数理解析研究所講究録
1106 巻 1999 年 116-129

LOGIC OF AWARENESS AND BELIEF WITH ITS APPLICATION TO ECONOMIC BEHAVIOR*

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1 INTRODUCTION

Since the pioneering contribution of Aumann(1976), game theorists and mathematical economists have investigated the foundation of game theory, especially the concept of common-knowledge (or common-belief) in different kinds of epistemic models. There are two important approaches among others: The first is the logical approach; the axiomatic models of knowledge and belief, and the second is the Bayesian approach of knowledge; the model of belief with probability 1. Bacharach(1985) and Samet(1990) adopted the first approach and Monderer and Samet(1989) adopted the second. They succeeded in extending the 'Agreeing to disagree' theorem of Aumann(1976); that is, it is impossible to agree to disagree if their posteriors are common-knowledge, even when they have different informations.

In every approach, the players in model have been explicitly or implicitly required to have *logically omniscient* ability; that is, they can deduce all the logical implications of their knowledge (or belief) and they know (or believe) every tautology. However real people are not complete reasoners and the recent idea of 'bounded rationality' suggests dropping the problematic assumption. In this connection Dekel, Lipman and Rustichini(1998) introduced a unawareness operator with axiom of plausibility and investigated the relation between unawareness and non-partitional possibility correspondences.

While in the economics literature knowledge and common-knowledge are treated in game-theoretical terms, in the logic literature these notions are analyzed in terms of semantics models (e.g. Kripke semantics.) Although these approaches seem to be different, the underling ideas are essentially same. Therefore there is a possibility of the logical reformulation of results concerning of common-knowledge in the economic literature such as Aumann's theorem as its main achievement. de Swart and Rauszer(1995) succeeded in giving the logical reconstruction of the proof of Aumann's theorem to the modal logic **S5** with common-knowledge.

Date: 7 May 1999

* This is a preliminary version. An abstract of this paper was presented in the R.I.M.S. Symposium of 'Languages, Algebras and Computer Systems,' Kyoto University, 8-10 March 1999. The paper in final form will be published elsewhere.

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The purpose of this lecture is to give a syntactical reformulation of the "Agreeing to disagree" theorem to a weak logic with awareness and common-belief, in which the players are required neither to be men of complete perception nor to have the complete ability of logical reasoning . We present the logic of 'Agreeing to disagree' that is an extension of the logic of awareness and common-belief, and we show the two results: First that the logics have a finite model property, and secondly that the formal sentence of Aumann's theorem is a theorem in the extended logic.

2 LOGIC OF AWARENESS AND COMMON-BELIEF

We present the logic of awareness and common-belief that is a generalization of the logic of common-belief by Lismon(1993).

Let us consider multi-modal logics for finitely many players $N = \{1, 2, \dots, n\}$. The *sentences* of the language form the least set containing each *atomic* sentence $\mathbf{P}_m (m = 0, 1, 2, \dots)$ closed under the following operations:

- nullary operators for *falsity* $\bot$ and for *truth* $\top$;
- unary and binary syntactic operations for *negation* $\neg$, *conditionality* $\rightarrow$ and *conjunction* $\wedge$, respectively;
- $2n + 3$ unary operations for *modality* $\Box_1, \Box_2, \dots, \Box_n, \Box_E, \Box_C, \odot_1, \odot_2, \dots, \odot_n, \odot_E$.

Other such operations are defined in terms of those in usual ways.

The intended interpretation of $\Box_i \varphi$ is the sentence that 'player i believes a sentence φ' whereas $\odot_i \varphi$ is interpreted as the sentence that 'i is aware of φ,' $\Box_E \varphi$ as 'everybody believes φ.' $\Box_C \varphi$ is interpreted as 'φ is commonly believed among all players,' and $\odot_E \varphi$ as 'everybody is aware of φ.'

An *N-modal logic* is a set L of sentences containing all truth-functional tautologies and closed under substitution and modus ponens. An N-modal logic L' is an *extension* of L if $L \subsetneqq L'$. A sentence φ in an N-modal logic L is a *theorem* of L, written by $\vdash_L \varphi$. Other proof-theoretical notions such as *L-deducibility, L-consistency, L-maximality* are defined in usual ways. (See, Chellas, 1980.)

Worthy to notice is that *Lindenbaum's lemma* is always true for any N-modal logic L; that is, every L-consistency set of sentences has an L-maximal extension. This is because L includes the ordinary propositional logic. As a consequence, we can observe that a *sentence is a theorem of L if and only if it is a member of every maximal set of sentences.* We denote by $|\varphi|_L$ the class of L-maximal sets of sentences containing the sentence φ; this is called the *proof set* of φ. We note that a sentence $\varphi \rightarrow \psi$ is a theorem of L if and only if $|\varphi|_L \subsetneqq |\psi|_L$.

Definition. A *system of awareness and common-belief* is an N-modal logic L closed under the $2n + 3$ rules of inference $\mathrm{RE}_\Box$, $\mathrm{RE}_\odot$ and containing the schemata

(Def$\Box_E$), (Def$\odot_E$) and (PL):

(RE$_\Box$) $$\frac{\varphi \longleftrightarrow \psi}{\Box_* \varphi \longleftrightarrow \Box_* \psi} \quad \text{for } * = 1, 2, \ldots, n, E, C;$$

(RE$_\odot$) $$\frac{\varphi \longleftrightarrow \psi}{\odot_* \varphi \longleftrightarrow \odot_* \psi} \quad \text{for } * = 1, 2, \ldots, n, E;$$

(Def$\Box_E$) $$\Box_E \varphi \longleftrightarrow \Box_1 \varphi \wedge \Box_2 \varphi \wedge \cdots \wedge \Box_n \varphi;$$

(FP) $$\Box_C \varphi \longleftrightarrow \Box_E(\varphi \wedge \Box_C \varphi);$$

(Def$\odot_E$) $$\odot_E \, \varphi \longleftrightarrow \odot_1 \varphi \wedge \odot_2 \varphi \wedge \cdots \wedge \odot_n \varphi;$$

(PL) $$\Box_i \varphi \vee \Box_i \neg \Box_i \varphi \longrightarrow \odot_i \varphi \qquad \text{for } i = 1, 2, \ldots, n.$$

By the *logic of awareness and common-belief* we mean the smallest system of awareness and common-belief, denoted by ACB.

Definition. A player i is said to have a *logically omniscient* ability in an N-modal logic if the system has the axiom and rules:

$$(\mathrm{N}_{\Box_i}) \quad \Box_i \top; \qquad\qquad (\mathrm{M}_{\Box_i}) \quad \frac{\varphi \longrightarrow \psi}{\Box_i \varphi \longrightarrow \Box_i \psi}.$$

Remark. Every player in the logic ACB is not to be required to have a logically omniscient ability.

2 AWARENESS STRUCTURE

We present the notion of awareness structure. By a *state-space* we mean a non-empty (perhaps, infinite) set.

Definition. *A belief structure* is a tuple $\langle \Omega, (B_i) \rangle$ in which Ω is a state-space and (B_i) is a class of i's *belief* operators on 2^Ω. The *mutual belief* operator is the operator B_E that assigns to each event F the intersection of $B_i F$ for all i of N; that is,

$$B_E F = \bigcap_{i \in N} B_i F \; .$$

The interpretation of $B_i F$ is the event that 'i believes F,' whereas $B_E F$ is interpreted as the event that 'everybody believes F'

A *common-belief* operator is an operator B_C on Ω satisfying the *fixed point property*:

FP $\quad B_C F \subseteq B_E(F \cap B_C F) \quad$ for every F of 2^Ω.

We say that an event E is *common-belief at* ω if ω belongs to $B_C E$.

The *canonical* common-belief operator is defined as follows: Construct the descending chain $\{B^m\}$ such that $B^0F := B_E F$; $\bar{B}^{m-1}F := B_E(F \cap B^{m-1}F)$; $B^m F := \bar{B}^{m-1}F \cap B^{m-1}F$. The common-belief operator B_C is given by the infimum of the chain:

$$B_C E = \bigcap_{m=0,1,2,\cdots} B^m E.$$

Therefore the canonical common-belief operator satisfies Axiom **FP**, and it yields the *iterated* notion of common-belief; that is, when ω occurs then for all players it is true that all players believe E and they believe that they believe E and ... and so on.

Definition. An *awareness structure* is a tuple $\langle \Omega, (A_i), (B_i) \rangle$ in which $\langle \Omega, (B_i) \rangle$ is a belief structure and (A_i) is a class of i's *awareness* operators on 2^Ω such that Axiom **PL** (axiom of plausibility) is valid:

PL $\quad B_i F \cup B_i(\Omega \setminus B_i F) \subseteq A_i F$.

The awareness structure is called *finite* if the state-space is a finite set.

The axiom **PL** due to Dekel, Lipman and Rustichini (1998) says that i is aware of F if he believes it or if he believes that he dose not believe it.

The *mutual awareness* operator is the operator A_E on 2^Ω that assigns to each event F the intersection of $A_i F$ for all i of N; that is,

$$A_E F = \bigcap_{i \in N} A_i F .$$

The interpretation of $A_i F$ is the event that 'i is aware of F,'where as $A_E F$ is interpreted as the event 'everybody is aware of F.'

3 MODEL AND TRUTH

A *model on* awareness structure is a tuple $\mathcal{B} =< \mathcal{A}, A_E, B_E, B_C, V >$, in which $\mathcal{A} =< N, \Omega, (A_i), (B_i) >$ is an awareness structure, A_E the mutual awareness operator, B_E the mutual belief operator, B_C a common-belief operator and a mapping V assigns either 0 or 1 to every $\omega \in \Omega$ and to every atomic formula $\mathbf{P}_m$. The model is said to be *finite* if the awareness structure is finite.

Definition. By $\models^{\mathcal{B}}_{\omega} \varphi$, we mean that a sentence φ is *true* at a state ω in a model $\mathcal{B}$. *Truth at a state* ω in a model $\mathcal{B} =< \mathcal{A}, A_E, B_E, B_C, V >$ is defined as follows:

(i) $\models^{\mathcal{B}}_{\omega} \mathbf{P}_m$ if and only if $V(\omega, \mathbf{P}_m) = 1$, for $m = 0, 1, 2, \ldots$;

(ii) $\models^{\mathcal{B}}_{\omega} \top$, and not $\models^{\mathcal{B}}_{\omega} \bot$;

(iii) $\models^{\mathcal{B}}_{\omega} \neg\varphi$ if and only if not $\models^{\mathcal{B}}_{\omega} \varphi$;

(iv) $\models^{\mathcal{B}}_{\omega} \varphi \longrightarrow \psi$ if and only if $\models^{\mathcal{B}}_{\omega} \varphi$ implies $\models^{\mathcal{B}}_{\omega} \psi$;

(v) $\models^{\mathcal{B}}_{\omega} \varphi \wedge \psi$ if and only if $\models^{\mathcal{B}}_{\omega} \varphi$ and $\models^{\mathcal{B}}_{\omega} \psi$;

(vi) $\models^{\mathcal{B}}_{\omega} \Box_i\varphi$ if and only if $\omega \in B_i(||\varphi||^{\mathcal{B}})$, for $i = 1, 2, \ldots, n$;

(vii) $\models^{\mathcal{B}}_{\omega} \odot_i\varphi$ if and only if $\omega \in A_i(||\varphi||^{\mathcal{B}})$, for $i = 1, 2, \ldots, n$;

(viii) $\models^{\mathcal{B}}_{\omega} \odot_E\varphi$ if and only if $\omega \in A_E(||\varphi||^{\mathcal{B}})$.

(ix) $\models^{\mathcal{B}}_{\omega} \Box_E\varphi$ if and only if $\omega \in B_E(||\varphi||^{\mathcal{B}})$;

(x) $\models^{\mathcal{B}}_{\omega} \Box_C\varphi$ if and only if $\omega \in B_C(||\varphi||^{\mathcal{B}})$;

We denote by $||\varphi||^{\mathcal{B}}$ the set of all the states in $\mathcal{B}$ at which φ is true; this is called the *truth set of* φ. We say that a sentence φ is *true in the model* $\mathcal{B}$ and write $\models^{\mathcal{B}} \varphi$ if $\models^{\mathcal{B}}_{\omega} \varphi$ for every state ω in $\mathcal{B}$. A sentence is said to be *valid in* an awareness structure $\mathcal{A}$ if it is true in every model on $\mathcal{A}$.

Definition. Let Σ be a set of sentences. We say that $\mathcal{B}$ is a *model for* Σ if every member of Σ is true in $\mathcal{B}$. An awareness structure $\mathcal{A}$ is said to be *for* Σ if every member of Σ is valid in $\mathcal{A}$.

Let $\boldsymbol{M}$ be a class of models on awareness structure. An N-modal logic L is *sound with respect to* $\boldsymbol{M}$ if every member of $\boldsymbol{M}$ is a model for L. It is *complete with respect to* $\boldsymbol{M}$ if every sentence valid in all members of $\boldsymbol{M}$ is a theorem of L. We say that L is *determined by* $\boldsymbol{M}$ if L is sound and complete with respect to $\boldsymbol{M}$.

Proposition 1. *Every system of awareness logic is sound with respect to the class* $\boldsymbol{C}$ *of all models on awareness structure. In particular, the logic of awareness and belief* ACB *is sound with respect to the class* $\boldsymbol{C}_{FIN}$ *of all finite models on awareness structure.*

Proof. Follows easily from the definition of an awareness structure. $\Box$

4 CANONICAL MODEL AND COMPLETENESS THEOREM

A *canonical model* for a system L of awareness and common-belief is a model $\mathcal{B}_L =< \mathcal{A}_L, A^L{}_E, B^L{}_E, B^L{}_C, V_L >$ for L where $\mathcal{A}_L =< N, \Omega_L, (A^L{}_i), (B^L{}_i) >$ is an awareness structure with the mutual belief operator $B^L{}_E$, the common-belief operator $B^L{}_C$ and the mutual awareness operator $A^L{}_E$, which consists of:

(i) Ω_L is the set of all the L-maximal sets of sentences;

(ii) $A^L{}_i$ is the operator on 2^{Ω_L} such that for every ω of Ω_L,

$$\odot_i\varphi \in \omega \qquad \text{if and only if} \qquad \omega \in A^L{}_i(|\varphi|_L);$$

(iii) ${A^L}_E$ is the operator on 2^{Ω_L} such that for every ω of Ω_L,

$$\odot_E\varphi \in \omega \qquad \text{if and only if} \qquad \omega \in {A^L}_E(|\varphi|_L);$$

(iv) ${B^L}_i$ is the operator on 2^{Ω_L} such that for every ω of Ω_L,

$$\Box_i\varphi \in \omega \qquad \text{if and only if} \qquad \omega \in {B^L}_i(|\varphi|_L);$$

(v) ${B^L}_E$ is the operator on 2^{Ω_L} such that for every ω of Ω_L,

$$\Box_E\varphi \in \omega \qquad \text{if and only if} \qquad \omega \in {B^L}_E(|\varphi|_L);$$

(vi) ${B^L}_C$ is the operator on 2^{Ω_L} such that for every ω of Ω_L,

$$\Box_C\varphi \in \omega \qquad \text{if and only if} \qquad \omega \in {B^L}_C(|\varphi|_L);$$

(vii) V_L is the mapping that assigns to every $\mathbf{P}_m$ and to every $\omega \in \Omega$ either 0 or 1 such that for $m = 0, 1, 2, \dots$,

$$\mathbf{P}_m \in \omega \qquad \text{if and only if} \qquad V_L(\omega, \mathbf{P}_m) = 1.$$

Where we should note that these operators B_*^L and A_*^L are well-defined in the sense that $B_*^L(|\varphi|_L) = B_*^L(|\psi|_L)$ and $A_*^L(|\varphi|_L) = A_*^L(|\psi|_L)$ whenever $|\varphi|_L = |\psi|_L$ by virtue of the rules ($\mathrm{RE}_\Box$) and ($\mathrm{RE}_\odot$).

We prove that

Proposition 2. *There is a canonical model for each system of awareness and common-belief.*

Proof. Let L be a system of awareness and common-belief. Let Ω_L be set by the same way as above and and V_L defined by $V_L(\omega, \mathbf{P}_m) = \{\omega \in \Omega^L | \mathbf{P}_m \in \omega\}$. We define the awareness structure $\mathcal{A}_L =< N, L, ({B^L}_i), ({A^L}_i) >$ such that ${B^L}_iF$ consists of all the states ω of Ω_L that for some sentence φ, $F = |\varphi|_L$ with $\Box_i\varphi \in \omega$, and ${A^L}_iF$ consists of all the state ω of Ω_L that for some sentence φ, $F = |\varphi|_L$ with $\odot_i\varphi \in \omega$. We can plainly observe that ${B^L}_i$ and ${A^L}_i$ respectively satisfy the above conditions (iv) and (ii). Let ${B^L}_E$ and ${A^L}_E$ be respectively the mutual belief operator and the mutual awareness operator . We can verify by (Def$\Box_E$) and (Def$\odot_E$) that ${B^L}_E$ and ${A^L}_E$ satisfy the conditions (v) and (iii) respectively; that is, ${B^L}_E(|\varphi|_L) = |\Box_E\varphi|_L, {A^L}_E(|\varphi|_L) = |\odot_E \varphi|_L$. In view of the plausibility axiom (PL) we can further verify that Axiom **PL** is valid. Let ${B^L}_C$ be the operator on Ω_L such that ${B^L}_CF$ consists of all the state ω of Ω_L that for some sentence φ, $F = |\varphi|_L$ with $\Box_C\varphi \in \omega$. In view of Axiom (FP) and the above definition of ${B^L}_C$ it is plainly observed that both the property **FP** and the condition (vi) are valid.

Therefore, $\mathcal{B}_L =< \mathcal{A}_L, A^L{}_E, B^L{}_E, B^L{}_C, V_L >$ is indeed a canonical model on an awareness structure $\mathcal{A}_L =< N, \Omega_L, (A^L{}_i)_i, (B^L{}_i) >$, in completing the proof. $\square$

The important result about a canonical model is the following:

Proposition 3. *Let $\mathcal{B}_L$ be a canonical model for a system L of awareness and common-belief. Then for every sentence φ,*

$$\models^{\mathcal{B}_L} \varphi \qquad \text{if and only if} \qquad \vdash_L \varphi \, .$$

Proof. By induction on the complexity of φ. For non-model cases, see Section 2.7 in Chellas(1980). For modal cases, adapt the proof of Theorem 9.5 with taking note of Exercise 7.10 in Chellas(1980). $\square$

We can now state a main result in this lecture:

Theorem 1. *The logic of awareness and common-belief ACB is determined by the class of all finite models on awareness structure $\boldsymbol{C}_{FIN}$.*

That is, the logic of awareness and common-belief has a finite model property.

Before proceeding to the proof we introduce the notion of the filtration of models. By this we can construct a finite awareness structure from a system of logic.

Filtration of a model $\mathcal{B}_L$.

Let L be a system of awareness and common-belief. For each set of sentences Γ, we define the equivalence relation $\equiv$ on Ω_L by

$$\omega \equiv \xi \qquad \text{if and only if for every sentence } \psi \text{ of } \Gamma,$$

$$\models^{\mathcal{B}_L}_{\omega} \psi \qquad \Longleftrightarrow \qquad \models^{\mathcal{B}_L}_{\xi} \psi \, .$$

Let $[\omega]$ denote the equivalence class of ω and $[X]$ the set of equivalence classes $[\omega]$ for all ω of X whenever X is a subset of Ω_L.

By the *filtration of $\mathcal{B}_L$ through* Γ, we mean the tuple

$$\mathcal{B}^{\Gamma}_L = \langle N, \Omega^{\Gamma}, (A^{\Gamma}_i), (B^{\Gamma}_i), A^{\Gamma}_E, B^{\Gamma}_E, B^{\Gamma}_C, V^{\Gamma} \rangle$$

such that:

(i) $\Omega^{\Gamma} = [\Omega_L]$;

(ii) For each $* = 1, 2, \ldots, n, E$,

$$A^{\Gamma}_*([F]) = [A_* F] \qquad \text{for every } [F] \subseteqq \Omega^{\Gamma};$$

(iii) For each $* = 1, 2, \ldots, n, E, C$, the operator B^{Γ}_* on $2^{\Omega^{\Gamma}}$ is given by

$$B^{\Gamma}_*([F]) = [B_* F] \qquad \text{for every } [F] \subseteqq \Omega^{\Gamma};$$

(iv) V^{Γ} is the mapping that assigns to every $\mathbf{P}_m$ and to every $[\omega] \in \Omega^{\Gamma}$ either 0 or 1 such that for $m = 0, 1, 2, \ldots$,

$$V^{\Gamma}([\omega], \mathbf{P}_m) = V_L(\omega, \mathbf{P}_m) \, .$$

It should be noticed that $\mathcal{B}_L^\Gamma$ is indeed a *finite* model on an awareness structure $\mathcal{A}_L^\Gamma =< N, \Omega^\Gamma, (A_i^\Gamma), (B_i^\Gamma) >$: For B_E^Γ and A_E^Γ respectively coincides with the intersection of all belief operators B_i^Γ and that of all awareness operators A_i^Γ. It is plainly observed that Axiom **PL** is true for A_i^Γ and B_i^Γ , furthermore that Axiom **FP** is also valid. Viewing that the number of all subsentences of φ is finite we can verify that Ω^Γ is a finite set.

We note further that for every state ω of Ω_L and for every subsentence ψ of φ, $\omega \in B_*^L(||\psi||^{\mathcal{B}_L})$ if and only if $[\omega] \in B_*^\Gamma(||\psi||^{\mathcal{B}_L{}^\Gamma})$ for each $* = 1, 2, \dots, n, E, C$ and note that $\omega \in A_*^L(||\psi||^{\mathcal{B}_L})$ if and only if $\omega \in A_*^\Gamma(||\psi||^{\mathcal{B}_L^\Gamma})$ for each $* = 1, 2, \dots, n, E$. This implies that for every sentence φ and for the filtration $\mathcal{B}_L{}^\Gamma$ of $\mathcal{B}_L$ through subsentences of φ,

$$\models^{\mathcal{B}_L} \varphi \qquad \text{if and only if} \qquad \models^{\mathcal{B}_L{}^\Gamma} \varphi \, .$$

Proof of Theorem 1. Soundness follows from Proposition 1. For completeness, we first observe that ACB is complete with respect to $\boldsymbol{C}$: For, in view of Proposition 3, this follows from the existence of a canonical model $\mathcal{B}_{ACB}$ for the system ACB by Proposition 2. Suppose that φ is not a theorem of ACB, so that by Proposition 3 it is false in the canonical model $\mathcal{B}_{ACB}$. Let $\mathcal{B}_{ACB}^\Gamma$ be the filtration of $\mathcal{B}_{ACB}$ through Γ the set of all subsentences of φ. Viewing the above equivalence about validity between in $\mathcal{B}_L$ and in $\mathcal{B}_L^\Gamma$ we can observe that φ is false in a finite model $\mathcal{B}_{ACB}^\Gamma$ on awareness structure $\mathcal{A}_{ACB}^\Gamma$, in completing the proof. □

6 EXTEMSION OF AUMANN'S THEOREM

While Aumann(1976) had showed his 'Agreeing to disagree' theorem in the partitional information model, Bacharach(1985) proved it in the Kripke semantics to the modal logic **S5**. We extend the theorem to the models for the logic ACB. We first introduce the following two concepts:

Definition. We say that an event F is *self-aware* of i if $F \subseteq A_iF$ and it is said to be *publicly aware* if $F \subseteq A_EF$. An event T is said to be *i's evident belief* if $T \subseteq B_iT$, and it is said to be *public belief* at state ω if $\omega \in T \subseteq B_ET$.

An event is public belief (or respectively, it is publicly aware) if whenever it occurs all players believe it (or they are all aware of it.) We can think of public belief as embodying the essence of what is involved in all players making their direct observations.

Definition. The *associated* information structure (P_i) is a class of the mappings P_i of Ω into 2^Ω in which P_i assigns to each ω the intersection of all the i's evident beliefs T to which ω belongs; that is,

$$P_i(\omega) = \bigcap_{T \in 2^\Omega} \{T \,|\, \omega \in T \subseteq B_iT\}.$$

(If there is no event T for which $\omega \in T \subseteq B_i T$ then we take $P_i(\omega)$ to be non-defined.) We call $P_i(\omega)$ the i's *evidence set* at ω.

An evidence set is interpreted as the basis for all i's evident beliefs since each i's evident belief T is decomposed into a union of all evidence sets contained in T.

Definition. A non-empty event H is said to be *P_i-invariant* if for every ξ of H, $P_i(\xi)$ is defined and is contained in H.

Remark. The *strong epistemic model* (Bacharach,1985)[1] can be interpreted as an awareness structure $\langle \Omega, (A_i), (B_i) \rangle$ such that B_i satisfies **N**, **K**, **T**, **4** and **5**, and A_i is the *trivial* awareness operator; i.e. $A_i(E) = \Omega$ for every $E \in 2^\Omega$. This says that an awareness structure is an extension of the strong epistemic model.

Example. Consider the following situation. Player 1 believes that "the earth is not flat and it moves around the sun," while Player 2 believes that "the earth is not flat and it does not moves around the sun"; furthermore these beliefs are evident and it is public belief that "the earth is not flat."

In this circumstances the logic ACB is given as follows: The language consists of two atomic sentences $\mathbf{P}_1, \mathbf{P}_2$ and modal operators $\Box_1, \Box_2, \odot_1, \odot_2$, where $\mathbf{P}_1$ represents the sentence "the earth is flat," $\mathbf{P}_2$ represents the sentence "the earth moves around the sun."

For this logic a model $\langle \Omega, (B_i), V \rangle$ will be constructed as follows: The state-space Ω consists of four states $\alpha, \beta, \gamma, \delta$, and V is the valuation such that

$$V(\alpha, \mathbf{P}_1) = 0, V(\beta, \mathbf{P}_1) = 0, V(\gamma, \mathbf{P}_1) = 1, V(\delta, \mathbf{P}_1) = 1;$$
$$V(\alpha, \mathbf{P}_2) = 1, V(\beta, \mathbf{P}_2) = 0, V(\gamma, \mathbf{P}_2) = 0, V(\delta, \mathbf{P}_2) = 1.$$

Whence a state α represents the proposition "the earth is not flat but it moves around the sun," state β represents the proposition "the earth is not flat and it does not moves around the sun." The belief operators are given by:

$$B_1(\{\emptyset\}) = \{\alpha, \gamma\}, B_1(\{\alpha, \beta\}) = \{\alpha, \beta\}, B_1(\Omega) = \{\alpha\} \text{ and } B_1(E) = \emptyset \text{ otherwise;}$$
$$B_2(\{\emptyset\}) = \{\beta, \gamma\}, B_2(\{\beta\}) = \Omega, B_2(\{\alpha, \beta\}) = \{\alpha, \beta\}, B_2(\Omega) = \{\beta\} \text{ and}$$
$$B_2(E) = \emptyset \text{ otherwise.}$$

The associated information structure is given by:

$$P_1(\alpha) = \{\alpha\}, P_1(\beta) = \{\alpha, \beta\} \text{ and } P_1(\omega) \text{ is not defined otherwise;}$$
$$P_2(\alpha) = \{\alpha, \beta\}, P_2(\beta) = \{\beta\} \text{ and } P_2(\omega) \text{ is not defined otherwise.}$$

[1] The strong epistemic model is a tuple $\langle \Omega, (K_i) \rangle$, in which Ω is a state-space and K_i is an i's knowledge operator satisfying the following axioms: For every E, F of 2^Ω,

N	$K_i\Omega = \Omega$;	**K**	$K_i(E \cap F) = K_iE \cap K_iF$;	**T**	$K_iF \subseteq F$;
4	$K_iF \subseteq K_iK_iF$;	**5**	$\Omega \setminus K_iF \subseteq K_i(\Omega \setminus K_iF)$.		

Let μ be the equal probability measure on Ω: $\mu(\omega) = 1/4$, and $q_i(X, \omega)$ the posterior of X at ω defined by $\mu(X|P_i(\omega))$. Accordingly we obtain that $q_2(\{\alpha\}, \alpha) = 1/2$; that is, player 2's posterior of the event $\{\alpha\}$ (the earth is not flat and it moves around the sun) is $1/2$ when player 2 believes that β is the true state and never believes that α is so ($B_2(\alpha) = \emptyset$), contrary to the spirit of the example. □

We improve on the definition of posterior as follows:

Definition. Let $\langle\Omega, (A_i), (B_i), \mu\rangle$ be an awareness structure with a common-prior μ. For every real number q_i, we denote

$$[q_i] = \{\omega \in \Omega \mid \mu(X \cap A_i(X) \ |P_i(\omega)) = q_i\}.$$

An interpretation of $\mu(X \cap A_i(X) \ |P_i(\omega))$ is the conditional probability of the i's *awareness section* of X under his evidence set at ω.

We say q_i to be the i's *posterior* of X at ω if ω belongs to $[q_i]$. We denote by q the profile $(q_i)_{i\in N}$. An event $[q]$ is the intersection of the sets $[q_i]$ for all i of N; that is,

$$[q] = \bigcap_{i\in N} [q_i].$$

For Example as above, letting $A_i(E) = B_i(E) \cup B_i(-B_i(E))$ we obtain that $A_2(\{\alpha\}) = \{\beta\}$. Therefore it follows that the player 2's improved posterior of $\{\alpha\}$ at state α is $\mu(\{\alpha\} \cap A_i(\{\alpha\}) \ |P_i(\alpha)) = 0$, as desired.

The following lemma is the key to proving Theorem 2.

Fundamental Lemma. *Let (P_i) be the associated information structure with a finite awareness structure and μ a common-prior. Let q_i be an i's posterior of an event X at a state ω. If there is an event H such that the following two properties* (a), (b) *are true then we obtain that*

$$\mu(X \cap A_i(X) \,|H) = q_i :$$

(a) *H is non-empty and it is P_i-invariant,*
(b) *H is contained in $[q_i]$.*

Proof. See Appendix in Matsuhisa and Usami(1999). □

We say that the players *commonly believe their posteriors* q_i of X at ω if $[q]$ is common-belief at ω; that is, $\omega \in B_C([q])$. We can prove the generalized version of Aumann's theorem:

Theorem 2.(Matsuhisa and Usami, 1999) *In a finite awareness structure with a common-prior, if all players commonly believe their posteriors q_i of a publicly* *aware event X at a state ω then they cannot agree to disagree; that is, $q_i = q_j$ for every i, j, even when they does not have logically omniscient ability.*

Proof. We set $[q] \cap B_C([q])$ by H. We note that H is P_i-invariant for every i. It follows that H satisfies the conditions (a) and (b) in Fundamental Lemma. Therefore $\mu(X|H) = \mu(X \cap A_i(X)\,|H) = q_i$ for every i. $\square$

Remark. The class of finite models for the logic ACB is not sound for the following rules: For $\tau = \Box_1, \Box_2. \cdots, \Box_n$,

$$(\mathrm{RM}_\tau) \qquad \frac{\varphi \longrightarrow \psi}{\tau\varphi \longrightarrow \tau\psi}\ ;$$

$$(\mathrm{RI}) \qquad \frac{\varphi \longrightarrow \Box_E\varphi}{\Box_E\varphi \longrightarrow \Box_C\varphi}\ :$$

In fact, putting $\varphi = \bot$, $\psi = \neg\mathbf{P}_1$, we can observe that the model $\langle \Omega, (B_i), V\rangle$ constructed in Example with the canonical common-belief operator B_C is actually a counter model for (RM_τ) and for (RI).

7 LOGIC OF 'AGREEING TO DISAGREE'

In order to give a syntactical reformulation of the extension of Aumann's theorem, we extend the logic ACB to the *logic of 'agreeing to disagree' AD* that consists of the symbols, the rules and the schemata of ACB in addition with

- **Symbols**:

Variables	$\mathbf{q}_1, \mathbf{q}_2, \cdots, \mathbf{q}_n$	: Real numbers;
Constants	**0**,**1**	: Zero, one
Predicates	$=, <$	: Equality, is lesser than
For $i \in N$,	ρ_i	: i's posterior

- **Terms and Sentences**:
  (i) The variables and the constants are *terms*;
  (ii) If s and t are two terms then $\mathrm{s} = \mathrm{t}$ and $\mathrm{s} < \mathrm{t}$ are *atomic* sentences;
  (iii) If φ is a sentence then $\rho_i(\varphi)$ is a term.
- **Axiom**(EA): $\varphi \to \odot_E\varphi$.

By $\rho_i(\varphi) = \mathbf{q}_i$ we mean the sentence that a player i's posterior of φ is $\mathbf{q}_i$. Axiom(EA) says that all players are aware of every sentence. Therefore, by

$$(\wedge_{i\in N}\rho_i(\varphi) = \mathbf{q}_i) \wedge \Box_C(\wedge_{i\in N}\rho_i(\varphi) = \mathbf{q}_i) \longrightarrow \wedge_{i,j\in N}(\mathbf{q}_i = \mathbf{q}_j).$$

we mean the sentence that when each player$i$'s posterior is $\mathbf{q}_i$ and if it is common-belief then all the posteriors are equal to each other; i.e. it is impossible to agree to disagree if all posteriors are common-belief among the players. The purpose of this section is to show that the sentence is a theorem in the logic AD.

Models for ACB are extended as follows:

Definition. A *model* for the logic of agreeing to disagree AD is a tuple $\mathcal{M} = \langle \mathcal{B}, \mu, v \rangle$, in which $\mathcal{B} = \langle N, \Omega, (A_i), (B_i), A_E, B_E, B_C, V \rangle$ is a model for ACB such that μ is a *probability measure* on 2^Ω, and v is *valuation* on $\{ \mathbf{q}_1, \mathbf{q}_2, \cdots, \mathbf{q}_n, \mathbf{0}, \mathbf{1} \}$ into $\mathbf{R}$ with $v(\mathbf{0}) = 0$ and $v(\mathbf{1}) = 1$.

We denote by $\boldsymbol{M}$ the set of all models $\mathcal{M}$ for AD, and by $\boldsymbol{M}_{FIN}$ the set of all *finite* models $\mathcal{M}$ for AD.

The variables and constants get their meaning via a valuation v, and *truth* in a model $\mathcal{M} = \langle \mathcal{B}, \mu, v \rangle$ is defined as follows:

Definition. An *extended valuation* $v_{\langle \mathcal{M}, \omega \rangle}$ is a mapping on the set of all terms of AD into $\mathbf{R}$ such that

- $v_{\langle \mathcal{M}, \omega \rangle}(\mathbf{q}_i) = v(\mathbf{q}_i)$, $v_{\langle \mathcal{M}, \omega \rangle}(\mathbf{0}) = 0$, $v_{\langle \mathcal{M}, \omega \rangle}(\mathbf{1}) = 1$;
- $v_{\langle \mathcal{M}, \omega \rangle}(\rho_i(\varphi)) = \mu(||\varphi||^{\mathcal{M}} \cap A_i(||\varphi||^{\mathcal{M}}) \; |P_i(\omega))$,

where (P_i) is the associated information structure with $\mathcal{B}$. By $\models^{\mathcal{M}}_{\omega} \varphi$ we mean that φ is *truth* at ω in a model $\mathcal{M}$. *Truth* at ω in a model $\mathcal{M} = \langle \mathcal{B}, \mu, v \rangle$ is defined as follows: For terms s and t,

(i) $\models^{\mathcal{M}}_{\omega} \mathrm{s} = \mathrm{t}$ if and only if $v_{\langle \mathcal{M}, \omega \rangle}(\mathrm{s}) = v_{\langle \mathcal{M}, \omega \rangle}(\mathrm{t})$;

(ii) $\models^{\mathcal{M}}_{\omega} \mathrm{s} < \mathrm{t}$ if and only if $v_{\langle \mathcal{M}, \omega \rangle}(\mathrm{s}) < v_{\langle \mathcal{M}, \omega \rangle}(\mathrm{t})$;

(iii) $\models^{\mathcal{M}}_{\omega} \mathbf{P}$ if and only if $V(\omega, \mathbf{P}) = 1$ for any atomic sentence $\mathbf{P}$;

(iv) $\models^{\mathcal{M}}_{\omega} \Box_* \varphi$ if and only if $\omega \in B_*(||\varphi||^{\mathcal{M}})$, for $* = 1, 2, \ldots, n, E, C$;

(v) $\models^{\mathcal{M}}_{\omega} \odot_* \varphi$ if and only if $\omega \in A_*(||\varphi||^{\mathcal{M}})$, for $* = 1, 2, \ldots, n, E$;

where $||\varphi||^{\mathcal{M}}$ is the truth set $\{\omega \in \Omega \mid \models^{\mathcal{M}}_{\omega} \varphi \}$ of φ.

Definition. A *canonical model* for the logic AD is a model $\mathcal{M}_C = \langle \mathcal{B}, \mu, v \rangle \in \boldsymbol{M}$ such that $\mathcal{B} = \langle \Omega, (A_i), (B_i), A_E, B_E, B_C, V \rangle$ is a canonical model for the logic ACB with μ a probability measure on Ω and v a valuation.

By Lindenbaum's lemma it can be verified that in a canonical model $\mathcal{M}_C$ as above, a sentence φ is a theorem if and only if $\varphi \in \Omega$. We denote by $|\varphi|_{AD}$ the set of all maximal consistent sets in AD containing φ..

Basic theorem. *A sentence φ is a theorem in the logic of agreeing to disagree AD if and only if it is valid in a canonical model for AD; i.e.,*

$$||\varphi||^{\mathcal{M}_C} = |\varphi|_{AD} \,.$$

Proof can be done in the same line of Proposition 3. $\square$

Definition. Let $\mathcal{M}$ be a model for the logic AD and Γ a finite subset of sentences in AD. a Γ-*filtration* of $\mathcal{M} = \langle \mathcal{B}, \mu, v \rangle$ is a tuple $\mathcal{M}^\Gamma = \langle \mathcal{B}^\Gamma, \mu^\Gamma, v^\Gamma \rangle \in \boldsymbol{M}_{FIN}$, in which $\mathcal{B}^\Gamma = \langle \Omega^\Gamma, (A_i^\Gamma), (B_i^\Gamma), A_E^\Gamma, B_E^\Gamma, B_C^\Gamma \rangle$ is the filtration of $\mathcal{B}$ through Γ, μ^Γ is the equal probability measure defined by $\mu^\Gamma([\omega]) = \frac{1}{|\Omega^\Gamma|}$, and $v^\Gamma = v$.

In view of the above definition it immediately follows that

Proposition 4. *A sentence φ is valid in $\mathcal{M}$ if and only if φ is valid in $\mathcal{M}^\Gamma$; that is,*

$$\models^{\mathcal{M}} \varphi \quad \textit{if and only if} \quad \models^{\mathcal{M}^\Gamma} \varphi. \quad \square$$

Let $\boldsymbol{C}$ be a subclass of $\boldsymbol{M}$. We denote $\models^{\boldsymbol{C}} \varphi$ when $\models_\omega^{\mathcal{M}} \varphi$ for all $\mathcal{M} \in \boldsymbol{C}$ and for all $\omega \in \mathcal{M}$. The following theorem is a main result in this lecture.

Theorem 3. *The logic of agreeing to disagree has a finite model property:*

That is, a sentence φ is a theorem in the logic AD if and only if φ is valid for all finite models for AD;

$$\vdash_{AD} \varphi \quad \text{if and only if} \quad \models^{\boldsymbol{M}_{FIN}} \varphi.$$

Proof. If $\vdash_{AD} \varphi$ it is plainly observed that $\models^{\boldsymbol{M}_{FIN}} \varphi$. The converse will be shown by the way of contradiction as follows: Suppose that some sentence φ is not a theorem in AD. In view of the basic theorem it follows that φ is not valid for a canonical model $\mathcal{M}_C$. Let Γ be the set of all subsentences of φ. By Proposition 4 we can observe that φ is not valid for a finite model $\mathcal{M}_C^\Gamma$, in contradiction. $\square$

The following result shows that a formal statement of "Agreeing to disagree" is a theorem in the logic AD.

Theorem 4. $\vdash_{AD} (\wedge_{i \in N} \rho_i(\varphi) = \mathbf{q}_i) \wedge \Box_C(\wedge_{i \in N} \rho_i(\varphi) = \mathbf{q}_i) \longrightarrow \wedge_{i,j \in N}(\mathbf{q}_i = \mathbf{q}_j)$.

Proof. In view of Theorem 2, it follows that for any finite model $\mathcal{M} = \langle \mathcal{B}, \mu, v \rangle$ in $\boldsymbol{M}_{FIN}$,

$$|| \wedge_{i \in N} \rho_i(\varphi) = \mathbf{q}_i ||^{\mathcal{M}} \cap B_C(|| \wedge_{i \in N} \rho_i(\varphi) = \mathbf{q}_i ||^{\mathcal{M}} \subseteq \bigcap_{i,j,\in N} || \mathbf{q}_i = \mathbf{q}_j ||^{\mathcal{M}}.$$

Viewing this result together with Theorem 3 we have shown Theorem 4 in completing the proof. $\square$

8 CONCLUDING REMARK

de Swart and Rauszer(1995) gave the logical reconstruction of the proof of Aumann's theorem by Kripke semantics of the modal logic **S5** with common-knowledge. In the same line we present the weaker logic AD without player's logically omniscient ability and show that AD has a finite model property. By virtue of this we observe that the formal sentence of Aumann's theorem is provable in AD.

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