \Gamma_{21}(k\psi_2 - \psi_1), \quad (4.10)$$

where

$$\Gamma_{12}(\psi_1 - k\psi_2) = \frac{\omega}{2\pi} \int_0^{2\pi/\omega} dt \mathbf{Z}_1(\psi_1 + k\omega t) \cdot \mathbf{g}_{12}(\psi_1 + k\omega t, \psi_2 + \omega t),$$

$$\Gamma_{21}(k\psi_2 - \psi_1) = \frac{\omega}{2\pi} \int_0^{2\pi/\omega} dt \mathbf{Z}_2(\psi_2 + \omega t) \cdot \mathbf{g}_{21}(\psi_2 + \omega t, \psi_1 + k\omega t).$$

This approximation is valid to order ϵ . Consequently, we obtain the evolutionary equations of the averaged phase variables θ_1 and θ_2 :

$$\frac{d\theta_1}{dt} = k\omega + \epsilon \Gamma_{12}(\theta_1 - k\theta_2), \quad (4.11)$$

$$\frac{d\theta_2}{dt} = \omega + \epsilon \Gamma_{21}(k\theta_2 - \theta_1). \quad (4.12)$$

We emphasize that the coupling functions Γ_i ($i=1,2$) depend on $\theta_1 - k\theta_2$ (or $k\theta_2 - \theta_1$) when the natural frequencies of the oscillators satisfy the resonant relation $k : 1$. Generalizing this to the $m : n$ case, we can state that Γ_i ($i=1,2$) depends on $n\theta_1 - m\theta_2$ (or $m\theta_2 - n\theta_1$). In the case of the irrational ratio, the interaction functions become constants independent of the phase variables after the averaging [35] and we do not consider this trivial case in the present paper.

Our discussion of resonant interactions is now extended to populations of phase oscillators. We consider two populations of phase oscillators with different average frequencies. We assume that both populations have inherent Lorentzian distributions of natural frequencies. The mean frequency ratio between the two populations is $2 : 1$ (i.e., $k = 2$ in Fig. 4.1):

$$g_{\text{fast}}(\omega) = \frac{D}{\pi} \frac{1}{(\omega - 2\Omega)^2 + D^2}, \quad (4.13)$$

$$g_{\text{slow}}(\omega) = \frac{D}{\pi} \frac{1}{(\omega - \Omega)^2 + D^2}, \quad (4.14)$$

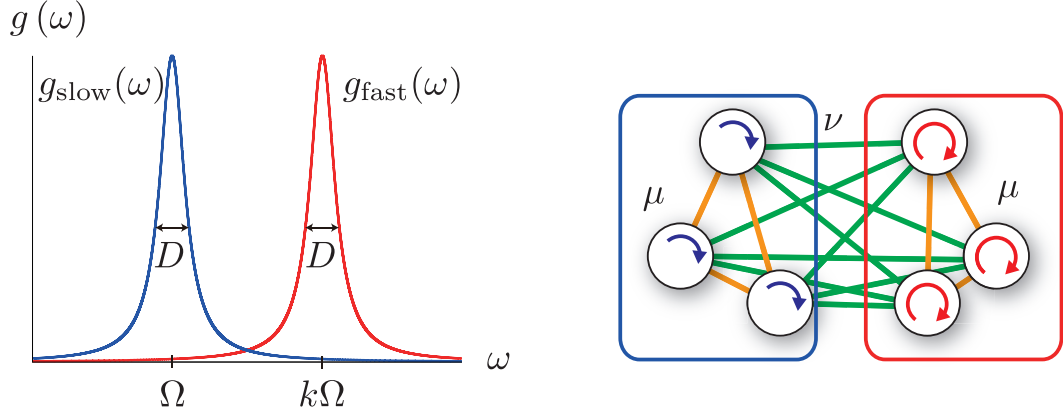


Figure 4.1: Left: Natural frequency distributions of the phase oscillators in the two populations whose mean frequencies satisfy the resonant relation $k : 1$. Right: The coupling strengths is $\mu = (1 + A)/2$ within the same population and $\nu = (1 - A)/2$ between the two populations.

where D is the common width of the distributions and Ω is the mean of the distribution in the slow population. As usual, we assume dominance of the first term in the Fourier series of the coupling function, and take $H(\theta) = \sin(\theta + \alpha)$ as in the Sakaguchi-Kuramoto model [92], where α is the phase lag parameter. This parameter can be estimated directly if Eqs. (4.1) and (4.2) are explicitly given, but in this work we will consider general situations and regard α as the model parameter. We thus consider the following model:

$$\frac{d\theta_i^{\text{fast}}}{dt} = \omega_i^{\text{fast}} + \frac{\mu}{N} \sum_{j=1}^N \sin(\theta_j^{\text{fast}} - \theta_i^{\text{fast}} - \alpha) + \frac{\nu}{N} \sum_{j=1}^N \sin(2\theta_j^{\text{slow}} - \theta_i^{\text{fast}} - \alpha), \quad (4.15)$$

$$\frac{d\theta_i^{\text{slow}}}{dt} = \omega_i^{\text{slow}} + \frac{\mu}{N} \sum_{j=1}^N \sin(\theta_j^{\text{slow}} - \theta_i^{\text{slow}} - \alpha) + \frac{\nu}{N} \sum_{j=1}^N \sin(\theta_j^{\text{fast}} - 2\theta_i^{\text{slow}} - \alpha), \quad (4.16)$$

where $\theta_i^{\text{fast/slow}}$ is the phase of oscillator i ($i = 1, \dots, N$) in the fast/slow population. Incorporating the phase reduction concept, we now consider the interactions between the oscillators. For this purpose, we introduce the parameter A and define $\mu = (1 + A)/2, \nu = (1 - A)/2$ as in [1]. In this setting, $\mu + \nu = 1$. The parameter A controls the ratio of the strengths of the interactions within each population and between the populations. When $0 < A < 1$, the interactions are stronger within the populations than across the populations; conversely, when $-1 < A < 0$ the interactions across the populations dominate. In this study we fix the phase lag parameter $\alpha = \pi/2 - 0.05$ as in [54]. Under this condition, chimera states will

appear, and a variety of dynamics with a reasonably general coverage are expected [1]. The general k case will be straightforwardly extended from the $k = 2$ case in Sec. 4.6.

4.3 Reduction to Low-dimensional Dynamics

Using the Ott-Antonsen ansatz [76, 77, 78], we will attempt to reduce the system represented by Eqs. (4.13)-(4.16) to a low-dimensional system in the limit $N \rightarrow \infty$. However, our model includes the specific interactions such as $\sin(2\theta_j^{\text{slow}} - \theta_i^{\text{fast}} - \alpha)$ in Eqs. (4.15) and (4.16) so that we cannot use the Ott-Antonsen ansatz straightforwardly as the original case in [76]. To handle the high-order Kuramoto model, we use a modified version of the Ott-Antonsen ansatz used by Skardal *et al.* [99]. However, the modified ansatz alone does not reduce the original dynamics to a low-dimensional system. To overcome this difficulty, we employ one additional assumption in which we replace $\sin(\theta_j^{\text{slow}} - \theta_i^{\text{slow}} - \alpha)$ with $\sin(2\theta_j^{\text{slow}} - 2\theta_i^{\text{slow}} - \alpha)$ in Eq. (4.16). Equation (4.16) becomes

$$\frac{d\theta_i^{\text{slow}}}{dt} = \omega_i^{\text{slow}} + \frac{\mu}{N} \sum_{j=1}^N \sin(2\theta_j^{\text{slow}} - 2\theta_i^{\text{slow}} - \alpha) + \frac{\nu}{N} \sum_{j=1}^N \sin(\theta_j^{\text{fast}} - 2\theta_i^{\text{slow}} - \alpha). \quad (4.17)$$

The dynamics can be reduced by applying the Ott-Antonsen ansatz to (4.15) and the modified model Eq. (4.17). In general, however, it is unlikely that the interaction terms among the slow oscillators contain no first Fourier mode. In Sec. 4.5, we will check the validity of the modified model under general conditions.

We now consider the continuum limit $N \rightarrow \infty$ in our modified model. The probability density functions (PDFs) of the fast and slow populations in the continuum limit are denoted as $f_{\text{fast}}(\theta, \omega, t)$ and $f_{\text{slow}}(\theta, \omega, t)$ respectively, where $f_j(\theta, \omega, t)d\theta d\omega$ is the fraction of oscillators with phase between θ and $\theta + d\theta$ and natural frequency between ω and $\omega + d\omega$ at time t in population $j(=\text{fast,slow})$. The common order parameter for the fast population is given by

$$\begin{aligned} z_{\text{fast}}(t) &= \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{j=1}^N e^{i\theta_j^{\text{fast}}} \\ &= \int_{-\infty}^{\infty} d\omega \int_0^{2\pi} d\theta f_{\text{fast}}(\theta, \omega, t) e^{i\theta}. \end{aligned} \quad (4.18)$$

For the slow population, we define the Daido order parameter [20] as

$$\begin{aligned} z_{\text{slow}}(t) &= \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{j=1}^N e^{2i\theta_j^{\text{slow}}} \\ &= \int_{-\infty}^{\infty} d\omega \int_0^{2\pi} d\theta f_{\text{slow}}(\theta, \omega, t) e^{2i\theta}. \end{aligned} \quad (4.19)$$

Following the Ott-Antonsen ansatz [76, 77, 78] and its variant [99], we expand the PDFs as two Fourier series:

$$f_{\text{fast}}(\theta, \omega, t) = \frac{g_{\text{fast}}(\omega)}{2\pi} \left[1 + \sum_{n=1}^{\infty} \left(a(\omega, t)^n e^{in\theta} + \text{c.c.} \right) \right], \quad (4.20)$$

$$f_{\text{slow}}(\theta, \omega, t) = \frac{g_{\text{slow}}(\omega)}{2\pi} \left[1 + \sum_{m=1}^{\infty} \left(b(\omega, t)^m e^{2im\theta} + \text{c.c.} \right) \right], \quad (4.21)$$

where c.c. stands for complex conjugate. The ansatz requires the conditions, $|a(\omega, t)| < 1$ and $|b(\omega, t)| < 1$ for the convergence of Eqs. (4.20) and (4.21). To conserve the total number of oscillators in each population, the following continuity equations should be satisfied:

$$\frac{\partial f_j}{\partial t} + \frac{\partial}{\partial \theta_j} (f_j \dot{\theta}_j) = 0 \quad (j = \text{fast, slow}),$$

Substituting Eqs. (4.20) and (4.21) into these continuity equations, we obtain the ordinal differential equations for a and b :

$$\frac{\partial a}{\partial t} + i\omega a + \frac{\mu}{2} (z_{\text{fast}} a^2 e^{-i\alpha} - \bar{z}_{\text{fast}} e^{i\alpha}) + \frac{\nu}{2} (z_{\text{slow}} a^2 e^{-i\alpha} - \bar{z}_{\text{slow}} e^{i\alpha}) = 0, \quad (4.22)$$

$$\frac{1}{2} \frac{\partial b}{\partial t} + i\omega b + \frac{\mu}{2} (z_{\text{slow}} b^2 e^{-i\alpha} - \bar{z}_{\text{slow}} e^{i\alpha}) + \frac{\nu}{2} (z_{\text{fast}} b^2 e^{-i\alpha} - \bar{z}_{\text{fast}} e^{i\alpha}) = 0. \quad (4.23)$$

From Eqs. (4.18) and (4.20), we immediately find that

$$z_{\text{fast}}(t) = \int_{-\infty}^{\infty} d\omega g_{\text{fast}}(\omega) \bar{a}(\omega, t). \quad (4.24)$$

Given that the natural frequency distribution is Lorentzian Eq. (4.13), we can write

$$z_{\text{fast}}(t) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} d\omega \left(\frac{1}{\omega - 2\Omega - iD} - \frac{1}{\omega - 2\Omega + iD} \right) \bar{a}(\omega, t). \quad (4.25)$$

Following [76] we assume that $\bar{a}(\omega, t)$ is analytic in $\text{Im } \omega > 0$. From the complex conjugate of Eq. (4.22), we know that $\partial \bar{a} / \partial t \sim -(\text{Im } \omega) \bar{a}$ as $\text{Im } \omega \rightarrow \infty$. As $|\bar{a}(\omega, t)| < 1$, we also know that $\bar{a}(\omega, t) \rightarrow 0$ as $\text{Im } \omega \rightarrow \infty$. Integrating the RHS of Eq. (4.25) over the upper semicircular contour in the complex plain, we obtain

$$z_{\text{fast}}(t) = \bar{a}(2\Omega + iD, t).$$

A similar analysis gives

$$z_{\text{slow}}(t) = \bar{b}(\Omega + iD, t).$$

Substituting $\omega = 2\Omega + iD$ and $\omega = \Omega + iD$ in the complex conjugate of Eqs. (4.22) and (4.23), respectively, we obtain the dynamics of the complex order parameters of the two populations:

$$\frac{dz_{\text{fast}}}{dt} = (-D + 2\Omega i) z_{\text{fast}} + \frac{e^{-i\alpha}}{2} (\mu z_{\text{fast}} + \nu z_{\text{slow}}) - \frac{e^{i\alpha}}{2} (\mu \bar{z}_{\text{fast}} + \nu \bar{z}_{\text{slow}}) z_{\text{fast}}^2, \quad (4.26)$$

$$\frac{dz_{\text{slow}}}{dt} = (-2D + 2\Omega i) z_{\text{slow}} + e^{-i\alpha} (\mu z_{\text{slow}} + \nu z_{\text{fast}}) - e^{i\alpha} (\mu \bar{z}_{\text{slow}} + \nu \bar{z}_{\text{fast}}) z_{\text{slow}}^2. \quad (4.27)$$

We now rewrite Eqs. (4.26) and (4.27) in terms of the polar coordinates $z_{\text{fast}} = r_{\text{fast}} e^{-i\phi_{\text{fast}}}$ and $z_{\text{slow}} = r_{\text{slow}} e^{-i\phi_{\text{slow}}}$ and their phase difference $\phi = \phi_{\text{fast}} - \phi_{\text{slow}}$. The resulting system comprises three ordinary differential equations with three degrees of freedom $(r_{\text{fast}}, r_{\text{slow}}, \phi)$:

$$\begin{aligned} \frac{dr_{\text{fast}}}{dt} = & -Dr_{\text{fast}} \\ & + \frac{1 - r_{\text{fast}}^2}{2} (\mu r_{\text{fast}} \cos \alpha + \nu r_{\text{slow}} \cos(\phi - \alpha)), \end{aligned} \quad (4.28)$$

$$\begin{aligned} \frac{dr_{\text{slow}}}{dt} = & -2Dr_{\text{slow}} \\ & + \frac{1 - r_{\text{slow}}^2}{2} (\mu r_{\text{slow}} \cos \alpha + \nu r_{\text{fast}} \cos(\phi + \alpha)), \end{aligned} \quad (4.29)$$

$$\begin{aligned} \frac{d\phi}{dt} = & \frac{1 + r_{\text{fast}}^2}{2} \left(\mu \sin \alpha - \nu \frac{r_{\text{slow}}}{r_{\text{fast}}} \sin(\phi - \alpha) \right) \\ & - (1 + r_{\text{slow}}^2) \left(\mu \sin \alpha + \nu \frac{r_{\text{fast}}}{r_{\text{slow}}} \sin(\phi + \alpha) \right). \end{aligned} \quad (4.30)$$

From these equations, we found that the reduced system evolves independently of Ω , the mean of natural frequency distribution of the oscillators in the slow population. Later, we will show that this independency holds in general k cases.

4.4 Results: Clustered Chimera States

In the previous section, we showed that, if we make a slight modification to Eq. (4.16) and by replacing $\sin(\theta_j^{\text{slow}} - \theta_i^{\text{slow}} - \alpha)$ with Eq. $\sin(2\theta_j^{\text{slow}} - 2\theta_i^{\text{slow}} - \alpha)$, this modified system with (4.15) in the continuum limit can be simplified to the reduced system (4.28-4.30) by applying the Ott-Antonsen ansatz. In this section, we numerically simulate the detailed dynamics of the modified and reduced systems. We set $D = 1.0 \times 10^{-3}$ and imposed the initial condition $r_{\text{fast}}, r_{\text{slow}} \simeq 1$ (Each θ_i was chosen from a uniform distribution in $[0, \pi/30]$), and each population comprised $N = 10^4$ oscillators. The initial order parameters in the reduced systems were set by substituting the conditions of the corresponding $N = 10^4$ modified systems.

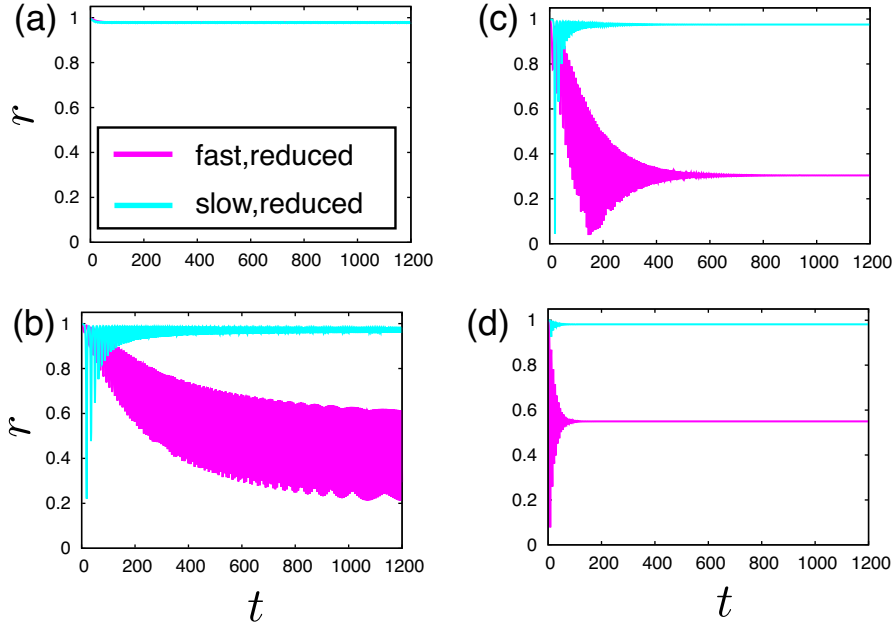


Figure 4.2: Time evolutions of the order parameters of the populations in the reduced system for (a) $A = 0.9$, (b) $A = 0.1$, (c) $A = -0.1$, and (d) $A = -0.9$.

Figure 4.2 plots the time evolutions of the order parameters of the reduced system for (a) $A = 0.9$, (b) $A = 0.1$, (c) $A = -0.1$, and (d) $A = -0.9$. In all cases, we can see the order parameters of the populations reach a steady state. Panels (a)-(c) reveal three different types of dynamics; coherent, breathing chimera and stable chimera. In the coherent state (Fig. 4.2 (a)), the order parameters of both populations approach almost 1, indicating that the fast and slow oscillators form synchronous clusters. In the chimera state, the order parameter of

one population becomes partially coherent while that of the other population is coherent [2]. In this case, the slow oscillators are mutually synchronized while the fast oscillators are not. In the breathing chimera state (Fig. 4.2 (b)), the order parameters oscillate. Conversely, in Fig. 4.2 (c), the order parameters go finally to the fixed point, which is called the stable chimera state [1]. Panels (c) and (d) exhibit similar order parameter behaviors but different phase distributions. These differences will be thoroughly explored in later simulations of the modified system.

Figure 4.3 plots the evolving order parameters and instantaneous phase distributions of the modified system with $N = 10^4$. In panels (a)-(d), the modified systems exhibit the same steady-state behavior as their counterpart reduced systems, but the transient behavior in Fig. 4.3 (d) differs from that Fig. 4.2 (d). In Fig. 4.2 (d) at $A = -0.9$, the reduced system goes to the steady state faster than the modified system with $N = 10^4$ does in Fig. 4.3 (d). This difference appears when the coupling between the populations is dominant.

Figure 4.3 also represents snapshots of the steady-state phase distributions of the two populations in the modified system. For $A = 0.1$, -0.1 and 0.9 (panels (a), (b), and (c), respectively), the fast population is partially coherent, whereas the slow population splits into two clusters with a phase difference π . Similar states, known as clustered chimera states, have been reported in a delay-coupled system [96].

Let us check the result correspondence in the modified and reduced systems. Panels (a) and (b) of Fig. 4.4 plot the averages and standard deviations, respectively, of the steady-state order parameters in the system reduced by the ansatz and the modified system with $N = 10^4$. When the intra-population coupling strengths are sufficiently strong, both populations settle into coherent states. Weakened coupling leads to stable or breathing chimera states.

These results were obtained under the initial condition $r_{\text{fast}}, r_{\text{slow}} \simeq 1$. The steady-state order parameters behaved similarly under other starting conditions (data not shown), suggesting that the clustered chimera states in our system are robust. However, the height ratio of two clusters in the asymptotic phase distributions of the slow oscillators does depend on the initial conditions, because the height difference have no effect on the phase dynamics in Eq. (4.17). Moreover, the transient behaviors crucially depend on the initial conditions in some parameter ranges.

4.5 Validity of Specific Assumption

For compatibility with the Ott-Antonsen ansatz, the above analysis imposed an unrealistic assumption on the interactions between the slow oscillators in Eq. (4.17). In real situations, the interaction term between the slow oscillators in Eq. (4.16) should contain the first mode in the Fourier series; consequently we need to reassess this assumption. Unfortunately, without replacing $\sin(\theta_j^{\text{slow}} - \theta_i^{\text{slow}} - \alpha)$ with $\sin(2\theta_j^{\text{slow}} - 2\theta_i^{\text{slow}} - \alpha)$, we cannot reduce the original system to a low-dimensional system through using the Ott-Antonsen ansatz. In this section, we investigate whether imposing the above assumption affects the dynamics of two oscillator populations.

After the above modification, the model equations (4.15) and (4.17) can be rewritten as

$$\frac{d\theta_i^{\text{fast}}}{dt} = \omega_i^{\text{fast}} + \mu \text{Im} \left[z_{\text{fast}}^{(N)} e^{-i(\theta_i^{\text{fast}} + \alpha)} \right] + \nu \text{Im} \left[z_{\text{slow}}^{(N)} e^{-i(\theta_i^{\text{fast}} + \alpha)} \right], \quad (4.31)$$

$$\frac{d\theta_i^{\text{slow}}}{dt} = \omega_i^{\text{slow}} + \mu \text{Im} \left[z_{\text{slow}}^{(N)} e^{-i(2\theta_i^{\text{slow}} + \alpha)} \right] + \nu \text{Im} \left[z_{\text{fast}}^{(N)} e^{-i(2\theta_i^{\text{slow}} + \alpha)} \right], \quad (4.32)$$

where $z_{\text{fast}}^{(N)}$ and $z_{\text{slow}}^{(N)}$ are the Kuramoto order parameter for the fast N oscillators and the Daido order parameter for the slow N oscillators, respectively. From these representation we see that all the interactions from the slow oscillators to the other can be described only through the Daido order parameter. As in Eq. (4.19), the Daido order parameter for $k = 2$ do not distinguish θ and $\theta + \pi$ in the phase variables of the slow oscillators. Therefore, with our modification the difference between θ and $\theta + \pi$ in the phase variables of the slow oscillators does not matter and it is sufficient to use the Daido order parameter to examine the synchrony of the slow oscillators. However, without our modification, the model (4.15) and (4.16) includes the interactions from the slow oscillators that cannot be represented only by the Daido order parameters, and thus we have to be careful for how such condition affects the dynamics.

To scrutinize the effect of our assumption, we numerically examine the dynamics of the original system (4.15) and (4.16). The natural frequency distributions of the populations are those of the modified system in Secs. 4.2-4.4. As the original system cannot be reduced to a low-dimensional system, we numerically simulate both systems with $N = 10^4$. As in Sec. 4.4, we set $D = 1.0 \times 10^{-3}$ and $\alpha = \pi/2 - 0.05$. The initial conditions were set to

$r_{\text{fast}}, r_{\text{slow}} \simeq 1$ (each θ_i was chosen from uniform distribution in $[0, \pi/30]$). The results for $A = 0$ (homogeneous coupling strengths in the system) are plotted in Fig. 4.5. In Fig. 4.5 (a) compares the time evolutions of the order parameters in the modified and original systems with $N = 10^4$. Panels (b) and (c) of this figure are snapshots of the steady-state phase distributions in the modified and original systems, respectively. In Fig. 4.5 (a), the order parameters of both systems reach similar steady states. However, closer inspection reveals that the asymptotic phase distributions of the slow oscillators differ between the two systems. Specifically, the slow population in the modified system splits into two clusters (Fig. 4.5 (b)) but is unimodal in the original system (Fig. 4.5 (c)). Therefore, the original system settles into a normal rather than a clustered chimera state. The same result emerged under all tested conditions. The averages and standard deviations of the steady-state order parameters in the reduced and original systems are compared in Fig. 4.6 (Note that the results of the reduced system are replicated from Fig. 4.4). The behaviors of the reduced and original systems are qualitatively similar. We checked that this similarity remains if D is sufficiently small.

If we use not the Daido order parameter but the Kuramoto order parameter, the results might apparently be changed. However, considering the coupling functions of the slow oscillators in the modified system, it is more natural to use the Daido order parameter for $k = 2$ rather than the Kuramoto order parameter. In terms of the synchrony with the Daido order parameter, the modified and original systems show similar properties.

Finally we remark on the different outcomes of the two systems. In Fig. 4.6, the asymptotic behaviors of the order parameters differ in certain ranges of the parameter A . The two models differ when A is small and positive; that is, when the intra-population interactions are slightly stronger than the inter-population interactions. To clarify this results, Fig. 4.7 (a) plots the evolved order parameter dynamics of the modified and original systems with $N = 10^4$ for $A = 0.2$. Fig. 4.7 (b) and (c) represent snapshots of the steady-state phase distributions. In this case, the specific assumption alters the steady-state distribution of the modified system in two ways. First it amplifies the peak of the fast oscillators relative to the original system. Second, the slow oscillators separate into two clusters in the modified system, but they form a single cluster in the original system.

4.6 General Resonant Cases

As mentioned above, our method is readily extensible from the resonant condition $2 : 1$ to the general integer resonant condition $k : 1$. Thus, we consider a model of oscillators of two populations under the resonant condition $k : 1$:

$$\frac{d\theta_i^{\text{fast}}}{dt} = \omega_i^{\text{fast}} + \frac{\mu}{N} \sum_{j=1}^N \sin(\theta_j^{\text{fast}} - \theta_i^{\text{fast}} - \alpha) + \frac{\nu}{N} \sum_{j=1}^N \sin(k\theta_j^{\text{slow}} - \theta_i^{\text{fast}} - \alpha), \quad (4.33)$$

$$\frac{d\theta_i^{\text{slow}}}{dt} = \omega_i^{\text{slow}} + \frac{\mu}{N} \sum_{j=1}^N \sin(\theta_j^{\text{slow}} - \theta_i^{\text{slow}} - \alpha) + \frac{\nu}{N} \sum_{j=1}^N \sin(\theta_j^{\text{fast}} - k\theta_i^{\text{slow}} - \alpha), \quad (4.34)$$

where the corresponding natural frequency distributions are assumed to obey the Lorentz distributions of Fig. 4.1. Note that the strength parameter A satisfies $\mu = (1 + A)/2$ and $\nu = (1 - A)/2$. Thus, by replacing the interaction term $\sin(\theta_j^{\text{slow}} - \theta_i^{\text{slow}} - \alpha)$ in Eq. (4.34) with $\sin(k\theta_j^{\text{slow}} - k\theta_i^{\text{slow}} - \alpha)$, we can reduce the original system of Eqs. (4.33) and (4.34) in the continuum limit $N \rightarrow \infty$. After this modification, the PDF of the slow oscillator population can be expressed as a Fourier series:

$$f_{\text{slow}}(\theta, \omega, t) = \frac{g_{\text{slow}}(\omega)}{2\pi} \left[1 + \sum_{m=1}^{\infty} \left(b(\omega, t)^m e^{ikm\theta} + \text{c.c.} \right) \right],$$

where we assume $|b(\omega, t)| < 1$ under the Ott-Antonsen ansatz. We also define the complex order parameter for the slow oscillator population z_{slow} :

$$z_{\text{slow}}(t) = \int_{-\infty}^{\infty} d\omega \int_0^{2\pi} d\theta f_{\text{slow}}(\theta, \omega, t) e^{ik\theta}. \quad (4.35)$$

Similarly to Sec 4.3, the population dynamics are finally described by a set of ordinary differential equations with three degrees of freedom:

$$\frac{dr_{\text{fast}}}{dt} = -Dr_{\text{fast}} \quad (4.36)$$

$$+ \frac{1 - r_{\text{fast}}^2}{2} (\mu r_{\text{fast}} \cos \alpha + \nu r_{\text{slow}} \cos(\phi - \alpha)),$$

$$\frac{dr_{\text{slow}}}{dt} = -kDr_{\text{slow}} \quad (4.37)$$

$$+ \frac{k(1 - r_{\text{slow}}^2)}{2} (\mu r_{\text{slow}} \cos \alpha + \nu r_{\text{fast}} \cos(\phi + \alpha)),$$

$$\frac{d\phi}{dt} = \frac{1 + r_{\text{fast}}^2}{2} \left(\mu \sin \alpha - \nu \frac{r_{\text{slow}}}{r_{\text{fast}}} \sin(\phi - \alpha) \right) \quad (4.38)$$

$$- \frac{k(1 + r_{\text{slow}}^2)}{2} \left(\mu \sin \alpha + \nu \frac{r_{\text{fast}}}{r_{\text{slow}}} \sin(\phi + \alpha) \right).$$

Here, we used the polar coordinates $z_{\text{fast}} = r_{\text{fast}}e^{-i\phi_{\text{fast}}}$, $z_{\text{slow}} = r_{\text{slow}}e^{-i\phi_{\text{slow}}}$ and denoted the phase difference $\phi = \phi_{\text{fast}} - \phi_{\text{slow}}$ as in Sec. 4.3. We remark that the system dynamics depend only on the ratio of the means of the natural frequencies, k . In other words, the absolute value of the mean frequencies Ω does not influence the collective behavior of the system.

Fig. 4.8 shows the averages and standard deviations of the steady-state order parameters in the reduced and original systems for various k (3, 5, 7, 11). As in Sec. 4.5, we set $N = 10^4$ and the initial conditions $r_{\text{fast}}, r_{\text{slow}} \simeq 1$. We also set $\alpha = \pi/2 - 0.05$ and $D = 1.0 \times 10^{-3}$. Qualitatively, the system (4.36)-(4.38) exhibits the same dynamics as the system (4.28)-(4.30), even at higher values of k .

However, the asymptotic behaviors differ between the reduced and original systems as k increases. The widening difference is especially apparent in the standard deviation. For large k , the order parameter of the slow oscillators tends to fluctuate with larger amplitudes in the original system, probably because of the imposed assumption. In typical Fourier series, the magnitudes of the low-frequency modes are more similar to the first Fourier mode amplitude than those of high-frequency modes.

Under the general condition $k : 1$, the modified system develops k -clustered chimera states, with k clusters of coherent slow oscillators. To our knowledge these states have not been previously reported. The slow oscillators in the k -clustered chimera states form k synchronous oscillator groups. The phase difference between the nearest clusters is approximately $2\pi/k$. The k -clustered chimera states can be considered as generalized versions of the

2-cluster ones. Like the standard chimera state, the k -clustered chimera states are classifiable as stable or breathing.

We numerically checked the result correspondence between the reduced and modified systems with large N and general k . Therefore, our proposed reduction is valid in the general $k : 1$ case.

4.7 Chapter Summary

Referring to phase reduction theory, we investigated the dynamics of multiple populations of phase oscillators. First, we assumed that the mean of frequency distribution in one population was twice faster than that in the other population. Applying the Ott-Antonsen ansatz [76, 99] and imposing an additional assumption, we reduced the original system to a low-dimensional system of ordinary differential equations describing the time evolution of the order parameters. The population of slow oscillators was treated by the Daido order parameter [20]. Clustered chimera states emerged when the inter-population coupling strength was relatively large.

We also investigated the general resonant condition. Our analysis was extensible from the simple resonant case $2 : 1$ to the general case $k : 1$, where k is any integer. We confirmed that the results for the case $2 : 1$ were qualitatively replicated in the general case. However, for large k , our additional assumption significantly altered the dynamics of the original system.

As a future work, we will investigate multifrequency systems completely. In other words, we can think about dynamics of systems with the more general resonant case $m : n$. We can apply our approach to the resonant case $m : n$, although it requests us one more assumption on the interactions within populations of fast oscillators.

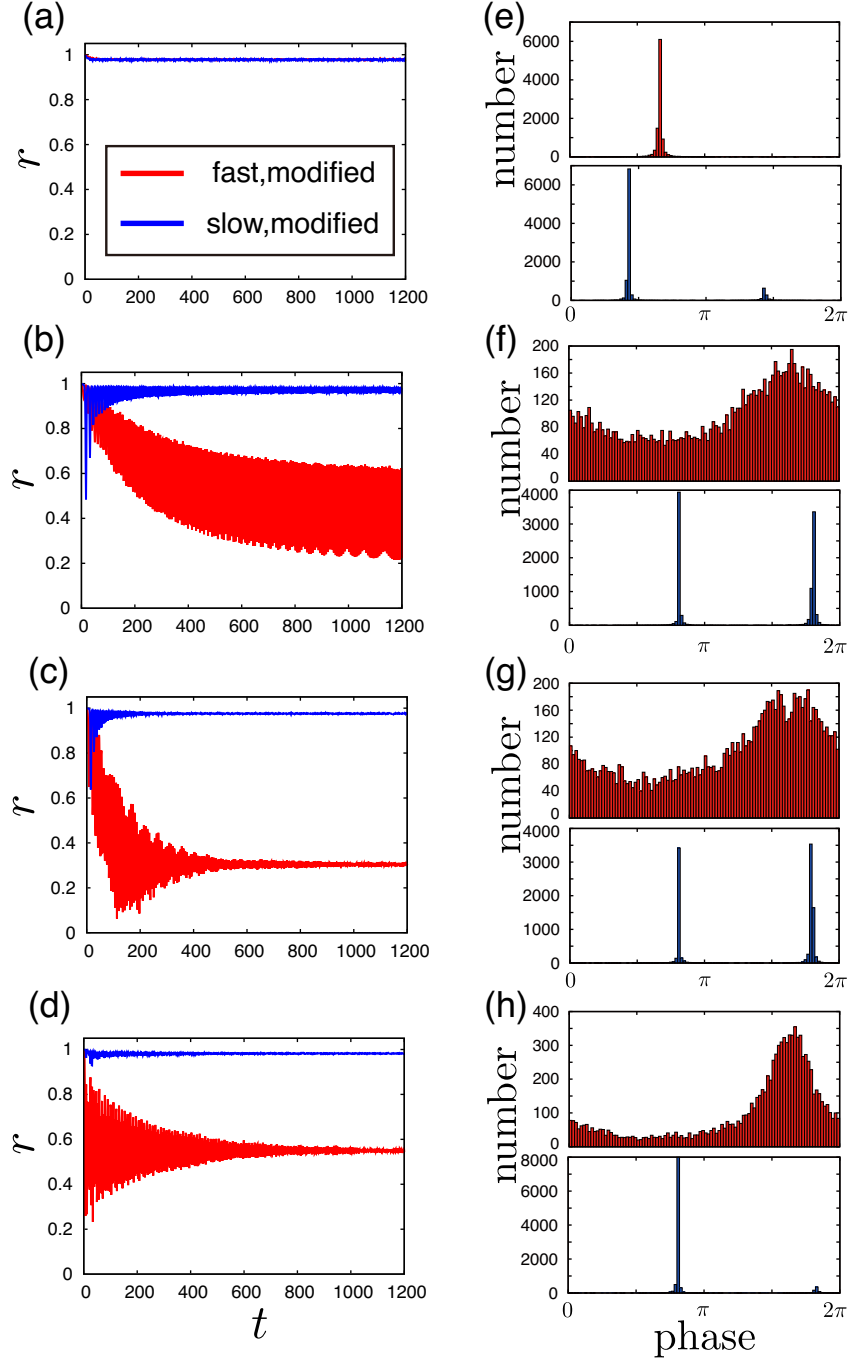


Figure 4.3: Time evolutions of the order parameters of the populations in the modified systems; (a) $A = 0.9$, (b) $A = 0.1$, (b) $A = -0.1$, and (d) $A = -0.9$. Right panels show corresponding snapshots of the steady-state phase distributions (Up: fast population; Down: slow population).

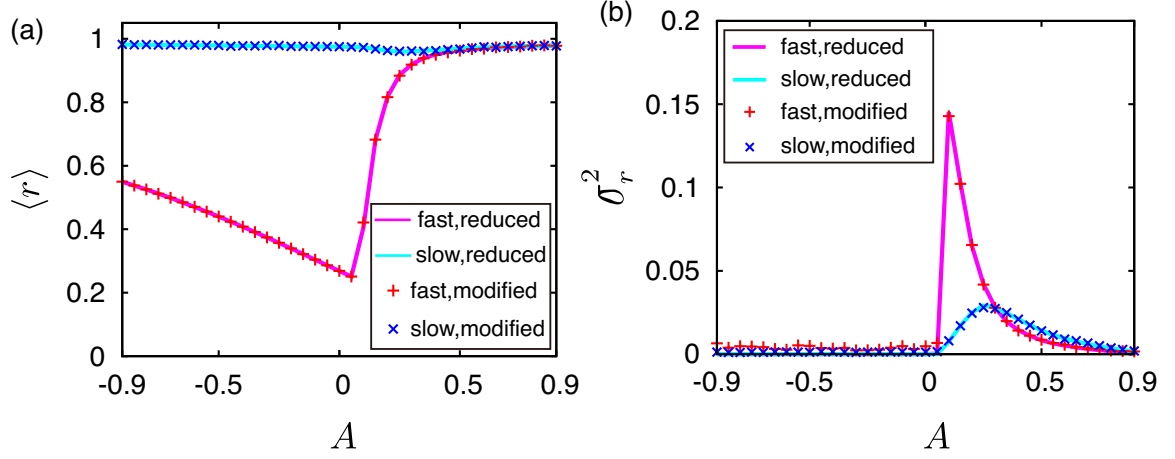


Figure 4.4: Comparisons between the average (a) and the standard deviation (b) of the steady-state order parameters in the modified system (with $N = 10^4$) and the reduced system.

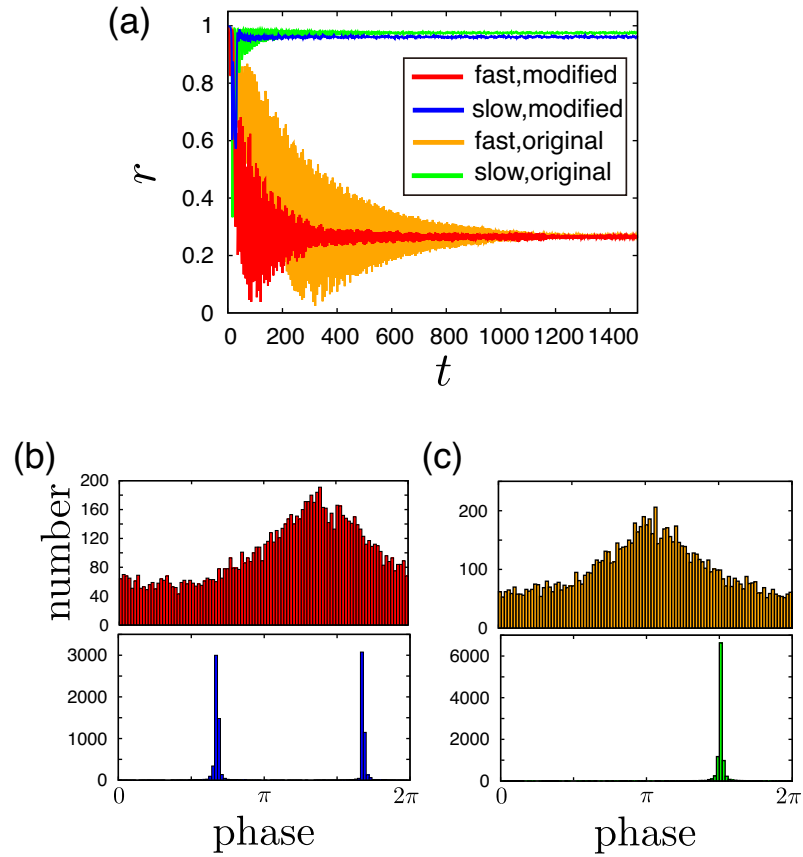


Figure 4.5: (a) Time evolutions of the order parameters in the modified and original systems with $N = 10^4$ for $A = 0$. Bottom panels are snapshots of the steady-state phase distributions in the modified (b) and original (c) systems for $A = 0$.

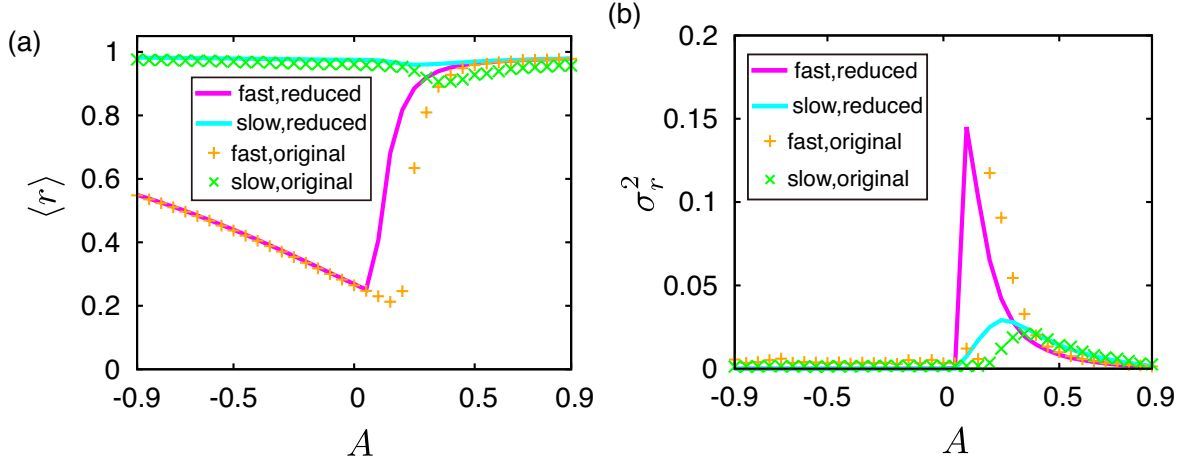


Figure 4.6: Comparison between averages (a) and standard deviations (b) of the steady-state order parameters in the reduced and original systems ($N = 10^4$).

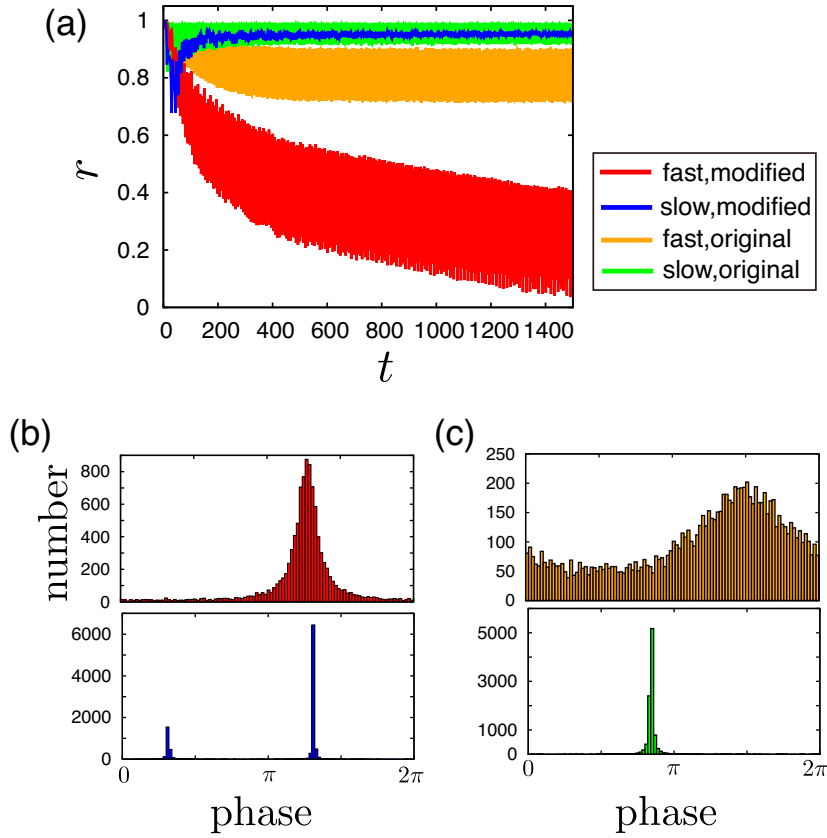


Figure 4.7: (a) Time evolutions of the order parameters in the modified and original models with $N = 10^4$ for $A = 0$. Bottom panels are snapshots of the steady-state phase distributions in (b) the modified system and (c) the original system for $A = 0.2$.

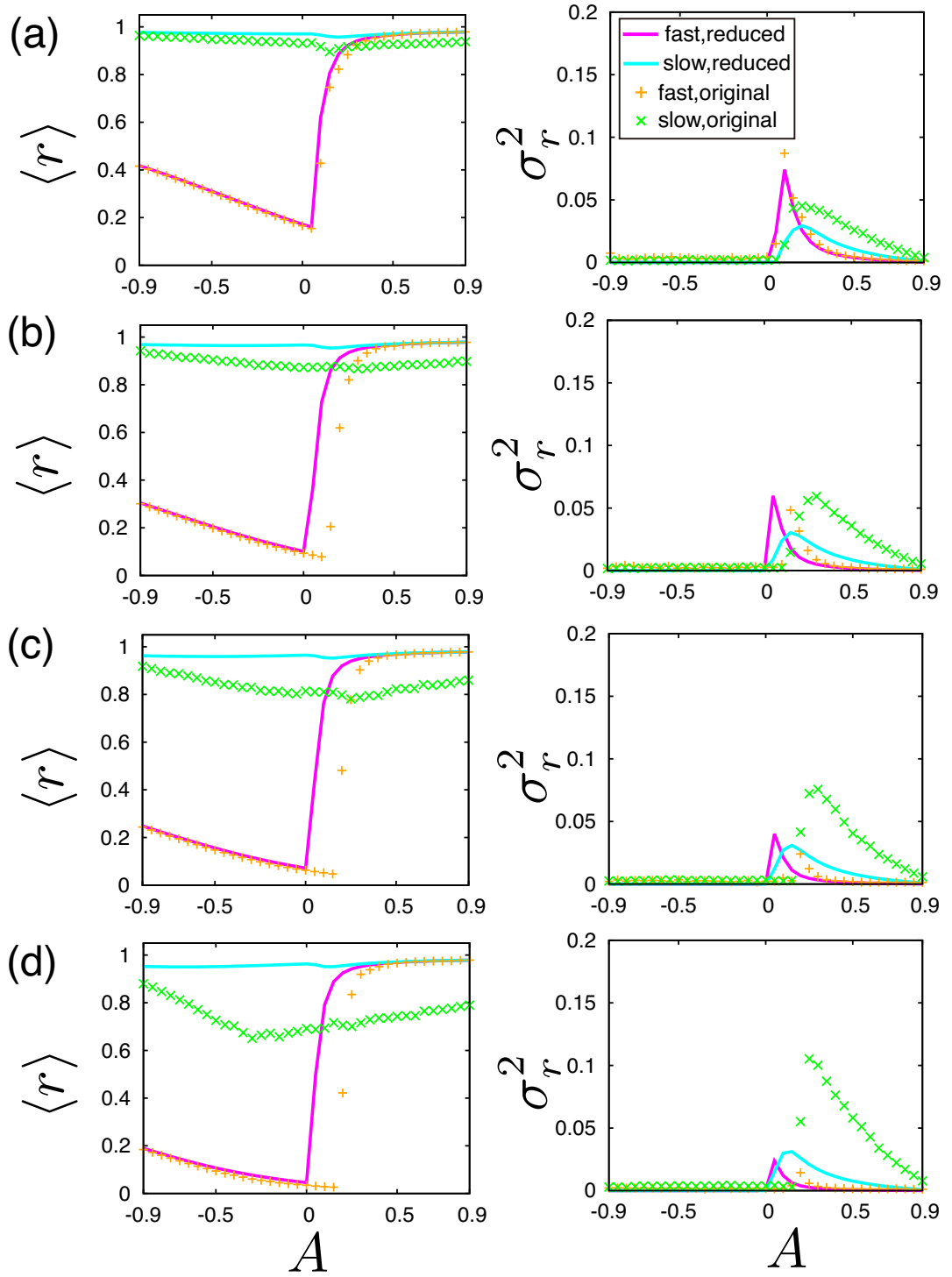


Figure 4.8: (Left) Averages and (Right) standard deviations of the steady-state order parameters in the reduced and original systems with $N = 10^4$. Top to bottom: (a) $k = 3$, (b) $k = 5$, (c) $k = 7$ and (d) $k = 11$.

Chapter 5

Review of Information Theory

Before moving on to the last topic, we review the information theory related to our research.

5.1 Probability Distributions on Circle

Generally, the information theory considers a random variable x drawn from a distribution $p(x)$, where x is continuous or discontinuous, and defined in real axis [18]. Especially, we here suppose that a random variable θ is continuous and defined on a circle, i.e., in the interval $[0, 2\pi)$. Then, we consider the information theory on the variable θ , which does not change formally from the ordinary case with x .

In this section we introduce examples of distributions on the circular variable θ . The probability density function $p(\theta)$ satisfies the normalized condition

$$\int_0^{2\pi} d\theta p(\theta) = 1, \quad (5.1)$$

and is always nonnegative for $[0, 2\pi)$. As typical distributions, I illustrate two familiar distributions for the circular random variable θ according to Ref. [40]: one is the von Mises (circular normal) distribution and the other is the wrapped normal distribution.

The von Mises distribution has the form of the density function as

$$p_{\text{vM}}(\theta) = \frac{1}{2\pi I_0(\kappa)} e^{\kappa \cos(\theta - \mu)}, \quad (5.2)$$

where the parameters $0 \leq \mu < 2\pi$ and $\kappa \geq 0$ decide the mode and the deviation, respectively. $I_0(\kappa)$ is the modified Bessel function and defined as

$$I_0(\kappa) = \frac{1}{2\pi} \int_0^{2\pi} d\theta e^{\kappa \cos \theta}. \quad (5.3)$$

The von Mises distribution has the maximum at the point $\theta = \mu$ and its form is symmetric unimodal.

A wrapped normal distribution is given by wrapping a normal distribution around the circle, and its density function has the form

$$p_{\text{WN}}(\theta) = \frac{1}{\sqrt{2\pi\sigma^2}} \sum_{m=-\infty}^{\infty} e^{-\frac{(\theta-\mu-2m\pi)^2}{2\sigma^2}}, \quad (5.4)$$

where μ represents the center and σ sets the deviation. The comparison between a original normal distributions and the corresponding wrapped normal distributions are represented in Fig. 5.1. We note that the wrapped normal distribution can be approximated to the von

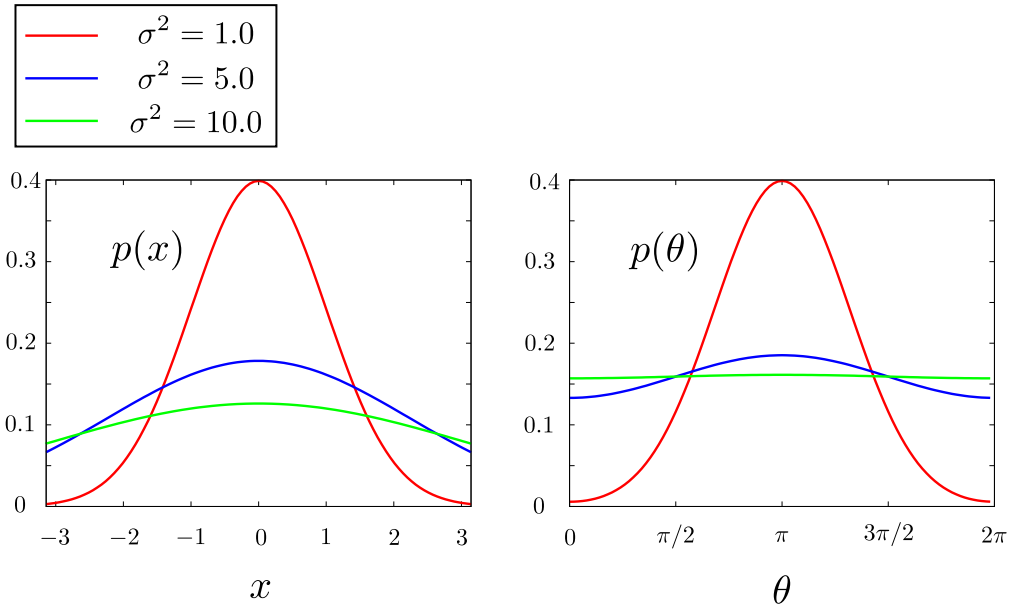


Figure 5.1: (Left) Normal distributions for $\sigma^2 = 1.0, 5.0, 10.0$ and (right) the corresponding wrapped normal distributions for $\sigma^2 = 1.0, 5.0, 10.0$.

Mises distribution with $\kappa = -\sigma^2/2$ when σ is small.

5.2 Shannon Entropy

We consider a circular random variable $\theta \in [0, 2\pi)$ drawn from a distribution $p(\theta)$. We here introduce a fundamental quantity for information, the entropy [18]. The entropy $H(\theta)$ is defined for θ by

$$H(p(\theta)) = - \int_0^{2\pi} d\theta p(\theta) \log p(\theta), \quad (5.5)$$

where we suppose that $0 \log 0 = 0$. This quantity represents the uncertainty of random variable θ . We also denote $H(p(\theta))$ as only $H(p)$ or $H(\theta)$ for simplicity.

With the constraints that $\langle \cos \theta \rangle$ and $\langle \sin \theta \rangle$ are constants, the distribution which maximizes $H(\theta)$ is the von Mises distribution [40].

5.3 Relative Entropy

We next introduce a quantity to measure the difference between two probability density functions $p(\theta)$ and $q(\theta)$. The relative entropy (Kullback-Leibler divergence) is defined by

$$D(p(\theta)||q(\theta)) = \int_0^{2\pi} d\theta p(\theta) \log \frac{p(\theta)}{q(\theta)}, \quad (5.6)$$

where we suppose that $0 \log \frac{0}{0} = 0$. We denote $D(p(\theta)||q(\theta))$ as only $D(p||q)$ for simplicity.

The relative entropy is nonnegative and becomes zero if and only if $p(\theta) = q(\theta)$ almost everywhere .

5.4 Mutual Information

We introduce a quantity to characterize the correlation between two random variables θ_1 and θ_2 , where they are drawn from different probability density functions $p_1(\theta_1)$ and $p_2(\theta_2)$, respectively. For the two variables θ_1 and θ_2 , which have the joint probability density function $p(\theta_1, \theta_2)$, the mutual information is defined by

$$I(\theta_1; \theta_2) = \int_0^{2\pi} d\theta_1 \int_0^{2\pi} d\theta_2 p(\theta_1, \theta_2) \log \frac{p(\theta_1, \theta_2)}{p_1(\theta_1) p_2(\theta_2)}. \quad (5.7)$$

5.5 Transfer Entropy

Finally, we introduce the transfer entropy [95, 13], which measures the directed information flow between two random processes $\{\theta_1(t)\}_{t=1}^{t=N}$ and $\{\theta_2(t)\}_{t=1}^{t=N}$, where time is discontinuous.

Transfer entropy is defined by

$$T_{1 \rightarrow 2}^\tau(t) = \int_0^{2\pi} d\theta_1(t) \int_0^{2\pi} d\theta_2(t) \int_0^{2\pi} d\theta_2(t+\tau) p[\theta_2(t+\tau), \theta_2(t), \theta_1(t)] \log \frac{p[\theta_2(t+\tau)|\theta_2(t), \theta_1(t)]}{p[\theta_2(t+\tau)|\theta_2(t)]} \quad (5.8)$$

$$= \left\langle \log \frac{p[\theta_2(t+\tau)|\theta_2(t), \theta_1(t)]}{p[\theta_2(t+\tau)|\theta_2(t)]} \right\rangle, \quad (5.9)$$

where we denote the expectation of f as $\langle f \rangle$. The transfer entropy can be written by the relative entropy as

$$T_{1 \rightarrow 2}^\tau(t) = D(p[\theta_2(t+\tau), \theta_2(t), \theta_1(t)] || p[\theta_2(t+\tau)|\theta_2(t)] p[\theta_1(t), \theta_2(t)]), \quad (5.10)$$

from which the transfer entropy must be nonnegative. In addition, the transfer entropy can be represented in terms of the mutual information and the entropy as

$$T_{1 \rightarrow 2}^\tau(t) = I(\theta_2(t+\tau); \theta_1(t) | \theta_2(t)) \quad (5.11)$$

$$= H(\theta_2(t+\tau) | \theta_2(t)) - H(\theta_2(t+\tau) | \theta_1(t), \theta_2(t)). \quad (5.12)$$

The transfer entropy compares the probability density functions $p[\theta_2(t+\tau)|\theta_2(t), \theta_1(t)]$ and $p[\theta_2(t+\tau)|\theta_2(t)]$, where the former represents the prediction of $\theta_2(t+\tau)$ from the knowledge of only $\theta_2(t)$ while the latter from $\theta_2(t)$ and $\theta_1(t)$. Therefore, the transfer entropy measures the effect of the present observation $\theta_1(t)$ in predicting the future $\theta_2(t+\tau)$. In other words, the transfer entropy represents the directed information flow from the present state of θ_1 to the future state of θ_2 .

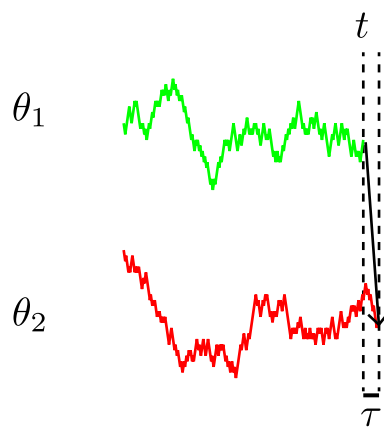


Figure 5.2: Transfer entropy quantifies the causality between two processes.

Chapter 6

Transfer Entropy in Coupled Oscillators

The content of this chapter will be published in [110].

6.1 Introduction

Rhythmic phenomena are observed ubiquitously in nature and rhythms play an important role in various fields [104, 90, 15]. Especially it was suggested that the rhythms are often correlated to functions in the brain [15]. Rhythmic systems are described mathematically by dynamical systems [90, 50, 117]. The dynamical systems have generally high dimensions, which makes it difficult to analyze. The phase reduction theory allows us to reduce perturbed limit-cycle systems to phase oscillator systems and this reduction helps us analyze the dynamical systems [50]. Thus, coupled phase oscillators play an important role as a paradigmatic model, which describes synchronization.

Most studies of coupled phase oscillators focus only on the dynamical properties as in the previous chapters. On the other hand, causality in dynamical systems recently has been shed light on [30, 107]. Detecting causality in dynamical systems is a significant problem and is expected to serve us to understand background mechanisms and control the systems. To reveal causality in time series, the transfer entropy was introduced from the point of view of information theory [95, 42, 13]. The transfer entropy measures directed information transfer between two observables and has been used to detect causality.

On the other hand, in the field of information thermodynamics the transfer entropy plays a crucial role in the different context [37, 38]. The transfer entropy was extended to continuous time systems and has been calculated in simple examples [6, 36].

In this chapter we consider the transfer entropy in coupled oscillator systems with continuous time. Then, as in the linear Gaussian case [38] we use the path integral approach. Moreover, we will discuss the relation between the types of coupling functions and the transfer entropy.

This chapter organizes as follows. In Sec. 6.2, the model setting is explained. We will formalize the transfer entropy in our model and estimate it in section 6.3. In Sec. 6.4, the calculated transfer entropy is represented. The summary of this chapter is described in section 6.5.

6.2 Model

To formulate the transfer entropy in coupled phase oscillators, we consider a pair of coupled oscillators with stochastic noises

$$\frac{d\theta_1}{dt} = \omega_1 - K_{12} \sin(\theta_1 - \theta_2 + \alpha) + \xi_1(t), \quad (6.1)$$

$$\frac{d\theta_2}{dt} = \omega_2 - K_{21} \sin(\theta_2 - \theta_1 + \alpha) + \xi_2(t), \quad (6.2)$$

as a simple example. The natural frequencies are ω_1, ω_2 , and the coupling strengths are K_{12} and K_{21} . The phase lag parameter α is supposed to be time-independent. We assume that the Gaussian noises $\xi_i(t)$ satisfy the conditions of the averages and deviations

$$\langle \xi_i(t) \rangle = 0, \quad \langle \xi_i(t) \xi_j(t') \rangle = 2D \delta_{ij} \delta(t - t'), \quad (6.3)$$

where the diffusion coefficient D characterizes the noise intensity.

6.3 Transfer Entropy Rate

In this section we consider the transfer entropy in the coupled phase oscillators. We introduced the transfer entropy for discontinuous time processes in section 6.3, but we deal with

continuous systems here. In Ref. [6], the transfer entropy rate was considered in continuous time processes and we will employ it. For this case the transfer entropy rate $dT_{1 \rightarrow 2}(t)/dt$ at time t is defined as

$$dT_{1 \rightarrow 2}(t) = \left\langle \log \frac{p[\theta_2(t+dt) | \theta_2(t), \theta_1(t)]}{p[\theta_2(t+dt) | \theta_2(t)]} \right\rangle \quad (6.4)$$

$$= \left\langle \log \frac{p[\theta_2(t+dt) | \theta_2(t), \theta_1(t)] p[\theta_2(t)]}{p[\theta_2(t+dt) | \theta_2(t)]} \right\rangle, \quad (6.5)$$

where dt represents infinitesimal limit. In order to obtain the transfer entropy rate we will calculate the probability density functions $p[\theta_2(t+dt) | \theta_2(t), \theta_1(t)]$, $p[\theta_2(t+dt) | \theta_2(t)]$ and $p[\theta_1(t)]$ under the evolutions (6.1) and (6.2) in the following. We here assume the case $\omega_1 = \omega_2 = \omega$ for simplicity, although the other case can be considered similarly.

The Fokker-Planck equation for the phase difference $\Theta = \theta_1 - \theta_2$ corresponding to the pair of the equations is [25]

$$\frac{\partial}{\partial t} p[\Theta, t] = -\frac{\partial}{\partial \Theta} \{(-K_{12} \sin(\Theta + \alpha) - K_{21} \sin(\Theta - \alpha)) p[\Theta, t]\} + \frac{\partial^2}{\partial \Theta^2} \{2D p[\Theta, t]\}. \quad (6.6)$$

The stationary solution $p[\Theta]$ to this equation is given by

$$p[\Theta] = \frac{1}{Z} \exp \left[\frac{K_{12} + K_{21}}{2D} \cos \Theta \cos \alpha \right], \quad (6.7)$$

where Z is the normalization constant as

$$Z = \int_0^{2\pi} d\Theta \exp \left[\frac{K_{12} + K_{21}}{2D} \cos \Theta \cos \alpha \right]. \quad (6.8)$$

Therefore, the joint probability density $p[\theta_1, \theta_2]$ is given by

$$p[\theta_1, \theta_2] = \frac{1}{2\pi Z} \exp \left[\frac{K_{12} + K_{21}}{2D} \cos(\theta_1 - \theta_2) \cos \alpha \right]. \quad (6.9)$$

The marginal distribution for θ_2 , $p[\theta_2]$, is obtained by marginalizing the joint distribution $p[\theta_1, \theta_2]$ as

$$\begin{aligned} p[\theta_2] &= \int_0^{2\pi} d\theta_1 p[\theta_1, \theta_2] \\ &= \frac{1}{2\pi Z} \int_0^{2\pi} d\theta_1 \exp \left[\frac{K_{12} + K_{21}}{2D} \cos \theta_1 \cos \alpha \right] \\ &= \frac{1}{2\pi}. \end{aligned} \tag{6.10}$$

Then, we consider next the conditional distribution $p[\theta_2(t+dt) | \theta_1(t), \theta_2(t)]$ carefully. If the variable $\theta_2(t+dt)$ were considered in the interval $(-\infty, \infty)$, we use the evolution equation (6.2) and the Onsager-Machlup function [91] and express $p[\theta_2(t+dt) | \theta_1(t), \theta_2(t)]$ as

$$\begin{aligned} p[\theta_2(t+dt) | \theta_1(t), \theta_2(t)] \\ = \frac{1}{\mathcal{N}(\theta_1(t), \theta_2(t))} \exp \left[-\frac{dt}{4D} \left(\frac{\theta_2(t+dt) - \theta_2(t)}{dt} - \omega_2 + K_{21} \sin(\theta_2(t) - \theta_1(t) + \alpha) \right)^2 \right], \end{aligned} \tag{6.11}$$

where $\mathcal{N}(\theta_1(t), \theta_2(t))$ is the normalized constant and we emphasize that it depends on $\theta_1(t)$ and $\theta_2(t)$. Remind that the variable $\theta_2(t+dt)$ is in fact defined on the circle $[0, 2\pi)$ and we wrap the normal distribution (6.11) and express $p[\theta_2(t+dt) | \theta_1(t), \theta_2(t)]$ as the infinite series:

$$\begin{aligned} p[\theta_2(t+dt) | \theta_1(t), \theta_2(t)] \\ = \frac{1}{\mathcal{N}(\theta_1(t), \theta_2(t))} \\ \sum_{k=-\infty}^{\infty} \exp \left[-\frac{dt}{4D} \left(\frac{\theta_2(t+dt) + 2k\pi - \theta_2(t)}{dt} - \omega_2 + K_{21} \sin(\theta_2(t) - \theta_1(t) + \alpha) \right)^2 \right]. \end{aligned} \tag{6.12}$$

Then, using Eqs. (6.9) and (6.12), we calculate the probability density function $p[\theta_2(t+dt), \theta_2(t)]$ as

$$\begin{aligned}
& p[\theta_2(t+dt), \theta_2(t)] \\
&= \int_0^{2\pi} d\theta_1(t) p[\theta_2(t+dt) | \theta_1(t), \theta_2(t)] p[\theta_1(t), \theta_2(t)] \\
&= Z \int_0^{2\pi} d\theta_1(t) \frac{1}{\mathcal{N}(\theta_1(t), \theta_2(t))} \exp \left[\frac{K_{12} + K_{21}}{2D} \cos(\theta_1(t) - \theta_2(t)) \cos \alpha \right. \\
&\quad \left. - \frac{dt}{4D} \left(\frac{\theta_2(t+dt) - \theta_2(t)}{dt} - \omega_2 + K_{21} \sin(\theta_2(t) - \theta_1(t) + \alpha) \right)^2 \right] \\
&= Z \int_0^{2\pi} d\theta \frac{1}{\mathcal{N}(\theta, 0)} \times \\
&\quad \exp \left[\frac{K_{12} + K_{21}}{2D} \cos \theta \cos \alpha - \frac{dt}{4D} \left(\frac{\theta_2(t+dt) - \theta_2(t)}{dt} - \omega_2 + K_{21} \sin(\theta + \alpha) \right)^2 \right]. \quad (6.13)
\end{aligned}$$

Consequently, we obtain $dT_{1 \rightarrow 2}(t)$ from the representation (6.5). Then, the transfer entropy rate can be calculated by dividing it by dt . In the following section, we will estimate this value numerically.

6.4 Results

In the previous section the representation of the transfer entropy rate was obtained. We numerically calculate the transfer entropy rate from the expression (6.5) and bare stochastic simulations in this section. We set $dt = 0.01$, $\omega = 1.0$ and $K_{12} = K_{21}$.

In Fig. 6.1, the transfer entropy rates for (6.1) and (6.2) are represented for $D = 0.1, 1.0$. The color represents the value of α , which decides the type of coupling functions. The lines are obtained by performing the integrals of Eq. (6.5). The points show the transfer entropy rates obtained from the simulations of the stochastic differential equations (6.1) and (6.2) by the Euler-Maruyama method, in which the time-series data from $t \in [5.0, 1.0 \times 10^8]$ was used. In our simulation the state spaces for the phase variables were separated into 200 bins, respectively. We can see the correspondence between the two results. The dependence on the coupling strength K is similar to the linear Gaussian case [38]. That is, the strong coupling induces the high transfer entropy rate. Comparing the results of two different noise strengths in Fig. 6.1, we can suppose that the noise decreases the transfer entropy rate. In fact, the

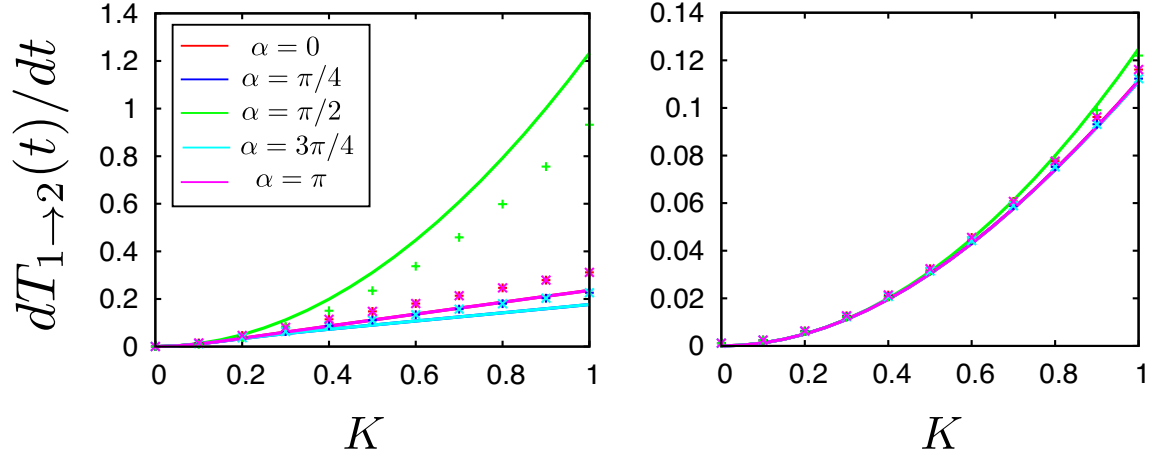


Figure 6.1: Transfer entropy rate for (left) $D = 0.1$ and (right) $D = 1.0$, where the lines represent the theoretical values obtained from (6.5) and the points represent the direct simulations with the stochastic differential equations (6.1) and (6.2), respectively. The line and point colors represent the α -values as shown in the inset.

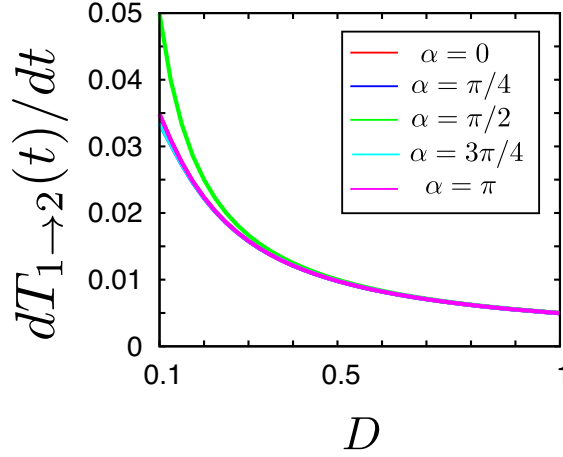


Figure 6.2: Transfer entropy rate for $K = 0.2$, where the theoretical line colors represent the α -values as in figure 6.1.

transfer entropy rate and the diffusion coefficient D for $K = 0.2$ are plotted in Fig. 6.2. This result suggests that the high noise strength decreases the transfer entropy rate. The noise dependency is also similar to the linear case.

We next focus on the types of coupling functions. In figure 6.3 we show the transfer entropy rate in the case where the noise strength parameter D and the coupling strength K is set as $(D, K) = (0.1, 1.0)$. The α value decides the joint probability density $p[\theta_1, \theta_2]$ as in equation (6.9). The peak at $\alpha = \pi/2$ is observed, which means that the synchronization does not imply the maximum of information transfer. The interaction with $\alpha = \pi/2$ induces

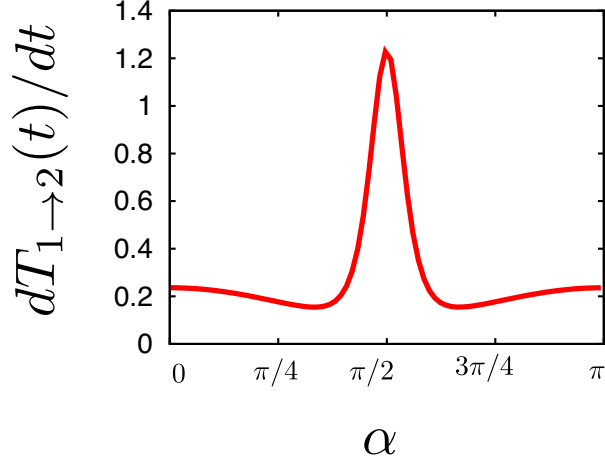


Figure 6.3: Transfer entropy rate for $(D, K) = (0.1, 1.0)$. The α value decides the type of the interaction between the oscillators.

the uniform distribution $p[\theta_1, \theta_2]$, which may cause the highest transfer entropy rate, since the uniformity increase averagely the effect of the interaction in (6.1) and (6.2).

6.5 Chapter Summary

In this chapter we calculated the transfer entropy rate in a pair of phase oscillators. The transfer entropy rate in continuous time systems is defined with infinitesimal dt [6] and we used the path integral approach to obtain its representation. The results from this formulation showed fairly the correspondence with the simulations of the stochastic differential equations. The dependence on the coupling strength and the noise strength is similar to the linear case [38].

Moreover, we paid attention to the types of coupling functions. The specific type of coupling function, which does not influence the dynamics of the phase difference, induces the peak of the transfer entropy rate. We speculate that this happens because this coupling function causes the uniform distribution as the joint distribution and this uniformity strengthens the effect from the other element in equations (6.1) and (6.2).

We would deal with the nonidentical cases where $\omega_1 \neq \omega_2$ elsewhere. In the identical case, the two oscillators are synchronized if the coupling strength has any value except zero. On the other hand, the difference in the natural frequencies generates the critical value K_c

which decides the existence of the fixed point of the phase difference equation, and there is the transition. We should reveal the relation between this transition and the transfer entropy.

Chapter 7

Conclusion

The thesis is concluded in this chapter. We summarize the results and suggestions obtained by our study.

7.1 Summary of Our Study

In this thesis, the phase reduction theory is employed and rhythmic systems are analyzed. At first, the breaking symmetry in the Kuramoto model, which is a paradigmatic model of coupled oscillators, is considered. The introduction of asymmetry in natural frequency distributions provides nonstandard transitions, which have not been observed in the Kuramoto model. The obtained bifurcation diagrams imply the roles of the asymmetry, for example, one of which is the addition of synchronization steps. Furthermore, the second theme considers the difference in natural frequency distributions across populations. From the phase reduction theory this difference brings specific coupling functions across the populations. We systematically investigate the dynamics of the populations with different natural frequency distributions by introducing the Ott-Antonsen ansatz and its variant with some modification. As a result, we can see the clustered chimera states, where clustered population and desynchronized population coexist. The effects by our modification to apply the Ott-Antonsen ansatz are evaluated. Finally, we employ the transfer entropy rate in coupled phase oscillators and discuss the relation between dynamics and information transfer. The expression of the transfer entropy rate is obtained by the path integral and this enables us to estimate the

information transfer. Our results imply that the synchronization does not always transfer the information best, although the strong coupling increases the information.

7.2 Future Works

In this section we exhibit some possibilities which our works give.

First, as we noted in chapter 3, it is a very interesting problem to consider the response in the Kuramoto model with asymmetric natural frequency distributions, since the analogy to the Hamiltonian mean-field model [118] gives us a suggestion on new phenomenon in coupled oscillator systems. Moreover, the comparison with symmetric cases [21] is sure to provide us deep understanding of the roles of the asymmetry.

Second perspective is related to the extension of the Ott-Antonsen ansatz. In chapter 4 we employ an additional assumption on the coupling function to apply the Ott-Antonsen ansatz and its variation straightforwardly to systems with multifrequencies. However, this assumption seems unnatural and alternative way to reduce the dynamics should be sought. This problem is related to the analysis of the Kuramoto model with first and second order interactions [46].

Moreover, we would tackle problems on the relation between asymmetry or multifrequency and computation in the brain. Many experiments reported the existences of the asymmetry and the multifrequency in the brain and suggested that they play some roles in the functions. [17, 79]. Our works have formulated theoretical frameworks to deal such systems, and further works to investigate their dynamics in terms of computation are needed.

In the last theme the transfer entropy was introduced to study information processing in coupled phase oscillators. On the other hand, the oscillators can show the interesting information processing which is related to associative memory [7]. We would like to reveal some relations between various types of information processing and dynamics.

Finally we remark that it is important to extend our attention to not only rhythms but also other types of dynamics in the brain. For example, it is known that theoretical population models consisting of simple neuronal elements show chaotic dynamics [101, 114]. These theoretical results suggest that realistic neuronal ensembles may exhibit chaos and

utilize its dynamics. The connection between chaos and computation must be a significant problem.

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