

CAUCHY PROBLEMS FOR MIXED-TYPE OPERATORS

KEISUKE UCHIKOSHI

Department of Mathematics,
National Defense Academy
Hashirimizu 1-10-20 Yokosuka, Japan
e-mail: uchikosh@cc.nda.ac.jp
打越敬祐 (防衛大)

ABSTRACT. We study a general theory of mixed-type operators containing the Tricomi operators, degenerate hyperbolic operators, and elliptic operators. We will give a necessary and sufficient condition for the Cauchy problems to be well-posed.

Let $P(x, D)$ be a microdifferential operator defined at $x^* = (0; 0, \dots, 0, \sqrt{-1}) \in \sqrt{-1}\mathbf{T}^*\mathbf{R}^n$ of order $m \geq 2$, written in the form

$$(1) \quad \begin{cases} P(x, D) = D_1^m + \sum_{0 \leq j \leq m-1} P_j(x, D') D_1^j, \\ \text{ord } P_j \leq m - j. \end{cases}$$

Here we have written $D' = (D_2, \dots, D_n)$. We also write as $D'' = (D_1, \dots, D_{n-1})$, $D''' = (D_2, \dots, D_{n-1})$. Let $\sigma_m(P)(x, \xi)$ be the principal symbol of $P(x, D)$. We assume that

$$(2) \quad \begin{cases} \text{if } x_1 = 0, \text{ then } \sigma_m(P) = \xi_n^m; \\ \text{if } x_1 \neq 0, \text{ then the equation } \sigma_m(P) = 0 \text{ has } m \text{ distinct roots} \\ \xi_1 = \varphi_1(x, \xi'), \dots, \varphi_m(x, \xi'). \end{cases}$$

We denote by $\mathcal{O}$ (resp. $\mathcal{O}_{(j)}$) the sheaf of holomorphic functions (resp. the sheaf of functions $f(x_1^{1/j}, x')$ such that $f(x)$ are holomorphic). Without loss of generality, we may assume that $\varphi_j(x, \xi') \in \mathcal{O}_{(m'), x^*}$ for some $m' \in \mathbf{N}$, that they are homogeneous in ξ' of degree 1, and that they vanish when $x_1 = 0$. From now on, we denote $\bar{\mathcal{O}} = \mathcal{O}_{(m')}$. It follows that

$$\begin{cases} \text{for some } q_j \in \mathbf{N}/m' \text{ and some } a_j(x, \xi') \in \bar{\mathcal{O}}_{x^*} \text{ we have} \\ \varphi_j(x, \xi') = x_1^{q_j} a_j(x, \xi'), \quad a_j(x^*) \neq 0 \quad (1 \leq j \leq m). \end{cases}$$

We also assume that

$$(3) \quad i \neq j \quad \implies \quad (q_i, a_i(x^*)) \neq (q_j, a_j(x^*)).$$

We denote by $\mathcal{C}$ (resp. $\mathcal{E}$) the sheaf of microfunctions (resp. microdifferential operators). Let us consider the Cauchy problem

$$(4) \quad \begin{cases} Pu = 0, \\ D_1^{j-1}u(0, x') = v_j(x'), \quad 1 \leq j \leq m, \end{cases}$$

where $u \in \mathcal{C}_{\mathbf{R}^n, x^*}$ and $v_j \in \mathcal{C}_{\mathbf{R}^{n-1}, x^{*'}} (x^{*'} = (0; 0, \dots, 0, \sqrt{-1}) \in \sqrt{-1}\mathbf{T}^*\mathbf{R}^{n-1})$. If $P(x, D)$ is microhyperbolic, (4) is well-posed for arbitrary initial values, as is well-known (See [3]). Otherwise (4) may be solvable for some initial values (e.g., for $v_1 = \dots = v_m = 0$), but may be unsolvable for other initial values. Therefore there arises a problem to know for which initial values (4) becomes solvable.

To give the main theorem we need to prepare some preliminaries. Let $A(x', D')$ be an both-side invertible $m \times m$ matrix whose components $A_{(\mu, \nu)}(x', D') \in \mathcal{E}_{x^*}^{\mathbf{R}}$ are independent of (x_1, D_1) . Here we denote by $\mathcal{E}^{\mathbf{R}}$ the sheaf of holomorphic microlocal operators (c.f. [1, 6]). We choose r rows of this matrix in an arbitrary way. To be clear, let $1 \leq j_1 < j_2 < \dots < j_r \leq m$ and choose the $j_1, \dots, j_r$ -th rows of A . Then we obtain an $r \times m$ matrix $A'(x', D')$ of holomorphic microlocal operators. We say that $v_1(x'), \dots, v_m(x') \in \mathcal{C}_{\mathbf{R}^{n-1}, x^{*'}}$ satisfy an r -relation if choosing some r rows of some $A(x', D')$ we have $A'(x', D')\vec{v}(x') = \vec{0}$. Here $\vec{v}$ denotes ${}^t(v_1, \dots, v_m)$. Note that even if $v_1(x'), \dots, v_m(x')$ satisfy an r -relation and another s -relation, it does not necessarily mean an $(r+s)$ -relation.

We next define a classification of the characteristic roots. Let $\theta \in \{0, \pi\}$. Let

$$(5) \quad (x, \xi') \in \mathbf{R}^n \times \mathbf{R}^{n-1}, \quad x_1 \neq 0, \quad \arg x_1 = \theta.$$

We define

$$\begin{aligned} M &= \{1, 2, \dots, m\}, \\ M_{0, \theta} &= \{\lambda \in M; \operatorname{Re}(x_1 \varphi_\lambda(x, \xi')) = 0, \text{ if } (x, \xi') \text{ satisfies (5)}\}, \\ M_{\pm, \theta} &= \{\lambda \in M; \pm \operatorname{Re}(x_1 \varphi_\lambda(x, \xi')) > 0, \text{ if } (x, \xi') \text{ satisfies (5)}\}, \\ M'_\theta &= M \setminus M_{0, \theta} \setminus M_{+, \theta} \setminus M_{-, \theta}. \end{aligned}$$

It is easy to see that $M_{0, \theta} \cup M_{+, \theta} \cup M_{-, \theta} \cup M'_\theta = M$ is a disjoint union.

Let $m_{0, \theta}$, $m_{\pm, \theta}$ be the number of the elements belonging to $M_{0, \theta}$, $M_{\pm, \theta}$, respectively. We assume that

$$(6) \quad M'_\theta = \emptyset, \quad \forall \theta \in \{0, \pi\}.$$

We also need a condition for the microfunctions. Let

$$\begin{aligned} \omega(r) &= \{(x, \xi) \in \sqrt{-1}\mathbf{T}^*\mathbf{R}^n; |x| < r, |\xi''| < r \operatorname{Im} \xi_n\}, \\ \omega'(r) &= \{(x', \xi') \in \sqrt{-1}\mathbf{T}^*\mathbf{R}^{n-1}; |x'| < r, |\xi'''| < r \operatorname{Im} \xi_n\}, \end{aligned}$$

and

$$\begin{aligned} \omega_0(r) &= \{(x, \xi) \in \omega(r); |x'| \leq r^{-1}|x_1|, |\xi''| \leq r^{-1}|x_1| \operatorname{Im} \xi_n\}, \\ \omega'_0(r) &= \{tx'^{*}; t > 0\}. \end{aligned}$$

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We define

$$C_0 = \varinjlim_{r>0} \Gamma_{\omega_0(r)}(C_{\mathbf{R}^n}, \omega_0(r)),$$

$$C'_0 = \varinjlim_{r>0} \Gamma_{\omega'_0(r)}(C_{\mathbf{R}^{n-1}}, \omega'_0(r)).$$

Then we have the following

Theorem. We assume (1) – (3) and (6). Let $v_1(x'), \dots, v_m(x') \in C'_0$. Then there exists an $m_{+,0}$ -relation and an $m_{+,\pi}$ -relation such that the Cauchy problem (4) has a solution $u \in C_0$ if, and only if, $v_1(x'), \dots, v_m(x')$ satisfy these relations.

We give some examples. At first we remind the reader of the well-known result for the operators of principal type.

Example 0 (Lewy-Mizohata operators). If $P_{\pm} = D_1 \pm \sqrt{-1}x_1D_n$, then we have $M_{\pm,\theta} = \{1\} (= M)$, $M_{\mp,\theta} = \emptyset$. The above theorem means that $P_-u = 0$, $u(0, x') = v(x')$ is solvable for any $v \in C'_0$ without any relations. In fact using the defining function we only need to let $u(x) = v(x''', x_n + \sqrt{-1}x_1^2/2)$. On the other hand, $P_+u = 0$, $u(0, x') = v(x')$ is solvable only for the case when $v(x')$ satisfies a one-relation. This means $v = 0$, and $u = 0$. It follows that $P_+u = 0 \Rightarrow u = 0$, i.e., P_+ is hypo-elliptic in C_0 (See [6]).

Lewy-Mizohata operators are the simplest case of our theory, and our theorem gives a similar result even for more complicated operators. The characteristic roots belonging to $M_{+,\theta}$ cause obstruction, and correspondingly the Cauchy data must satisfy so many relations. Let us see the case $m = 2$.

Example 1 (microhyperbolic operators). Let $P(x, D) = D_1^2 - x_1^2D_n^2 + P'(x, D)$, $\text{ord } P' \leq 1$. Without loss of generality, we may assume that P' is a polynomial in D_1 of degree 1. Since $\varphi_1(x, \xi') = x_1\xi_n$, $\varphi_2(x, \xi') = -x_1\xi_n$, and $\arg \xi_n = \pi/2$, it is easy to see that $M_{0,\theta} = \{1, 2\}$, $M_{\pm,\theta} = \emptyset$ for $\theta \in \{0, \pi\}$. It follows that (4) is solvable for arbitrary $v_1(x'), v_2(x') \in C'_0$ without any relations (See [3]).

Example 2 (Tricomi operators). Let $P(x, D) = D_1^2 - x_1D_n^2 + P'(x, D)$, $\text{ord } P' \leq 1$. We have $\varphi_1(x, \xi') = \sqrt{x_1}\xi_n$, $\varphi_2(x, \xi') = -\sqrt{x_1}\xi_n$. It follows that $M_{0,0} = \{1, 2\}$, $m_{+,0} = 0$, and that $M_{+,\pi} = \{1\}$, $M_{-,\pi} = \{2\}$, $m_{+,\pi} = 1$. It follows that there exists a 1-relation, and (4) is solvable if, and only if, the Cauchy data satisfy this relation. We can understand this phenomenon as follows. Let $\omega \subset \sqrt{-1}\mathbf{T}^*\mathbf{R}^n$ be a small neighborhood of x^* , and let $\omega^\theta = \{(x, \xi) \in \omega; x_1 \neq 0, \arg x_1 = \theta\}$, $\theta \in \{0, \pi\}$. At first we consider an elliptic boundary value problem in ω_π , giving one boundary datum on $\{x_1 = 0\}$. Then we can always extend this solution to the hyperbolic region ω_0 . This case was considered also by [4].

Example 3 (hypoelliptic operators). Let $P(x, D) = D_1^2 + x_1^2D_n^2 + P'(x, D)$, $\text{ord } P' \leq 1$. Since $\varphi_1(x, \xi') = \sqrt{-1}x_1\xi_n$, $\varphi_2(x, \xi') = -\sqrt{-1}x_1\xi_n$, it is easy to see that $M_{-,\theta} = \{1\}$, $M_{+,\theta} = \{2\}$, $m_{+,\theta} = 1$ for $\theta \in \{0, \pi\}$. There exist an $m_{+,0}$ -relation and an $m_{+,\pi}$ -relation such that the Cauchy problem (4) uniquely has a solution $u \in C_0$ if, and only if, $v_1(x'), \dots, v_m(x') \in C'_0$ satisfy both of these relations. In most cases two 1-relations mean a 2-relation, but this is not always true. If this is true (4) is solvable only in the case $v_1 = v_2 = 0$, and $u = 0$. In other words, $Pu = 0$ does not have any

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non-trivial solutions. It is well-known that this is true if the principal symbol $\sigma_1(P')$ of the lower order term satisfies $\xi_n^{-1}\sigma_1(P') \notin \{\sqrt{-1}, \sqrt{-13}, \sqrt{-15}, \dots\}$ (See [2,5]).

Of course our result applies for higher order operators, too.

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