

Galois Connections arising in Clone Theory

Hajime Machida[†] and Jovanka Pantović[‡]

[†] International Christian University, Tokyo, Japan

[‡] University of Novi Sad, Serbia

Abstract

Galois connections appear in various areas in mathematics and computer science. In this article a brief review is presented on Galois connections arising in clone theory.

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1 Basic Notions

For a set S let $\mathcal{P}(S)$ denote the power set of S . For non-empty sets A and B let φ and ψ be mappings from $\mathcal{P}(A)$ to $\mathcal{P}(B)$ and $\mathcal{P}(B)$ to $\mathcal{P}(A)$, respectively:

$$\varphi : \mathcal{P}(A) \longrightarrow \mathcal{P}(B), \quad \psi : \mathcal{P}(B) \longrightarrow \mathcal{P}(A)$$

A pair (φ, ψ) of mappings is a *Galois connection* between A and B if φ and ψ satisfy the following conditions for any $X_1, X_2, X \in \mathcal{P}(A)$ and $Y_1, Y_2, Y \in \mathcal{P}(B)$.

- (1) $X_1 \subseteq X_2 \implies \varphi(X_1) \supseteq \varphi(X_2)$
 $Y_1 \subseteq Y_2 \implies \psi(Y_1) \supseteq \psi(Y_2)$
- (2) $X \subseteq (\psi \circ \varphi)(X), \quad Y \subseteq (\varphi \circ \psi)(Y)$
- (3) $(\varphi \circ \psi \circ \varphi)(X) = \varphi(X), \quad (\psi \circ \varphi \circ \psi)(Y) = \psi(Y)$

Here, notice that these three conditions are not independent in a sense that Condition (3) follows from Conditions (1) and (2). In fact, the first inclusion in Condition (2) and the first implication in Condition (1), by letting $X_1 = X$ and $X_2 = (\psi \circ \varphi)(X)$, gives us $\varphi(X) \supseteq (\varphi \circ \psi \circ \varphi)(X)$ and the second inclusion in Condition (2), applied to $Y = \varphi(X)$, yields $\varphi(X) \supseteq (\varphi \circ \psi \circ \varphi)(X)$, resulting in the equality $(\varphi \circ \psi \circ \varphi)(X) = \varphi(X)$. The second equality in Condition (3) is obtained analogously.

Note: For a non-empty set A let η be a mapping from $\mathcal{P}(A)$ into itself, i.e., $\eta : \mathcal{P}(A) \longrightarrow \mathcal{P}(A)$. The mapping η is a *closure operator* on A if it satisfies the following three conditions for any $X_1, X_2, X \in \mathcal{P}(A)$.

- (1) $X_1 \subseteq X_2 \implies \eta(X_1) \subseteq \eta(X_2)$
- (2) $X \subseteq \eta(X)$
- (3) $\eta(\eta(X)) = \eta(X)$

It is well-known that a Galois connection induces closure operators. Namely, for a Galois connection (φ, ψ) between A and B , the compositions

$$\psi \circ \varphi : \mathcal{P}(A) \longrightarrow \mathcal{P}(A) \quad \text{and} \quad \varphi \circ \psi : \mathcal{P}(B) \longrightarrow \mathcal{P}(B)$$

are closure operators on A and B , respectively.

For a Galois connection (φ, ψ) between A and B , a subset $X \in \mathcal{P}(A)$, or $Y \in \mathcal{P}(B)$, is a *Galois closed set* if it satisfies $X = (\psi \circ \varphi)(X)$, or $Y = (\varphi \circ \psi)(Y)$, respectively.

For a non-empty set A and $n > 0$, the set of n -variable functions on A , i.e., maps from A^n into A , is denoted by $\mathcal{O}_A^{(n)}$. The set of all functions on A is denoted by $\mathcal{O}_A$, i.e., $\mathcal{O}_A = \bigcup_{n=1}^{\infty} \mathcal{O}_A^{(n)}$. For $1 \leq i \leq n$ the *projection* e_i^n on A is a function in $\mathcal{O}_A^{(n)}$ which always takes the value of the i -th variable. Denote by $\mathcal{J}_A$ the set of projections on A . A subset C of $\mathcal{O}_A$ is a *clone* on A if C contains $\mathcal{J}_A$ and is closed under (functional) composition.

For a non-empty set A and $m > 0$, $A^m (= A \times \cdots \times A)$ is the direct product of m copies of A . A subset ρ of A^m is called an *m -ary relation* on A , i.e., ρ is a relation on A if $\rho \subseteq A^m$ for some $m > 0$. Let $\mathcal{R}_A^{(m)}$ denote the set of all m -ary relations on A and $\mathcal{R}_A$ denote the set of all finitary relations on A , i.e., $\mathcal{R}_A = \bigcup_{m=1}^{\infty} \mathcal{R}_A^{(m)}$.

2 Galois Connections

2.1 Clones and Relations

For a function $f \in \mathcal{O}_A^{(n)}$ and a relation $\rho \in \mathcal{R}_A^{(m)}$, we say that f *preserves* ρ , or ρ is an *invariant relation* of f , if

$$\begin{pmatrix} f(a_{11}, a_{12}, \dots, a_{1n}) \\ f(a_{21}, a_{22}, \dots, a_{2n}) \\ \dots \\ f(a_{m1}, a_{m2}, \dots, a_{mn}) \end{pmatrix}$$

belongs to ρ whenever ${}^t(a_{11} \ a_{21} \ \dots \ a_{m1})$, ${}^t(a_{12} \ a_{22} \ \dots \ a_{m2})$, $\dots$, ${}^t(a_{1n} \ a_{2n} \ \dots \ a_{mn})$ all belong to ρ . For $\rho \in \mathcal{R}_A$, the set of functions in $\mathcal{O}_A$ which preserve ρ is called the *polymorph* of ρ and denoted by $\text{Pol } \rho$.

Define mappings

$$\text{Pol} : \mathcal{P}(\mathcal{R}_A) \longrightarrow \mathcal{P}(\mathcal{O}_A) \quad \text{and} \quad \text{Inv} : \mathcal{P}(\mathcal{O}_A) \longrightarrow \mathcal{P}(\mathcal{R}_A)$$

by

$$\text{Pol}(R) = \{ f \in \mathcal{O}_A \mid (\forall \rho \in R) f \text{ preserves } \rho \}$$

and

$$\text{Inv}(F) = \{ \rho \in \mathcal{R}_A \mid (\forall f \in F) f \text{ preserves } \rho \}$$

for all $R \in \mathcal{P}(\mathcal{R}_A)$ and $F \in \mathcal{P}(\mathcal{O}_A)$. To rephrase, $\text{Pol}(R) = \bigcap_{\rho \in R} \text{Pol } \rho$.

Clearly, the pair (Pol, Inv) is a Galois connection between $\mathcal{R}_A$ and $\mathcal{O}_A$. This is the best known, and most typical, Galois connection in clone theory.

We define the following operations on $\mathcal{R}_A$. (Here, by operations we mean set-theoretical operations.) The operations ζ, τ and pr are unary operations and the operations $\cap$ and $\times$ are binary operations.

- (1) For $\rho \in \mathcal{R}_A^{(1)}$ and $\rho = \emptyset$, $\zeta\rho = \tau\rho = \rho$ and $pr\rho = \emptyset$.

For $\rho \in \mathcal{R}_A^{(m)}$ where $m \geq 2$,

$$\begin{aligned}\zeta\rho &= \{(a_1, a_2, \dots, a_m) \in A^m \mid (a_m, a_1, \dots, a_{m-1}) \in \rho\}, \\ \tau\rho &= \{(a_1, a_2, a_3, \dots, a_m) \in A^m \mid (a_2, a_1, a_3, \dots, a_m) \in \rho\}, \\ pr\rho &= \{(a_2, \dots, a_m) \in A^{m-1} \mid (\exists a_1 \in A)(a_1, a_2, \dots, a_m) \in \rho\}.\end{aligned}$$

- (2) For $\rho_1, \rho_2 \in \mathcal{R}_A^{(m)}$ where $m > 0$,

$$\rho_1 \cap \rho_2 = \{(a_1, \dots, a_m) \in A^m \mid (a_1, \dots, a_m) \in \rho_1 \text{ and } (a_1, \dots, a_m) \in \rho_2\}$$

- (3) For $\rho_1 \in \mathcal{R}_A^{(m)}$ and $\rho_2 \in \mathcal{R}_A^{(m')}$ where $m, m' > 0$,

$$\rho_1 \times \rho_2 = \{(a_1, \dots, a_m, b_1, \dots, b_{m'}) \in A^{m+m'} \mid (a_1, \dots, a_m) \in \rho_1 \text{ and } (b_1, \dots, b_{m'}) \in \rho_2\}.$$

Moreover, we define the diagonal relation Δ_A of arity 2 by

$$\Delta_A = \{(a, a) \mid a \in A\}.$$

A subset R of $\mathcal{R}_A$ is a *co-clone* (or, *relational clone*) on A if R contains Δ_A and is closed under all of the operations $\zeta, \tau, pr, \cap$ and $\times$.

It is easy to see that $\text{Pol}(R)$ is a clone for any R in $\mathcal{P}(\mathcal{R}_A)$ and $\text{Inv}(F)$ is a co-clone for any F in $\mathcal{P}(\mathcal{O}_A)$. The following remarkable result was established independently by several authors (e.g., [1]).

Theorem 2.1 *Let A be a finite set with $|A| > 1$.*

- (1) *For any $R \in \mathcal{P}(\mathcal{R}_A)$, if R is a co-clone then $\text{Inv}(\text{Pol}(R)) = R$.*
- (2) *For any $F \in \mathcal{P}(\mathcal{O}_A)$, if F is a clone then $\text{Pol}(\text{Inv}(F)) = F$.*

In other words, clones and co-clones are Galois closed sets of the Galois connection (Pol, Inv) .

2.2 Centralizers and Monoids

For functions $f \in \mathcal{O}_A^{(n)}$ and $g \in \mathcal{O}_A^{(m)}$ we say that f *commutes* with g , or f and g *commute*, if the following holds for every $m \times n$ matrix M over A with rows $r_1, \dots, r_m$ and columns $c_1, \dots, c_n$.

$$f(g({}^t c_1), \dots, g({}^t c_n)) = g(f(r_1), \dots, f(r_m))$$

We write $f \perp g$ when f commutes with g . The relation $\perp$ is a symmetric relation on $\mathcal{O}_A$.

As a special case, let $m = 1$ and $n \geq 1$. Then, for $f \in \mathcal{O}_k^{(n)}$ and $g \in \mathcal{O}_k^{(1)}$, f commutes with g if

$$f(g(x_1), \dots, g(x_n)) = g(f(x_1, \dots, x_n))$$

holds for all $(x_1, \dots, x_n) \in A^n$. Let $\mathcal{A} = (A; F)$ be an algebra. By definition, $g \in \mathcal{O}_A^{(1)}$ is an *endomorphism* of $\mathcal{A}$ if and only if $f \perp g$ holds for every $f \in F$. Denote by $\text{End}(\mathcal{A})$ the set of

endomorphisms of $\mathcal{A}$, i.e., $\text{End}(\mathcal{A}) = \{g \in \mathcal{O}_A^{(1)} \mid f \perp g \text{ for } \forall f \in F\}$.

For $F \subseteq \mathcal{O}_A$ the *centralizer* F^* of F is defined by

$$F^* = \{g \in \mathcal{O}_A \mid g \perp f \text{ for all } f \in F\}.$$

For any subset $F \subseteq \mathcal{O}_A$ the centralizer F^* is easily verified to be a clone. When $F = \{f\}$ we often write f^* instead of F^* . We also write F^{**} for $(F^*)^*$.

Two types of Galois connections can be defined with respect to the centralizers. First, let φ and ψ be the same mapping $\varphi (= \psi) : \mathcal{P}(\mathcal{O}_A) \longrightarrow \mathcal{P}(\mathcal{O}_A)$ defined by

$$\varphi(F) (= \psi(F)) = F^*$$

for all $F \in \mathcal{P}(\mathcal{O}_A)$. Then, clearly, the pair (φ, ψ) is a Galois connection between $\mathcal{O}_A$ and itself. Hence the map $F \longmapsto F^{**}$ is a closure operator on $\mathcal{O}_A$.

The second type of a Galois connection relates the centralizers to the monoids. As is well-known, a non-empty subset M of $\mathcal{O}_A^{(1)}$ is a (transformation) *monoid* on A if it is closed under composition and contains the identity id . The whole set $\mathcal{O}_A^{(1)}$ is the largest monoid on A and the singleton $\{id\}$ is the smallest monoid on A . Denote by $\mathcal{M}_A$ the set of monoids on A . For a clone C on A the unary part $C^{(1)}$ of C , i.e., $C^{(1)} = C \cap \mathcal{O}_A^{(1)}$, is a monoid. In particular, for any centralizer F^* the unary part of F^* , i.e., $F^* \cap \mathcal{O}_A^{(1)}$, is a monoid.

Let us define the mappings φ and ψ

$$\varphi : \mathcal{P}(\mathcal{O}_A^{(1)}) \longrightarrow \mathcal{P}(\mathcal{O}_A) \quad \text{and} \quad \psi : \mathcal{P}(\mathcal{O}_A) \longrightarrow \mathcal{P}(\mathcal{O}_A^{(1)})$$

by

$$\varphi(M) = M^* \quad \text{for all } M \in \mathcal{P}(\mathcal{O}_A^{(1)})$$

and

$$\psi(F) = F^* \cap \mathcal{O}_A^{(1)} \quad \text{for all } F \in \mathcal{P}(\mathcal{O}_A).$$

Then the pair (φ, ψ) of mappings is a Galois connection between $\mathcal{O}_A^{(1)}$ and $\mathcal{O}_A$. Moreover, notice that $\varphi(M)$ is always a clone on A and $\psi(F)$ is always a monoid on A .

For $M \subseteq \mathcal{O}_A^{(1)}$, M is a *centralizing monoid* if M satisfies the equation

$$M = M^{**} \cap \mathcal{O}_A^{(1)}.$$

In other words, a monoid M on A is a centralizing monoid if M satisfies $(\psi \circ \varphi)(M) = M$, that is, a centralizing monoid M is a Galois closed set of a Galois connection (φ, ψ) .

Lemma 2.2 For $M \subseteq \mathcal{O}_A^{(1)}$ the following conditions are equivalent.

- (1) M is a centralizing monoid.
- (2) For some subset $F \subseteq \mathcal{O}_A$, $M = F^* \cap \mathcal{O}_A^{(1)}$
- (3) For some algebra $\mathcal{A} = (A; F)$, $M = \text{End}(\mathcal{A})$

Note that Lemma 2.2 (2) asserts that a centralizing monoid is the unary part of some centralizer.

Concerning the images of monoids under φ , we have the following theorem ([10]). Let A be a finite set with $|A| > 2$. For a monoid M on A , define properties I and II in the following way:

I (Partial separation property)

For all $a, b, c, d \in A$, if $\{a, b\} \neq \{c, d\}$ and $c \neq d$ then M contains $f (= f_{cd}^{ab})$ which satisfies

$$f(a) = f(b) \quad \text{and} \quad f(c) \neq f(d).$$

II (Fixed-point-free property)

For every $i \in A$, M contains g_i which satisfies $g_i(i) \neq i$.

Then we obtain a sufficient condition for $\varphi(M)$ to be the least clone.

Theorem 2.3 ([10]) *For a monoid M on A , if M satisfies Properties I and II then $\varphi(M) (= M^*)$ is $\mathcal{J}_A$, i.e., the clone of projections.*

2.3 Hyperclones and Relations

For a set A let $\mathcal{P}_A^*$ denote the set of non-empty subsets of A , i.e., $\mathcal{P}_A^* = \mathcal{P}(A) \setminus \{\emptyset\}$. An n -ary hyperoperation f on A is a mapping from A^n to $\mathcal{P}_A^*$. For $n \geq 1$ let $\mathcal{H}_A^{(n)}$ be the set of n -ary hyperoperations on A , and $\mathcal{H}_A$ be the set of all hyperoperations on A , i.e., $\mathcal{H}_A = \bigcup_{n \geq 1} \mathcal{H}_A^{(n)}$. For $1 \leq i \leq n$, an i -th n -ary (hyper-) projection $\widehat{e}_i^n$ on A is the n -ary hyperoperation defined by $\widehat{e}_i^n(x_1, \dots, x_i, \dots, x_n) = \{x_i\}$ for all $(x_1, \dots, x_n) \in A^n$. For $f \in \mathcal{H}_A^{(n)}$ and $g_1, \dots, g_n \in \mathcal{H}_A^{(m)}$ where $m, n > 0$, the composition $f(g_1, \dots, g_n)$ of f and $g_1, \dots, g_n$ is defined in a natural way by

$$f(g_1, \dots, g_n)(x_1, \dots, x_m) = \bigcup \{ f(y_1, \dots, y_n) \mid y_i \in g_i(x_1, \dots, x_m) \text{ for } 1 \leq \forall i \leq n \}.$$

A hyperclone on A is a set of hyperoperations on A which is closed under composition and contains all (hyper-) projections.

Galois connections between $\mathcal{H}_A$ and $\mathcal{R}_A$ have been studied in three different ways. Here we denote them by $(dPol, dInv)$, $(mPol, mInv)$ and $(hPol, hInv)$. The first one, $(dPol, dInv)$, is independently due to F. Börner ([3]) and B. A. Romov ([16]), the second one, $(mPol, mInv)$, due to T. Drescher and R. Pöschel ([8]) and the third one, $(hPol, hInv)$, due to I. G. Rosenberg ([17]) and H. Machida and J. Pantović ([9]).

Let $m > 0$ be an integer. Recall that $\mathcal{R}_A^{(m)}$ denotes the set of all m -ary relations on A . Let us denote by $\mathcal{P}^*(\mathcal{R}_A^{(m)})$ the set of non-empty subsets of $\mathcal{R}_A^{(m)}$, i.e., $\mathcal{P}^*(\mathcal{R}_A^{(m)}) = \mathcal{P}_{\mathcal{R}_A^{(m)}}^*$ in the above notation.

Let $\rho \in \mathcal{R}_A^{(m)}$ be an m (> 0)-ary relation on A . We define ρ_d , ρ_m and ρ_h in $\mathcal{P}^*(\mathcal{R}_A^{(m)})$ as follows:

$$\begin{aligned} \rho_d &= \{ (A_1, \dots, A_m) \mid A_1 \times \dots \times A_m \subseteq \rho \} \\ \rho_m &= \{ (A_1, \dots, A_m) \mid \forall \ell \in \{1, \dots, m\} \quad \forall a \in A_\ell \\ &\quad (A_1 \times \dots \times A_{\ell-1} \times \{a\} \times A_{\ell+1} \times \dots \times A_m) \cap \rho \neq \emptyset \} \\ \rho_h &= \{ (A_1, \dots, A_m) \mid (A_1 \times \dots \times A_m) \cap \rho \neq \emptyset \} \end{aligned}$$

Let x be either of d , m or h . For a hyperoperation $f \in \mathcal{H}_A^{(n)}$ and a relation $\rho \in \mathcal{R}_A^{(m)}$, f is said to x -preserve ρ if

$$\begin{pmatrix} f(a_{11}, a_{12}, \dots, a_{1n}) \\ f(a_{21}, a_{22}, \dots, a_{2n}) \\ \dots \\ f(a_{m1}, a_{m2}, \dots, a_{mn}) \end{pmatrix}$$

belongs to ρ_x whenever ${}^t(a_{11} a_{21} \dots a_{m1}), {}^t(a_{12} a_{22} \dots a_{m2}), \dots, {}^t(a_{1n} a_{2n} \dots a_{mn})$ all belong to ρ . Let $xPol \rho$ denote the set of all hyperoperations on A that x -preserve ρ .

Define mappings

$$xPol : \mathcal{P}(\mathcal{R}_A) \longrightarrow \mathcal{P}(\mathcal{H}_A) \quad \text{and} \quad xInv : \mathcal{P}(\mathcal{H}_A) \longrightarrow \mathcal{P}(\mathcal{R}_A)$$

by

$$xPol(R) = \{ f \in \mathcal{H}_A \mid (\forall \rho \in R) f \text{ } x\text{-preserves } \rho \}$$

and

$$xInv(F) = \{ \rho \in \mathcal{R}_A \mid (\forall f \in F) f \text{ } x\text{-preserves } \rho \}$$

for all $R \in \mathcal{P}(\mathcal{R}_A)$ and $F \in \mathcal{P}(\mathcal{H}_A)$. Equivalently, $xPol(R) = \bigcap_{\rho \in R} xPol \rho$.

It is easy to see that, for each x in $\{d, m, h\}$, the pair $(xPol, xInv)$ is a Galois connection between $\mathcal{R}_A$ and $\mathcal{H}_A$. However, it should be remarked that for any x in $\{d, m, h\}$, the invariant set $xInv F$ is, in general, not a co-clone on A ([6]).

Concerning the inclusion relations among $dPol \rho$, $mPol \rho$ and $hPol \rho$ we have the following results ([6]). The first part is an immediate consequence of the inclusion $\rho_d \subseteq \rho_m \subseteq \rho_h$.

Theorem 2.4 (a) For any relation $\rho \in \mathcal{R}_A$, it holds that $dPol \rho \subseteq mPol \rho \subseteq hPol \rho$.

(b) There exists $\rho_1, \rho_2 \in \mathcal{R}_A$ which satisfy $dPol \rho_1 \subset mPol \rho_1$ and $mPol \rho_2 \subset hPol \rho_2$.

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