

MULTIPLICITY DISTANCE OF KNOTS

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ABSTRACT. We define multiplicity and multiplicity distance in a category. We define neighbourhood category of knots and discuss its multiplicity distance.

§1. Multiplicity in a category

Let $\mathcal{C} = (\mathcal{O}, \{\text{Hom}(X, Y)\}_{X, Y \in \mathcal{O}}, \circ)$ be a category. Namely $\mathcal{O}$ is a set of objects and $\text{Hom}(X, Y)$ is the set of morphisms from an object X to an object Y .

$\forall \varphi \in \text{Hom}(X, Y), \forall \psi \in \text{Hom}(Y, Z), \psi \circ \varphi \in \text{Hom}(X, Z)$ is defined such that the following (1) (2) and (3) hold.

(Notation: $\varphi \in \text{Hom}(X, Y) \Leftrightarrow \varphi : X \rightarrow Y$)

(1) $\forall \varphi : X \rightarrow Y, \forall \psi : Y \rightarrow Z, \forall \tau : Z \rightarrow W, (\varphi \circ \psi) \circ \tau = \varphi \circ (\psi \circ \tau)$

(2) $\forall X \in \mathcal{O}, \exists \text{id}_X : X \rightarrow X$ s.t. $\forall \varphi : X \rightarrow Y, \varphi \circ \text{id}_X = \varphi$, and $\forall \psi : Y \rightarrow X, \text{id}_X \circ \psi = \psi$,

(3) $\text{Hom}(X, Y) \cap \text{Hom}(Z, W) \neq \emptyset$

$\Rightarrow X = Z$ and $Y = W$.

Definition 1.

$$m : \bigcup_{(X, Y) \in \mathcal{O} \times \mathcal{O}} \text{Hom}(X, Y) \rightarrow \mathbb{R}_{\geq 1} \cup \{\infty\}$$

is a *multiplicity* on $\mathcal{C}$

$\stackrel{\text{def}}{\Leftrightarrow}$ (1) $\forall X \in \mathcal{O}, m(\text{id}_X) = 1$, and

(2) $\forall \varphi : X \rightarrow Y, \forall \psi : Y \rightarrow Z, m(\psi \circ \varphi) \leq m(\varphi)m(\psi)$.

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Here we think $\forall x \in \mathbb{R}_{\geq 1}, x \leq \infty, x \cdot \infty = \infty \cdot x = \infty, \infty \leq \infty$ and $\infty \cdot \infty = \infty$.

Definition 2. $X, Y \in \mathcal{O}$

$$m(X : Y) = \inf\{m(\varphi) \mid \varphi \in \text{Hom}(X, Y)\}$$

$m(X : Y)$: *multiplicity of X over Y with respect to m .*

Proposition 3. m : *multiplicity on $\mathcal{C}$*

$$(1) \forall X \in \mathcal{O}, m(X : X) = 1,$$

$$(2) \forall X, Y, Z \in \mathcal{O}, m(X : Z) \leq m(X : Y)m(Y : Z).$$

m has *finite multiplicity property (FMP)*

$$\stackrel{\text{def}}{\iff} \forall X, Y \in \mathcal{O}, m(X : Y) < \infty.$$

Definition 4.

$$d_m(X, Y) = \log_e(m(X : Y)m(Y : X))$$

$d_m(X, Y)$: *m -distance of X and Y*

Proposition 5. m : *multiplicity on $\mathcal{C}$ with FMP*

$(\mathcal{O}, d_m)$ is a *pseudo metric space i.e.*

$$(D1') d_m(X, Y) \geq 0, d_m(X, X) = 0,$$

$$(D2) d_m(X, Y) = d_m(Y, X),$$

$$(D3) d_m(X, Z) \leq d_m(X, Y) + d_m(Y, Z).$$

Example 6. $\mathcal{C}$: subcategory of the category of sets and maps (SET)

$$X, Y \in \mathcal{O}, f : X \rightarrow Y$$

$$m(f) = \sup\{|f^{-1}(y)| \mid y \in Y\}$$

m : *map multiplicity*

Proposition 7. *map multiplicity is a multiplicity, i.e.*

$$(1) \forall X \in \mathcal{O}, m(\text{id}_X) = 1, \text{ and}$$

$$(2) \forall f : X \rightarrow Y, \forall g : Y \rightarrow Z, m(g \circ f) \leq m(f)m(g).$$

Example 8. R : PID, M, N : R -modules finitely generated over R .

$r(M)$: the minimal number of generators of M over R .

$f : M \rightarrow N$: R -linear map

$$m_{\ker}(f) = e^{r(\ker f)}$$

$$m_{\text{coker}}(f) = e^{r(\text{coker } f)}$$

$$(\text{coker } f = N/f(M))$$

Proposition 9. (1) $m_{\ker}$ is a multiplicity.

(2) m_{coker} is a multiplicity.

When R is a field, M and N are finite dimensional vector spaces over R , and

$$d_{m_{\ker}}(M, N) = d_{m_{\text{coker}}}(M, N) = |\dim M - \dim N|.$$

§2. Neighbourhood category of knots

$\mathcal{K}$: unoriented knot types in $\mathbb{S}^3$, $K_1, K_2 \in \mathcal{K}$.

$f = (V_2, k_1)$ is a morphism from K_1 to K_2

$\stackrel{\text{def}}{\iff} V_2 \subset \mathbb{S}^3$: knotted solid torus with knot type K_2 ,

$k_1 \subset \text{int} V_2$: knot in $\mathbb{S}^3$ with knot type K_1 .

$(V, k) = (V', k') \stackrel{\text{def}}{\iff} \exists h : \mathbb{S}^3 \rightarrow \mathbb{S}^3$ ori. preserving homeomorphism s.t.

$$h(V) = V' \text{ and } h(k) = k'$$

$\text{Hom}(K_1, K_2)$: the set of morphisms from K_1 to K_2 .

$f : K_1 \rightarrow K_2, g : K_2 \rightarrow K_3, f = (V_2, k_1), g = (V_3, k_2)$

V'_2 : regular neighbourhood of k_2 with $V'_2 \subset \text{int}V_3$, k'_1 : knot in V'_2
s.t. $(V'_2, k'_1) = (V_2, k_1)$.

We define $g \circ f = (V_3, k'_1)$.

Proposition 10. $\mathcal{C}_K = (\mathcal{K}, \{\text{Hom}(K, J)\}_{K, J \in \mathcal{K}}, \circ)$ is a category.

$\mathcal{C}_K$: neighbourhood category of Knots

$\text{id}_K : K \rightarrow K$ is given by $\text{id}_K = (V, k)$ where V is a regular neighbourhood of a knot k with knot type K .

§3. Multiplicity distance of knots

Definition 11. $f : K_1 \rightarrow K_2$: morphism, $f = (V_2, k_1)$,

$h : V_2 \rightarrow \mathbb{S}^1 \times \mathbb{D}^2$: homeomorphism, $\pi : \mathbb{S}^1 \times \mathbb{D}^2 \rightarrow \mathbb{S}^1$: natural projection.

h is generic $\stackrel{\text{def}}{\iff} \pi \circ h|_{k_1} : k_1 \rightarrow \mathbb{S}^1$ is a Morse map,

i.e. it has only finitely many critical points
in different levels.

$$m(h) = \max\{ |(\pi \circ h|_{k_1})^{-1}(y)| \mid y \in \mathbb{S}^1 \}$$

$$m(f) = \min\{m(h) \mid h : V_2 \rightarrow \mathbb{S}^1 \times \mathbb{D}^2 \text{ generic homeomorphism}\}$$

Proposition 12. m is a multiplicity on $\mathcal{C}_K$.

Proposition 13. (1) $f : K_1 \rightarrow K_2$, $m(f) = 1 \Rightarrow K_1 = K_2$.

(2) $m(K_1 : K_2) = 1 \Leftrightarrow K_1 = K_2$.

Proposition 14. $(\mathcal{K}, d_m)$ is an unbounded metric space.

d_m : multiplicity distance of knots

Definition 15. (Ozawa [1])

k : knot in $\mathbb{S}^3 - \{(0, 0, 0, 1), (0, 0, 0, -1)\}$,

$$\pi : \mathbb{S}^3 - \{(0, 0, 0, 1), (0, 0, 0, -1)\} \cong \mathbb{S}^2 \times \mathbb{R} \rightarrow \mathbb{R}$$

: natural projection, $\pi|_k : k \rightarrow \mathbb{R}$: Morse function.

$$\text{trunk}(k) = \max\{|\pi^{-1}(y)| \mid y \in \mathbb{R}\}$$

$$\text{trunk}(K) = \min\{\text{trunk}(k) \mid \text{knot type of } k \text{ is } K\}$$

Proposition 16. (Ozawa [1]) $\forall n \in \mathbb{N}, \exists K \in \mathcal{K}$ s.t. $\text{trunk}(K) \geq n$.

Proposition 17. (1) $K, J \in \mathcal{K}$,

$$\frac{\text{trunk}(K)}{\text{trunk}(J)} \leq m(K : J) \leq \text{trunk}(K).$$

(2) $K, J \in \mathcal{K}$, J is not a companion of K ,

$$\Rightarrow m(K : J) = \text{trunk}(K).$$

Proposition 18. $K, J \in \mathcal{K}$,

$$(1) m(K : 0_1) \leq \text{braid}(K),$$

$$(2) m(K : 0_1) \leq 2\text{bridge}(K) - 1,$$

$$(3) m(K : J) = 2 \Rightarrow K \text{ is a } (2, p)\text{-cable of } J, \text{ or } K = 0_1.$$

Corollary 19. $K \in \mathcal{K}$,

$$m(K : 0_1) \geq \frac{\text{trunk}(K)}{\text{trunk}(0_1)} = \frac{\text{trunk}(K)}{2}.$$

$$K \neq 0_1,$$

$$d_m(K, 0_1) \geq \log_e \frac{\text{trunk}(K)}{2} \cdot 2 = \log_e \text{trunk}(K).$$

$\{d_m(K, 0_1) \mid K \in \mathcal{K}\}$ is unbounded.

Definition 20. $K \in \mathcal{K}$, $m(K) = m(K : 0_1)$.

$m(K)$: multiplicity of K , $m(K) \in \mathbb{N}$.

Remark 21. $m(K) \leq n \Leftrightarrow d_m(K, 0_1) \leq \log_e 2n$.

Proposition 22. $K \in \mathcal{K}$,

$$(1) m(K) = 1 \Leftrightarrow K = 0_1,$$

$$(2) m(K) = 2 \Leftrightarrow K \text{ is a } (2, p)\text{-torus knot},$$

- (3) $m(K) = 3 \Leftrightarrow K \neq 0_1$, K is not a $(2, p)$ -torus knot, and
- (a) K is a closed 3-braid, or
 - (b) K is a connected sum of some 2-bridge knots.

Proposition 23. K :Montesinos knot $\Rightarrow m(K) \leq 4$.

The detail will appear in [2].

REFERENCES

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