

UNIFORM L^2 -STABILITY FOR THE BOLTZMANN EQUATION

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ABSTRACT. We discuss a recent progress on the uniform L^2 -stability for the Boltzmann equation in a close-to-Maxwellian regime.

1. INTRODUCTION

The purpose of this article is to present a recent formulation [6] on the uniform L^2 -stability for the Boltzmann equation near a global Maxwellian. Consider the Boltzmann equation describing the phase space evolution of a distribution function $F = F(x, \xi, t)$ of moderately dilute gas particles with the physical position $x \in \Omega$ and the velocity $\xi \in \mathbb{R}^3$ at time $t \in \mathbb{R}_+$:

$$(1.1) \quad \begin{aligned} \partial_t F + \xi \cdot \nabla_x F &= Q(F, F), \quad x \in \Omega, \xi \in \mathbb{R}^3, t > 0, \\ F(x, \xi, 0) &= F^{in}(x, \xi), \end{aligned}$$

where $Q(F, F)$ is a quadratic collision operator whose explicit form will be defined below.

Let (ξ', ξ'_*) be the post-collisional velocities defined in terms of pre-collisional velocities (ξ, ξ_*) and $\omega \in \mathbb{S}^2_+$:

$$(1.2) \quad \xi' = \xi - [(\xi - \xi_*) \cdot \omega] \omega \quad \text{and} \quad \xi'_* = \xi_* + [(\xi - \xi_*) \cdot \omega] \omega.$$

In this case, the collision operator is given by the following form:

$$(1.3) \quad Q(F, F)(\xi) \equiv \frac{1}{\kappa} \iint_{\mathbb{R}^3 \times \mathbb{S}^2_+} q(\xi - \xi_*, \omega) (F' F'_* - F F_*) d\omega d\xi_*.$$

Here κ is the Knudsen number which is the ratio between the mean free path and the characteristic length of the flow, $\mathbb{S}^2_+ \equiv \{\omega \in \mathbb{S}^2 : (\xi - \xi_*) \cdot \omega > 0\}$, and we used standard abbreviated notations:

$$F' \equiv F(x, \xi', t), \quad F'_* \equiv F(x, \xi'_*, t), \quad F \equiv F(x, \xi, t) \quad \text{and} \quad F_* \equiv F(x, \xi_*, t).$$

We assume that the collision kernel $q(\cdot, \cdot)$ satisfies the inverse power law and the angular cut-off assumption:

$$q(\xi - \xi_*, \omega) = |\xi - \xi_*|^\gamma b_\gamma(\theta), \quad -\frac{3}{2} < \gamma \leq 1 \quad \text{and} \quad \frac{b_\gamma(\theta)}{\cos \theta} \leq b_* < \infty,$$

where θ is the angle between $\xi - \xi_*$ and ω :

$$\theta \equiv \cos^{-1} \left(\frac{(\xi - \xi_*) \cdot \omega}{|\xi - \xi_*|} \right).$$

The spatial domain Ω is assumed to be either whole space $\mathbb{R}^3$ or a torus $\mathbb{T}^3 = \mathbb{R}^3 / \mathbb{L}^3$ ($\mathbb{L}$: any 3-dimensional lattice in $\mathbb{R}^3$) to focus on the initial value problem. Throughout the paper, we shall restrict ourselves to the Boltzmann equation in a Maxwellian regime, and denote by C the generic constant independent of time t .

In a global maxwellian regime, there are many literatures available for the existence theory of solutions and convergence toward a global maxwellian (see [2, 3] for a detailed survey). We next briefly review only the global existence theory of solutions to (1.1). In [10], Ukai first established the global existence of mild solutions to the Boltzmann equation for hard potential and hard sphere models combining a spectral analysis and a bootstrapping argument. Later Caflisch[1] and Ukai-Asano [11] further extended Ukai's seminal work to the moderately soft potentials $\gamma \in (-1, 0]$ on a periodic domain and whole space respectively. For the general case of $\gamma \in (-3, 0]$, the global existence of classical solutions was finally settled by Guo [5] employing an energy method. A global existence theory in an energy space $H_x^s(L_\xi^2)$ ($s \geq 8$) became available only in recent years due to Liu-Yang-Yu [8] and Guo [4]. In particular, Liu, Yang and Yu in [7] introduced a macro-microscopic decomposition of the solution so that the Boltzmann equation can be rewritten as a new fluid type system and an equation for a non-fluid component. Hence the existence theory for (1.1) in a global Maxwellian regime is now in a good shape for small perturbations.

The rest of this paper is organized as follows. In Section 2, we review the basic properties of the linearized collision operator and micro-macro decomposition of a solution and the Boltzmann equation, and key trilinear estimates for the stability analysis. In Section 3, we discuss a priori uniform L^2 -stability estimates [6] for the Boltzmann equation with moderately soft potentials $-\frac{3}{2} < \gamma \leq 1$.

Notations: Throughout the paper, we use various local and global norms on Ω , $\mathbb{R}_\xi^3$ and $\Omega \times \mathbb{R}_\xi^3$. Let $h = g(x, t, \xi)$ be a measurable function on $\Omega \times \mathbb{R}_t \times \mathbb{R}_\xi^3$. Below, $p, q \in [1, \infty]$:

$$\begin{aligned} \|h(x, t)\|_{L_\xi^q} &\equiv \begin{cases} \left(\int_{\mathbb{R}^3} |f(x, \xi, t)|^q d\xi \right)^{\frac{1}{q}}, & 1 \leq q < \infty, \\ \text{esssup}_{\xi \in \mathbb{R}^3} |f(x, \xi, t)|, & q = \infty, \end{cases} \\ \|h(t)\|_{L_x^p(L_\xi^q)} &\equiv \begin{cases} \left(\int_{\mathbb{R}^3} \|h(x, t)\|_{L_\xi^q}^p dx \right)^{\frac{1}{p}}, & 1 \leq p < \infty, \\ \text{esssup}_{x \in \mathbb{R}^3} \|h(x, t)\|_{L_\xi^q}, & p = \infty, \end{cases} \quad \|h(t)\|_{L^p} \equiv \|h(t)\|_{L_x^p(L_\xi^p)}. \end{aligned}$$

2. PRELIMINARIES

In this section, we review the basic properties of collision operators around a global Maxwellian, and micro-macro decomposition introduced in [7, 8]. Consider the Boltzmann equation

$$\begin{aligned} \partial_t F + \xi \cdot \nabla_x F &= Q(F, F), \quad x \in \Omega, \xi \in \mathbb{R}^3, t \in \mathbb{R}_+, \\ F(0, x, \xi) &= F_0(x, \xi). \end{aligned}$$

We now introduce a symmetric bilinear operator $Q[F, G]$ associated with $Q(F, F)$:

$$Q[F, G](\xi) \equiv \frac{1}{2\kappa} \iint_{\mathbb{R}^3 \times \mathbb{S}_+^2} q(\xi - \xi_*, \omega) (F'G'_* + F_*G' - FG_* - F_*G) d\omega d\xi_*.$$

Then it is easy to see that

$$Q[F, F] \equiv Q(F, F).$$

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2.1. The Boltzmann equation near M . In this part, we study the linearization of the Boltzmann equation around a global Maxwellian. We first introduce the perturbation f as

$$(2.1) \quad F = M + M^{\frac{1}{2}}f, \quad M \equiv \frac{1}{\sqrt{(2\pi)^3}} e^{-\frac{|\xi|^2}{2}}.$$

Then the perturbation f satisfies the linearized Boltzmann equation:

$$(2.2) \quad \partial_t f + \xi \cdot \nabla_x f = L(f) + \Gamma(f, f),$$

where $L(\cdot)$ and $\Gamma(\cdot, \cdot)$ are linear and nonlinear collision operators

$$L(f) \equiv 2M^{-\frac{1}{2}}Q[M, M^{\frac{1}{2}}f] \quad \text{and} \quad \Gamma(f, f) \equiv M^{-\frac{1}{2}}Q[M^{\frac{1}{2}}f, M^{\frac{1}{2}}f].$$

We formally define a quadratic form $\Gamma[\cdot, \cdot]$ associated with $\Gamma(\cdot, \cdot)$:

$$\Gamma[g, h] \equiv M^{-\frac{1}{2}}Q[M^{\frac{1}{2}}g, M^{\frac{1}{2}}h].$$

Proposition 2.1. [2] *For the Boltzmann equation (2.2), there exist positive constants $\nu_1 = \nu_1(\gamma)$, $\nu_2 = \nu_2(\gamma)$, σ , k_1, k_2, k_3, k_4 such that*

(1) *L has the decomposition*

$$L = -\nu(\xi)I + K,$$

where I is an identity operator and $\nu(\xi)$ is a collision frequency satisfying

$$\nu_1 \langle \xi \rangle^\gamma \leq \nu(\xi) \leq \nu_2 \langle \xi \rangle^\gamma, \quad \langle \xi \rangle = 1 + |\xi|, \quad \xi \in \mathbb{R}^3,$$

and K is a compact operator.

(2) *L is a non-positive self-adjoint operator on L_ξ^2 with the estimate*

$$\langle Lh, h \rangle \leq -\sigma \langle \nu^{\frac{1}{2}} \mathbf{P}_1 h, \mathbf{P}_1 h \rangle.$$

where $\langle \cdot, \cdot \rangle$ is a usual L^2 -inner product.

2.2. Micro-macro decomposition. In this part, we briefly present the micro-macro decomposition which enable us to see the multi-scale nature of the Boltzmann equation. This beautiful idea of decompose the solution and the Boltzmann equation to see its corresponding fluid part and non-fluid part directly at a time was introduced by Liu and Yu in [7] to the study of the positivity of Boltzmann shock. This micro-macro decomposition will play a key role in our L^2 -stability analysis for hard potential case in Section 3.2.

The linear collision operator L defines an unbounded symmetric operator on L_ξ^2 :

$$L_\xi^2 \equiv (L_\xi^2(\mathbb{R}^3), \langle \cdot, \cdot \rangle) \quad \text{and} \quad \langle f, g \rangle \equiv \int_{\mathbb{R}^3} f(\xi)g(\xi)d\xi \quad \text{for } f, g \in L_\xi^2.$$

The null space $\mathcal{N}$ of L is a five-dimensional vector space spanned by an orthonormal basis $\{\chi_i\}_{i=0}^4$:

$$\mathcal{N} \equiv \text{span}\{\chi_0, \chi_1, \chi_2, \chi_3, \chi_4\},$$

$$\chi_0 = M^{\frac{1}{2}}, \quad \chi_i = \xi_i M^{\frac{1}{2}}, \quad \chi_4 = \frac{1}{\sqrt{6}}(|\xi|^2 - 3)M^{\frac{1}{2}}, \quad \langle \chi_i, \chi_j \rangle = \delta_{ij}, \quad i = 1, 2, 3.$$

We decompose Hilbert space L_ξ^2 as a direct sum of $\mathcal{N}$ and its orthogonal component $\mathcal{N}^\perp$, and we denote by $\mathbf{P}_0$ the projection on this null space and $\mathbf{P}_1$ the complementary projection:

$$\begin{cases} f = \mathbf{P}_0 f + \mathbf{P}_1 f = f_0 + f_1, \\ f_0 = \mathbf{P}_0 f \equiv \rho(x, t)\chi_0 + \sum_{i=1}^3 m_i(x, t)\chi_i + e(x, t)\chi_4, \\ \rho(x, t) = \langle f, \chi_0 \rangle, \quad m_i(x, t) = \langle f, \chi_i \rangle \quad (i = 1, 2, 3), \quad e(x, t) = \langle f, \chi_4 \rangle, \\ f_1 = \mathbf{P}_1 f = f - f_0, \end{cases}$$

We next present trilinear estimates for nonlinear term $\Gamma[f + g, f - g](f - g)$. The property of $\Gamma[f + g, f - g] \in \mathcal{N}^\perp$ and Cauchy-Schwarz yield the following estimates.

Lemma 2.1. [6] *Let $-\frac{3}{2} \leq \gamma \leq 1$, and f, g be measurable functions in $\mathbb{R}_x^3 \times \mathbb{R}_\xi^3$ satisfying*

$$\|\nu^{\frac{1}{2}}(f + g)\|_{L_x^\infty(L_\xi^2)} < \infty, \quad \|f - g\|_{L^2} + \|\nu^{\frac{1}{2}}\mathbf{P}_1(f - g)\|_{L^2} < \infty.$$

Then there exists a positive constant C independent of t such that

$$\begin{aligned} (i) \quad & -\frac{3}{2} \leq \gamma \leq 0; \\ & \left| \int_{\mathbb{R}^3} \langle \Gamma[f + g, f - g], f - g \rangle(x) dx \right| \\ & \leq C \left(\|\nu^{\frac{1}{2}}f(t)\|_{L_x^\infty(L_\xi^2)}^2 + \|\nu^{\frac{1}{2}}g(t)\|_{L_x^\infty(L_\xi^2)}^2 \right) \|f(t) - g(t)\|_{L^2}^2 + \frac{\sigma}{2} \|\nu^{\frac{1}{2}}\mathbf{P}_1(f(t) - g(t))\|_{L^2}^2. \\ (ii) \quad & 0 < \gamma \leq 1; \\ & \left| \int_{\mathbb{R}^3} \langle \Gamma[f + g, f - g], f - g \rangle(x) dx \right| \\ & \leq C \left(\|\nu^{\frac{1}{2}}f(t)\|_{L_x^\infty(L_\xi^2)}^2 + \|\nu^{\frac{1}{2}}g(t)\|_{L_x^\infty(L_\xi^2)}^2 \right) \|f(t) - g(t)\|_{L^2}^2 \\ & + \left[C(\|f(t)\|_{L_x^\infty(L_\xi^2)} + \|g(t)\|_{L_x^\infty(L_\xi^2)}) + \frac{\sigma}{2} \right] \|\nu^{\frac{1}{2}}\mathbf{P}_1(f(t) - g(t))\|_{L^2}^2. \end{aligned}$$

3. A PRIORI UNIFORM L^2 -STABILITY

In this section, we briefly present a priori uniform L^2 -stability estimates. For details, we refer to [6]. Let f and g be two classical solutions to the Boltzmann equation (2.2) and $f, g \in L^\infty(\mathbb{R}_+; L_{x,\xi}^2 \cap L_x^\infty(L_\xi^2))$. Then f and g satisfy

$$(3.1) \quad \partial_t f + \xi \cdot \nabla_x f = L(f) + \Gamma(f, f),$$

$$(3.2) \quad \partial_t g + \xi \cdot \nabla_x g = L(g) + \Gamma(g, g).$$

We subtract (3.2) from (3.1) and multiply $(f - g)$ to both sides to find

$$(3.3) \quad \partial_t |f - g|^2 + \xi \cdot \nabla_x |f - g|^2 = L(f - g)(f - g) + \Gamma[f + g, f - g](f - g).$$

We now integrate (3.3) with respect to (x, ξ) using the boundary condition and Proposition 2.1 to see

$$\begin{aligned} (3.4) \quad \frac{d}{dt} \|f(t) - g(t)\|_{L^2}^2 &= \int_{\Omega} \langle L(f - g), f - g \rangle dx + \int_{\Omega} \langle \Gamma[f + g, f - g], f - g \rangle dx \\ &\leq -\sigma \|\nu^{\frac{1}{2}}\mathbf{P}_1(f(t) - g(t))\|_{L^2}^2 + \left| \int_{\Omega} \langle \Gamma[f + g, f - g], f - g \rangle dx \right|. \end{aligned}$$

We set the uniform L^2 -stability criterion as follows.

$$(3.5) \quad \int_0^\infty \left(\|\nu^{\frac{1}{2}}f(s)\|_{L_x^\infty(L_\xi^2)}^2 + \|\nu^{\frac{1}{2}}g(s)\|_{L_x^\infty(L_\xi^2)}^2 \right) ds < \infty.$$

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3.1. Soft potential and Maxwellian molecule: $-\frac{3}{2} < \gamma \leq 0$. Suppose two smooth perturbations f and g satisfy the stability condition (3.5). In (3.4), we use Lemma 2.1 to derive a Gronwall type inequality:

$$\begin{aligned} \frac{d}{dt} \|f(t) - g(t)\|_{L^2}^2 &\leq -\frac{\sigma}{2} \|\nu^{\frac{1}{2}} \mathbf{P}_1(f(t) - g(t))\|_{L^2}^2 \\ &\quad + C \left(\|\nu^{\frac{1}{2}} f(t)\|_{L_x^\infty(L_\xi^2)}^2 + \|\nu^{\frac{1}{2}} g(t)\|_{L_x^\infty(L_\xi^2)}^2 \right) \|f(t) - g(t)\|_{L^2}^2. \end{aligned}$$

Then Gronwall's lemma yields

$$\begin{aligned} \|f(t) - g(t)\|_{L^2}^2 &+ \frac{\sigma}{2} \int_0^t \|\nu^{\frac{1}{2}} \mathbf{P}_1(f(s) - g(s))\|_{L^2}^2 ds \\ &\leq \exp \left[C \int_0^t \left(\|\nu^{\frac{1}{2}} f(s)\|_{L_x^\infty(L_\xi^2)}^2 + \|\nu^{\frac{1}{2}} g(s)\|_{L_x^\infty(L_\xi^2)}^2 \right) dt \right] \|f^{in} - g^{in}\|_{L^2}^2 \\ &\leq C \|f^{in} - g^{in}\|_{L^2}^2. \end{aligned}$$

This yields the uniform L^2 -stability estimate.

Theorem 3.1. [6] *For $\gamma \in (-\frac{3}{2}, 0]$ and let F and G be two classical solutions to (1.1) in $L^\infty(\mathbb{R}^+; L^2(M^{-\frac{1}{2}} d\xi dx) \cap L_x^\infty(L^2(M^{-\frac{1}{2}} d\xi)))$ corresponding to initial data F^{in}, G^{in} respectively. Suppose the smooth perturbations f and g satisfy the condition (3.5). Then we have*

$$\sup_{0 \leq t < \infty} \|F(t) - G(t)\|_{L^2(M^{-1/2} d\xi dx)} \leq C \|F^{in} - G^{in}\|_{L^2(M^{-1/2} d\xi dx)},$$

where C is a positive constant independent of t .

Remark 3.1. *As a direct application of the above theorem, the classical solutions in [1, 5, 11] are uniformly L^2 -stable.*

3.2. Hard potential and hard sphere model: $0 < \gamma \leq 1$. Suppose two smooth perturbations f and g satisfy the stability condition (3.5) and the smallness condition:

$$(3.6) \quad \|f(t)\|_{L_x^\infty(L_\xi^2)} + \|g(t)\|_{L_x^\infty(L_\xi^2)} \ll \frac{\sigma}{4}.$$

In (3.4), we use Lemma 2.1 to get

$$\begin{aligned} (3.7) \quad \frac{d}{dt} \|f(t) - g(t)\|_{L^2}^2 &\leq C \left(\|\nu^{\frac{1}{2}} f(t)\|_{L_x^\infty(L_\xi^2)}^2 + \|\nu^{\frac{1}{2}} g(t)\|_{L_x^\infty(L_\xi^2)}^2 \right) \|f(t) - g(t)\|_{L^2}^2 \\ &\quad + \left[-\frac{\sigma}{2} + C \left(\|f(t)\|_{L_x^\infty(L_\xi^2)}^2 + \|g(t)\|_{L_x^\infty(L_\xi^2)}^2 \right) \right] \|\nu^{\frac{1}{2}} \mathbf{P}_1(f(t) - g(t))\|_{L^2}^2. \end{aligned}$$

We use (3.7) to find

$$\begin{aligned} \frac{d}{dt} \|f(t) - g(t)\|_{L^2}^2 &\leq C \left(\|\nu^{\frac{1}{2}} f(t)\|_{L_x^\infty(L_\xi^2)}^2 + \|\nu^{\frac{1}{2}} g(t)\|_{L_x^\infty(L_\xi^2)}^2 \right) \|f(t) - g(t)\|_{L^2}^2 \\ &\quad - \frac{\sigma}{4} \|\nu^{\frac{1}{2}} \mathbf{P}_1(f(t) - g(t))\|_{L^2}^2. \end{aligned}$$

Then Gronwall's lemma yield the following stability estimate.

Theorem 3.2. [6] *For $\gamma \in (0, 1]$ and let F and G be two small classical solutions to (1.1) in $L^\infty(\mathbb{R}^+; L^2(M^{-\frac{1}{2}} d\xi dx) \cap L_x^\infty(L^2(M^{-\frac{1}{2}} d\xi)))$ corresponding to small initial data F^{in}, G^{in}*

respectively. Suppose the smooth perturbations f and g satisfy (3.5) and (3.6). Then we have

$$\sup_{0 \leq t < \infty} \|F(t) - G(t)\|_{L^2(M^{-1/2}d\xi dx)} \leq C \|F^{in} - G^{in}\|_{L^2(M^{-1/2}d\xi dx)},$$

where C is a positive constant independent of t .

Remark 3.2. As a direct application of this theorem, the classical solutions in [12] are uniformly L^2 -stable.

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