

Jørgensen groups of parabolic type II

(Countable infinite case)

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ABSTRACT. This paper is the second part of the study on Jørgensen groups of parabolic type. We will show that there are countable infinite many Jørgensen groups of parabolic type on a certain cylinder in this case.

1. Introduction.

1.1. It is one of the most important problems in the theory of Kleinian groups to decide whether or not a subgroup G of the Möbius transformation group is discrete. For this problem there are two important and useful theorems: One is Poincaré's polyhedron theorem, which is a sufficient condition for G to be discrete. The other is Jørgensen's inequality, which is a necessary condition for a two-generator Möbius transformation group $\langle A, B \rangle$ to be discrete. Here we will consider extreme discrete groups (Jørgensen groups) for Jørgensen's inequality. This paper is the second part

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of a series of studies on Jørgensen groups (cf. Li - Oichi - Sato [4]).

1.2. Let Möb denote the set of all linear fractional transformations (Möbius transformations)

$$A(z) = \frac{az + b}{cz + d}$$

of the extended complex plane $\hat{\mathbf{C}} = \mathbf{C} \cup \{\infty\}$, where a, b, c, d are complex numbers and the determinant $ad - bc = 1$. There is an isomorphism between Möb and $PSL(2, \mathbf{C})$. We always write elements of Möb as matrices with determinant 1 in this paper. We recall that $\text{Möb} (= PSL(2, \mathbf{C}))$ acts on the upper half space H^3 of $\mathbf{R}^3$ as the group of conformal isometries of hyperbolic 3-space.

In this paper we use a Kleinian group in the same meaning as a discrete group. Namely, a Kleinian group is a discrete subgroup of Möb . A Kleinian group G is of *the first kind* if the limit set $\Lambda(G)$ of G is all of the extended complex plane $\hat{\mathbf{C}}$ and it is of *the second kind* otherwise. A subgroup G of Möb is said to be *elementary* if there exists a finite G -orbit in $\mathbf{R}^3$.

1.3. The *trace* $\text{tr}(A)$ of the matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad (ad - bc = 1)$$

in $SL(2, \mathbf{C})$ is defined by $\text{tr}(A) = a + d$. We remark that the trace of an element of $\text{Möb} (= PSL(2, \mathbf{C}))$ is not well-defined, but Jørgensen number (defined later) is still well-defined after choosing matrix representatives.

1.4. Let A^* and B^* be matrices in $SL(2, \mathbf{C})$ representing the Möbius transformations A and B , respectively. As A^* and B^* are determined by A and B to within a factor of -1 , we see that the commutator $A^*B^*(A^*)^{-1}(B^*)^{-1}$ (resp. $(A^*)^2$) are uniquely determined by A and B (resp. A). Thus we may write $\text{tr}(ABA^{-1}B^{-1}) = \text{tr}(A^*B^*(A^*)^{-1}(B^*)^{-1})$ and $\text{tr}^2(A) = \text{tr}^2(A^*)$.

In 1976 Jørgensen obtained the following important theorem, which gives a necessary condition for a non-elementary Möbius transformation group $G = \langle A, B \rangle$ to be discrete.

Theorem A (Jørgensen [1]). *Suppose that the Möbius transformations A and B generate a non-elementary discrete group. Then*

$$J(A, B) := |\operatorname{tr}^2(A) - 4| + |\operatorname{tr}(ABA^{-1}B^{-1}) - 2| \geq 1.$$

The lower bound 1 is best possible.

1.5. DEFINITION 1. Let A and B be Möbius transformations. The *Jørgensen number* $J(A, B)$ for the ordered pair (A, B) is defined by

$$J(A, B) := |\operatorname{tr}^2(A) - 4| + |\operatorname{tr}(ABA^{-1}B^{-1}) - 2|.$$

DEFINITION 2. A subgroup G of Möb is called a *Jørgensen group* if G satisfies the following four conditions:

- (1) G is a two-generator group.
- (2) G is a discrete group.
- (3) G is a non-elementary group.
- (4) There exist generators A and B of G such that $J(A, B) = 1$.

1.6 Jørgensen and Kiikka showed the following.

Theorem B (Jørgensen-Kiikka [2]). *Let $\langle A, B \rangle$ be a non-elementary discrete group with $J(A, B) = 1$. Then A is elliptic of order at least seven or A is parabolic.*

If $\langle A, B \rangle$ is a Jørgensen group such that A is parabolic and $J(A, B) = 1$, then we call it a *Jørgensen group of parabolic type*. There are infinite many Jørgensen groups of parabolic type (Jørgensen-Lascurain-Pignataro [3], Sato [6]).

Now it gives rise to the following problem.

Problem 1. Find all Jørgensen groups of parabolic type.

1.7. Let $\langle A, B \rangle$ be a marked two-generator group such that A is parabolic. Then we can normalize A and B as follows:

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \text{ and } B_{\sigma, \mu} = \begin{pmatrix} \mu\sigma & \mu^2\sigma - 1/\sigma \\ \sigma & \mu\sigma \end{pmatrix}$$

where $\sigma \in \mathbb{C} \setminus \{0\}$ and $\mu \in \mathbb{C}$. See [4] for this normalization.

We denote by $G_{\sigma, \mu}$ the marked group generated by A and $B_{\sigma, \mu}$: $G_{\sigma, \mu} = \langle A, B_{\sigma, \mu} \rangle$. We say that $(\sigma, \mu) \in \mathbb{C} \setminus \{0\} \times \mathbb{C}$ is the point representing a marked group $G_{\sigma, \mu}$ and that $G_{\sigma, \mu}$ is the marked group corresponding to a point (σ, μ) .

1.8. In [6], Sato considered the case of $\mu = ik$ ($k \in \mathbb{R}$). Namely, he considered marked two-generator group $G_{\sigma, ik} = \langle A, B_{\sigma, ik} \rangle$ generated by

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \text{ and } B_{\sigma, ik} = \begin{pmatrix} ik\sigma & -k^2\sigma - 1/\sigma \\ \sigma & ik\sigma \end{pmatrix}$$

where $\sigma \in \mathbb{C} \setminus \{0\}$ and $k \in \mathbb{R}$.

Now we have the following conjecture.

CONJECTURE. For any Jørgensen group G of parabolic type there exists a marked group $G_{\sigma, ik}$ ($\sigma \in \mathbb{C} \setminus \{0\}, k \in \mathbb{R}$) such that $G_{\sigma, ik}$ is conjugate to G .

If this conjecture is true, then it is sufficient to consider the case of $\mu = ik$ in order to find all Jørgensen groups of parabolic type. In this paper we only consider the case of $\mu = ik$.

1.9. Let C be the following cylinder:

$$C = \{(\sigma, ik) \mid |\sigma| = 1, k \in \mathbb{R}\}.$$

Theorem C (Sato [6]). *If a marked two-generator group $G_{\sigma, ik}$ ($\sigma \in \mathbb{C} \setminus \{0\}, k \in$*

$\mathbf{R})$ is a Jørgensen group, then the point (σ, ik) representing $G_{\sigma, ik}$ lies on the cylinder C .

By Theorem C, if (θ, k) is a point on the cylinder C , then we can set $\sigma = -ie^{i\theta}$ ($0 \leq \theta \leq 2\pi$). If a point $(-ie^{i\theta}, ik)$ on the cylinder C represents a Jørgensen group, then we say that the group is a Jørgensen group of parabolic type (θ, k) .

Now it gives rise to the following problem.

Problem 2. Find all Jørgensen groups of parabolic type (θ, k) .

We divide Jørgensen groups of this type into three parts as follows:

Part 1. $|k| \leq \sqrt{3}/2$, $0 \leq \theta \leq 2\pi$ (finite case).

Part 2. $\sqrt{3}/2 < |k| \leq 1$, $0 \leq \theta \leq 2\pi$ (countable infinite case).

Part 3. $1 < |k|$, $0 \leq \theta \leq 2\pi$ (uncountable infinite case).

By some lemmas in [6], it suffices to consider the case of $0 \leq \theta \leq \pi/2$ and $k \geq 0$ in order to find Jørgensen groups of parabolic type (θ, k) .

In the previous paper [4] we find all Jørgensen groups in the case where $0 \leq \theta \leq \pi/2$ and $0 \leq k \leq \sqrt{3}/2$, that is, we obtain the following theorem.

Theorem D (finite case) (Li - Oichi - Sato [4]). (i) *There are sixteen Jørgensen groups in $D = \{(\theta, k) \in \mathbf{R} \mid 0 \leq \theta \leq \pi/2, 0 \leq k \leq \sqrt{3}/2\}$.*

(ii) *Nine of them are Kleinian groups of the first kind and seven groups are of the second kind.*

1.10. For a sufficient condition for a subgroup of the Möbius transformation group to be discrete, the following theorem is well-known.

Theorem E (Poincaré's Polyhedron Theorem (Maskit [5, p.73])).

Let P be a polyhedron with side pairing transformations satisfying the following conditions (1) through (6). Then, G , the group generated by the side pairing trans-

formations, is discrete and P is a fundamental polyhedron for G , and the reflection relations and cycle relations form a complete set of relations for G :

(1) For each side s of P , there is a side s' and there is an element $g_s \in G$ satisfying $g_s(s) = s'$ and $g_{s'} = g_s^{-1}$.

(2) $g_s(P) \cap P = \emptyset$.

(3) For every point $z \in P^*$, $p^{-1}(z)$ is a finite set. Here P^* is the space of equivalence classes so that the projection $p : \bar{P}$ (the closure of P) $\rightarrow P^*$ is continuous and open.

(4) Let e be an edge and let h be the cycle transformation at e . Then for each edge e , there is a positive integer t such that $h^t = 1$.

(5) Let $\{e_1, e_2, \dots, e_m\}$ be any cycle of edges of P and let $\alpha(e_k)$ ($k = 1, 2, \dots, m$) be the angle measured from inside P at the edge e_k . Let q be the smallest positive integer such that $h^q = 1$, where h is the cycle transformation at e_1 . Then the equality

$$\sum_{k=1}^m \alpha(e_k) = 2\pi/q$$

holds.

(6) P^* is complete.

2. Theorems.

In this section we will state that we find all Jørgensen groups in Part 2, that is, we obtain the following theorems. The proofs will appear elsewhere.

Main Theorem. *There are countable infinite many Jørgensen groups on the region $\{(\theta, k) \mid 0 \leq \theta \leq \pi/2, \sqrt{3}/2 < k \leq 1\}$.*

For simplicity we write $B_{\theta, k}$ for $B_{-ie^{i\theta}, ik}$.

This theorem consists of the following Theorem 1 through Theorem 6.

Let A and $B_{\theta,k}$ ($k \in \mathbf{R}, 0 \leq \theta \leq \pi/2$) be the following matrices:

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \text{ and } B_{\theta,k} = \begin{pmatrix} ke^{i\theta} & ie^{-i\theta}(k^2e^{2i\theta} - 1) \\ -ie^{i\theta} & ke^{i\theta} \end{pmatrix}$$

We can prove these theorems by using Jørgensen's inequality and Poincaré's polyhedron theorem.

Theorem 1 (Li - Oichi - Sato [4]). *Let $G_{\theta,k} = \langle A, B_{\theta,k} \rangle$ be the group generated by A and $B_{\theta,k}$. If $0 < \theta < \pi/6$, $\pi/6 < \theta < \pi/4$, $\pi/4 < \theta < \pi/3$, $\pi/3 < \theta < \pi/2$, then $G_{\theta,k} = \langle A, B_{\theta,k} \rangle$ are not Kleinian groups and so not Jørgensen groups for $k \in \mathbf{R}$.*

Theorem 2 (the case of $\theta = 0$). *Let*

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \text{ and } B_k := B_{0,k} = \begin{pmatrix} k & i(k^2 - 1) \\ -i & k \end{pmatrix} \quad (k \in \mathbf{R}),$$

and let $G_k = \langle A, B_k \rangle$. Then the following hold.

- (i) *In the case where $\cos(\pi/2m) < k < \cos(\pi/(2m+2))$ and $k \neq \cos(\pi/(2m+1))$ ($m = 3, 4, \dots$), G_k are not Kleinian groups and not Jørgensen groups.*
- (ii) *In the case of $k = 1$, G_k is a Kleinian group of the second kind and a Jørgensen group, and $\Omega(G_k)/G_k$ is a union of two Riemann surfaces with signature $(0; 2, 3, \infty)$.*
- (iii) *In the case of $k = \cos(\pi/n)$ ($n = 7, 8, \dots$), G_k is a Kleinian group of the second kind and a Jørgensen group, and $\Omega(G_k)/G_k$ is a union of two Riemann surfaces with signatures $(0; 2, 3, n)$ and $(0; 2, 3, \infty)$.*

Theorem 3 (the case of $\theta = \pi/6$). *Let*

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \text{ and } B_k := B_{\pi/6,k} = \begin{pmatrix} ke^{\pi i/6} & i(k^2e^{\pi i/6} - e^{-\pi i/6}) \\ -ie^{\pi i/6} & ke^{\pi i/6} \end{pmatrix} \quad (k \in \mathbf{R}),$$

and let $G_k = \langle A, B_k \rangle$. Then G_k are not Kleinian groups and not Jørgensen groups for k with $\sqrt{3}/2 < k \leq 1$.

Theorem 4 (the case of $\theta = \pi/4$). Let

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \text{ and } B_k := B_{\pi/4, k} = \begin{pmatrix} ke^{\pi i/4} & i(k^2 e^{\pi i/4} - e^{-\pi i/4}) \\ -ie^{\pi i/4} & ke^{\pi i/4} \end{pmatrix} \quad (k \in \mathbf{R}),$$

and let $G_k = \langle A, B_k \rangle$. Then the following hold.

(i) In case of $\sqrt{3}/2 < k < 1$, G_k are not Kleinian groups and not Jørgensen groups.

(ii) In the case of $k = 1$, G_k is a Kleinian group of the first kind and a Jørgensen group. The volume $V(G_{\pi/4, 1})$ of the 3-orbifold for $G_{\pi/4, 1}$ is

$$V(G_{\pi/4, 1}) = 8[2L(\pi/4) - L(\pi/12) - L(5\pi/12)],$$

where $L(\theta)$ is the Lobachevskii function:

$$L(\theta) = - \int_0^\theta \log |2 \sin u| du.$$

Theorem 5 (the case of $\theta = \pi/3$). Let

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \text{ and } B_k := B_{\pi/3, k} = \begin{pmatrix} ke^{\pi i/3} & i(k^2 e^{\pi i/3} - e^{-\pi i/3}) \\ -ie^{\pi i/3} & ke^{\pi i/3} \end{pmatrix} \quad (k \in \mathbf{R}),$$

and let $G_k = \langle A, B_k \rangle$. Then G_k are not Kleinian groups and not Jørgensen groups for k with $\sqrt{3}/2 < k \leq 1$.

Theorem 6 (the case of $\theta = \pi/2$). Let

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \text{ and } B_k := B_{\pi/2, k} = \begin{pmatrix} ik & -(k^2 + 1) \\ 1 & ik \end{pmatrix} \quad (k \in \mathbf{R}),$$

and let $G_k = \langle A, B_k \rangle$. Then the following hold.

- (i) In the case where $\cos(\pi/(2n-1)) < k < \cos(\pi/(2n+1))$ and $k \neq \cos(\pi/2n)$ ($n = 3, 4, \dots$), G_k are not Kleinian groups and not Jørgensen groups.
- (ii) In the case of $k = 1$, G_k is a Kleinian group of the second kind and a Jørgensen group, and $\Omega(G_k)/G_k$ is a Riemann surface with signature $(0; 3, 3, \infty)$.
- (iii) In the case of $k = \cos(\pi/n)$ ($n = 7, 8, \dots$), G_k are Kleinian groups of the second kind and Jørgensen groups, and $\Omega(G_k)/G_k$ is a Riemann surface with signature $(0; 3, 3, n)$.

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