数理解析研究所講究録 1393 巻 2004 年 79-81

On the SL(3,R) action on 4-sphere

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Abstract

We construct the smooth $SL(3,\mathbf{R})$ actions on S^4. That is we solve the problem of F.Uchida ([6] and (P2) in [7]).

1 Introduction

In 1981 ([5]), F. Uchida gave the example of the orthogonal $SO(4)$-action on the 6-sphere S^6 which is not extendable to any continuous $SL(4,\mathbf{R})$-action. In 1985 ([6]), he studied the action ψ of $SO(3)$ on the 4-sphere S^4 which coming from the adjoint action on the vector space $sym(3)\simeq\mathbf{R}^5$, where $sym(3)=\{A\in M_3(\mathbf{R}):trace(A)=0,A=A^t\}$ and $S^4=\{A\in sym(3):trace(A^tA)=1\}$. Then he constructed a continuous $SL(3,\mathbf{R})$-action on S^4 which extends this $SO(3)$-action ψ. However this action is not smooth. It is still open whether this $SO(3)$-action can be extended to a smooth $SL(3,\mathbf{R})$-action or not. In this paper we construct the smooth $SL(3;\mathbf{R})$-action on S^4 which extends ψ.

2 Structure

First, remember the orbit structure of the $SO(3)$-action ψ on S^4. The orbit space of this action is closed interval $[-\frac{1}{3\sqrt{6}},\frac{1}{3\sqrt{6}}]$. Put the projection $\pi:S^4\to[-\frac{1}{3\sqrt{6}},\frac{1}{3\sqrt{6}}]$ which induced from a determinant of matrix. We can easily check $\pi^{-1}(\pm\frac{1}{3\sqrt{6}})$ are the singular orbits $\mathbf{R}P(2)$ and other orbits are the principal orbits $SO(3)/\mathbf{Z}_2\oplus\mathbf{Z}_2$.

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3 Construction

Let us construct the smooth action. Consider the natural smooth $SL(3,\mathbf{R})$-action on $\mathbf{C}P(2) \simeq \mathbf{C}^3 - \{0\}/\mathbf{C}^*$. Then the restricted $SO(3)$-action has just two singular orbits S^2 and $\mathbf{R}P(2)$ and the other orbits are principal orbit $SO(3)/\mathbf{Z}_2$. Moreover this action commute with complex conjugation. Remember the quotient space of $\mathbf{C}P(2)$ by complex conjugation is S^4 ([2]). Hence this action induces the smooth $SL(3,\mathbf{R})$-action Ψ on S^4 because of the commutativity $SL(3,\mathbf{R})$-action and the complex conjugation on $\mathbf{C}P(2)$. Consider the restriction Ψ to $SO(3)$. Then this restricted action has just two singular orbits $\mathbf{R}P(2)$ and the other orbits are principal orbit $SO(3)/\mathbf{Z}_2 \oplus \mathbf{Z}_2$. From the classification ([1]), the effective $SO(3)$-action on S^4 which has this orbit structure is unique up to equivariant diffeomorphic. So this restricted action is equivariant diffeomorphic to ψ. Therefor the $SL(3,\mathbf{R})$-action Ψ is smooth and extended action of ψ.

4 Classification problem

Does $SO(3)$-action ψ extend to a smooth $SL(3,\mathbf{R})$-action ? This problem was solved in this paper. The answer was Yes. To solve this problem, we can consider the classification problem about a smooth $SL(3,\mathbf{R})$-action on S^4.

In 1974 ([3]), C. R. Schneider succeeded in the classification of the real analytic $SL(2,\mathbf{R})$-action on S^2. In 1979 ([4]), F. Uchida classified the real analytic $SL(n,\mathbf{R})$-action on S^n for $n \geq 3$. Moreover, in 1981 ([5]), he succeeded in the classification of the real analytic $SL(n,\mathbf{R})$-action on S^m for $5 \leq n \leq m \leq 2n-2$. As is well known, $SL(n,\mathbf{R})$-action on S^{n-1} is unique up to equivalence and $SL(n,\mathbf{R})$-action on S^m for $m \leq n-2$ is trivial. Therefore the case $(n,m) = (3,4),(4,5),(4,6)$ and $m \geq 2n-1$ (for $SL(n,\mathbf{R})$-action on S^m) are still open. The author would like to solve the case $(n,m) = (3,4),(4,5),(4,6)$ near the future.

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