

Fox's $\mathbb{Z}$ -colorings of braids and related topics

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1 Introduction

This article is an announcement of a forthcoming paper [4]. We refer the reader to [4] for more details.

Fox's coloring is one of the fundamental tools in knot theory. For example, the $\mathbb{Z}/p\mathbb{Z}$ -coloring number not only distinguishes various pairs of knots but also gives a lower bound for the unknotting number of a knot [5], and the $\mathbb{Z}$ -coloring is useful for classifying rational tangles [3].

The $\mathbb{Z}/p\mathbb{Z}$ -colorings of m -braids relate to the Hurwitz action of the m -braid group on $(\mathbb{Z}/p\mathbb{Z})^m$, and Berger [1] determines the orbits of this action. In this article we consider $\mathbb{Z}$ -colorings of m -braids corresponding to the Hurwitz action of the m -braid group on $\mathbb{Z}^m$. For two elements $v, w \in \mathbb{Z}^m$, we write $v \sim w$ if there is an m -braid admitting a $\mathbb{Z}$ -coloring such that the top and bottom points receive v and w , respectively. See Figure 1.1. Equivalently, $v \sim w$ holds if they belong to the same orbit. The main aim of this article is to characterize this equivalence relation $\sim$ on $\mathbb{Z}^m$ by introducing several invariants.

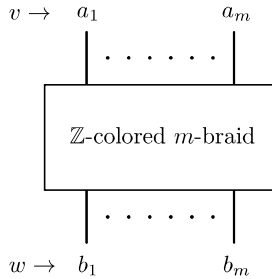


Figure 1.1: $v = (a_1, \dots, a_m) \sim w = (b_1, \dots, b_m) \in \mathbb{Z}^m$

An (m, m) -tangle is an m -braid without monotone property. It is permitted that an (m, m) -tangle has a finite number of loops. Considering $\mathbb{Z}$ -colorings of (m, m) -tangles, we define two coarser equivalence relations on $\mathbb{Z}^m$ than $\sim$ defined above. We write $v \stackrel{\mathcal{T}_0}{\sim} w$ (or $v \stackrel{\mathcal{T}}{\sim} w$) if there is an (m, m) -tangle without loops (or possibly with some loops) admitting a $\mathbb{Z}$ -coloring such that the top and bottom points receive v and w , respectively.

A *pure m -braid* (or an *m -string link*) is an m -braid (or an (m, m) -tangle) such that the i th top point connects to the i th bottom point by a string for any $i = 1, \dots, m$. Then we introduce three equivalence relations on $\mathbb{Z}^m$ such that $v \stackrel{\mathcal{P}}{\sim} w$ ($v \stackrel{\mathcal{L}_0}{\sim} w$, or $v \stackrel{\mathcal{L}}{\sim} w$) if there is a pure m -braid (an m -string link without loops, or an m -string link possibly with some loops) admitting a $\mathbb{Z}$ -coloring such that the top and bottom points receive v and w , respectively.

By definition, the six equivalence relations on $\mathbb{Z}^m$ have a relationship as shown in Table 1.1. For example, if $v \stackrel{\mathcal{P}}{\sim} w$ holds, that is, v and w are connected by a $\mathbb{Z}$ -colored pure m -braid, then we have $v \sim w$ and $v \stackrel{\mathcal{L}_0}{\sim} w$ by regarding the pure m -braid as just an m -braid and an m -string link without loops.

Table 1.1: A relationship among the six equivalence relations

$$\begin{array}{ccccc} v \sim w & \Rightarrow & v \stackrel{\mathcal{T}_0}{\sim} w & \Rightarrow & v \stackrel{\mathcal{T}}{\sim} w \\ \uparrow & & \uparrow & & \uparrow \\ v \stackrel{\mathcal{P}}{\sim} w & \Rightarrow & v \stackrel{\mathcal{L}_0}{\sim} w & \Rightarrow & v \stackrel{\mathcal{L}}{\sim} w \end{array}$$

Moreover, by permitting virtual crossings, we consider six equivalence relations $\stackrel{v}{\sim}$ (connected by a $\mathbb{Z}$ -colored virtual m -braid), $\stackrel{v\mathcal{T}_0}{\sim}$ (connected by a $\mathbb{Z}$ -colored virtual (m, m) -tangle without loops), $\stackrel{v\mathcal{T}}{\sim}$ (connected by a $\mathbb{Z}$ -colored virtual (m, m) -tangle possibly with loops), $\stackrel{v\mathcal{P}}{\sim}$ (connected by a $\mathbb{Z}$ -colored virtual pure m -braid), $\stackrel{v\mathcal{L}_0}{\sim}$ (connected by a $\mathbb{Z}$ -colored virtual m -string link without loops) and $\stackrel{v\mathcal{L}}{\sim}$ (connected by a $\mathbb{Z}$ -colored virtual m -string link possibly with loops).

For an element $v = (a_1, \dots, a_m) \in \mathbb{Z}^m$, we set

- $\Delta(v) = \sum_{i=1}^m (-1)^{i-1} a_i \in \mathbb{Z}$,
- $d(v) = \gcd\{a_2 - a_1, \dots, a_m - a_1\} \geq 0$,
- $d_2(v) = 2^s$ for $d(v) = 2^s t > 0$ with $s \geq 0$ and t odd, and $d_2(v) = 0$ for $d(v) = 0$,
- $M(v)_N = \{a_1, \dots, a_m\} \pmod{N}$ as a multi-set, and
- $\vec{M}(v)_N = (a_1, \dots, a_m) \pmod{N}$ as an ordered set.

Theorem 1.1. *The twelve equivalence relations on $\mathbb{Z}^m$ ($m \geq 2$) as above are characterized as shown in Table 1.2.*

For example, $v \sim w$ if and only if $\Delta(v) = \Delta(w)$, $d(v) = d(w)$ and $M(v)_{2d(v)} = M(w)_{2d(w)}$. Also, $v \stackrel{\mathcal{L}}{\sim} w$ if and only if $\Delta(v) = \Delta(w)$ and $\vec{M}(v)_2 = \vec{M}(w)_2$.

2 Preliminaries

For an integer $m \geq 2$, let $\mathcal{B}_m$ be the m -braid group with the standard generators $\sigma_1, \dots, \sigma_{m-1}$. The set $\mathbb{Z}^m = \{(a_1, \dots, a_m) \mid a_1, \dots, a_m \in \mathbb{Z}\}$ has the Hurwitz action

Table 1.2: Results

	classical case						virtual case					
	$\sim$	$\mathcal{T}_0$	$\mathcal{T}$	$\mathcal{P}$	$\mathcal{L}_0$	$\mathcal{L}$	v	$v\mathcal{T}_0$	$v\mathcal{T}$	$v\mathcal{P}$	$v\mathcal{L}_0$	$v\mathcal{L}$
$\Delta(v) = \Delta(w)$	○	○	○	○	○	○						
$d(v) = d(w)$	○			○			○			○		
$d_2(v) = d_2(w)$		○			○			○			○	
$M(v)_{2d} = M(w)_{2d}$	○						○					
$M(v)_{2d_2} = M(w)_{2d_2}$		○						○				
$M(v)_2 = M(w)_2$			○						○			
$\vec{M}(v)_{2d} = \vec{M}(w)_{2d}$				○						○		
$\vec{M}(v)_{2d_2} = \vec{M}(w)_{2d_2}$					○						○	
$\vec{M}(v)_2 = \vec{M}(w)_2$						○						○

of $\mathcal{B}_m$ from the right defined by

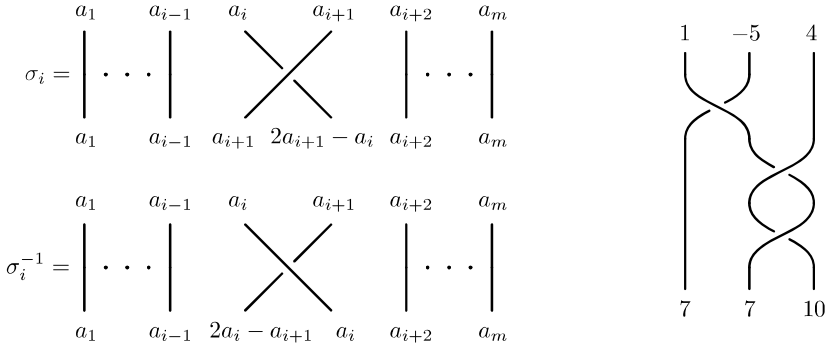
$$v \cdot \sigma_i = (a_1, \dots, a_{i-1}, a_{i+1}, 2a_{i+1} - a_i, a_{i+2}, \dots, a_m) \text{ and}$$

$$v \cdot \sigma_i^{-1} = (a_1, \dots, a_{i-1}, 2a_i - a_{i+1}, a_i, a_{i+2}, \dots, a_m)$$

for an element $v = (a_1, \dots, a_m) \in \mathbb{Z}^m$. See the left of Figure 2.1. We say that two elements $v, w \in \mathbb{Z}^m$ are (*Hurwitz*) *equivalent* if there is an m -braid $\beta \in \mathcal{B}_m$ such that $v \cdot \beta = w$, and denote it by $v \sim w$. The right figure shows that

$$(1, -5, 4) \cdot (\sigma_1^{-1} \sigma_2^2) = (7, 7, 10) \in \mathbb{Z}^3,$$

and hence $(1, -5, 4) \sim (7, 7, 10)$.

Figure 2.1: The Hurwitz action of $\mathcal{B}_m$ on $\mathbb{Z}^m$

An element $v \in \mathbb{Z}^m$ is called *trivial* if $v = a \cdot \mathbf{1} = (a, \dots, a)$ for some $a \in \mathbb{Z}$, where $\mathbf{1} = (1, \dots, 1)$. By definition, if $v \sim w$ and v is trivial, then we have $v = w$; in other words, the orbit of a trivial element v consists of v only.

For an element $v = (a_1, \dots, a_m) \in \mathbb{Z}^m$, we will consider two integers defined by

$$\Delta(v) = \sum_{i=1}^m (-1)^{i-1} a_i \text{ and}$$

$$d(v) = \gcd\{a_i - a_j \mid 1 \leq i \neq j \leq m\} = \gcd\{a_i - a_1 \mid 2 \leq i \leq m\} \geq 0.$$

We remark that $v \in \mathbb{Z}^m$ is trivial if and only if $d(v) = 0$. In what follows, “ $a \equiv b \pmod{0}$ ” means “ $a = b \in \mathbb{Z}$ ” for convenience. For example, the element $v = (1, -5, 4) \in \mathbb{Z}^3$ has

$$\Delta(v) = 1 - (-5) + 4 = 10 \text{ and } d(v) = \gcd\{-6, 3\} = 3.$$

Lemma 2.1. *For an element $v = (a_1, \dots, a_m) \in \mathbb{Z}^m$, we have the following.*

- (i) $a_1 \equiv \dots \equiv a_m \pmod{d(v)}$.
- (ii) If m is odd, then $\Delta(v) \equiv a_1 \pmod{d(v)}$.
- (iii) If m is even, then $\Delta(v) \equiv 0 \pmod{d(v)}$.

Let $\mathcal{S}_m$ be the symmetric group on $\{1, \dots, m\}$. For an m -braid $\beta \in \mathcal{B}_m$, we denote by $\pi_\beta \in \mathcal{S}_m$ the permutation associated with β ; that is, β connects each i th top point to the $\pi_\beta(i)$ th bottom point ($i = 1, \dots, m$). For a multi-subset X of $\mathbb{Z}$ and an integer $N \geq 2$, we denote by X_N the multi-subset of $\mathbb{Z}/N\mathbb{Z}$ consisting of the congruence classes of all integers in X modulo N . We also use the symbol $X_0 = X$ for convenience. For $v = (a_1, \dots, a_m) \in \mathbb{Z}^m$, we set $M(v) = \{a_1, \dots, a_m\}$ as a multi-subset of $\mathbb{Z}$.

Now we consider the multi-set $M(v)_{2d(v)} \subset \mathbb{Z}/2d(v)\mathbb{Z}$. If $v = a \cdot \mathbf{1} = (a, \dots, a) \in \mathbb{Z}^m$ is trivial, then we have $d(v) = 0$ and

$$M(v)_0 = \underbrace{\{a, \dots, a\}}_m \subset \mathbb{Z}.$$

If $v \in \mathbb{Z}^m$ is nontrivial, then it follows from Lemma 2.1(i) that

$$M(v)_{2d(v)} = \underbrace{\{r, \dots, r\}}_p, \underbrace{\{r + d(v), \dots, r + d(v)\}}_{m-p} \}_{2d(v)}$$

for some $0 \leq r < d(v)$ and $1 \leq p \leq m - 1$. For example, the element $v = (1, -5, 4)$ has

$$d(v) = 3 \text{ and } M(v)_6 = \{1, 1, 4\}_6.$$

Lemma 2.2. *Let $v = (a_1, \dots, a_m)$ and $w = (b_1, \dots, b_m) \in \mathbb{Z}^m$ be elements satisfying $v \cdot \beta = w$ for an m -braid $\beta \in \mathcal{B}_m$. Then we have the following.*

- (i) $\Delta(v) = \Delta(w)$.
- (ii) $d(v) = d(w)$.
- (iii) $b_{\pi_\beta(k)} \equiv a_k \pmod{2d(v)}$ for any $k = 1, \dots, m$.
- (iv) $M(v)_{2d(v)} = M(w)_{2d(v)}$.

For example, since $w = (7, 7, 10)$ is equivalent to $v = (1, -5, 4)$, we have

$$\Delta(w) = \Delta(v) = 10, \quad d(w) = d(v) = 3 \text{ and } M(w)_6 = M(v)_6 = \{1, 1, 4\}_6$$

by Lemma 2.2(i), (ii) and (iv).

3 The case $m \geq 3$ odd

Throughout this section we consider the equivalence relation $\sim$ on $\mathbb{Z}^m$ for $m = 2k - 1 \geq 3$. The following theorem provides a classification of $\mathbb{Z}^{2k-1}$ under $\sim$.

Theorem 3.1. *For two elements $v, w \in \mathbb{Z}^{2k-1}$, the following are equivalent.*

- (i) $v \sim w$.
- (ii) $\Delta(v) = \Delta(w)$, $d(v) = d(w)$ and $M(v)_{2d(v)} = M(w)_{2d(w)}$.

For example, $v = (2, -4, 11, 8, -1)$ and $w = (5, 5, 2, 2, 8) \in \mathbb{Z}^5$ satisfy

$$\Delta(v) = \Delta(w) = 8, \quad d(v) = d(w) = 3 \quad \text{and} \quad M(v)_6 = M(w)_6 = \{2, 2, 2, 5, 5\}_6.$$

Therefore we have $v \sim w$ by Theorem 3.1.

In the case $m = 2k - 1 = 3$, the multi-set $M(v)_{2d(v)}$ can be uniquely determined by $\Delta(v)$ and $d(v)$. In fact, we have

$$M(v)_{2d(v)} = \{\Delta(v), \Delta(v) + d(v), \Delta(v) + d(v)\}_{2d(v)},$$

which includes the case where v is trivial with $d(v) = 0$. Therefore we see that $v \sim w \in \mathbb{Z}^3$ if and only if $\Delta(v) = \Delta(w)$ and $d(v) = d(w)$ only.

To prove Theorem 3.1, we prepare Lemmas 3.2–3.6 and Proposition 3.7.

For an element $v = (a_1, a_2, a_3) \in \mathbb{Z}^3$, we set

$$|v| = \max\{a_1, a_2, a_3\} - \min\{a_1, a_2, a_3\} \geq 0.$$

Lemma 3.2. *Let $v = (a_1, a_2, a_3)$ be an element in $\mathbb{Z}^3$. If a_1, a_2 and a_3 are mutually distinct, then v is equivalent to some $w \in \mathbb{Z}^3$ with $|w| < |v|$.*

Lemma 3.3. *Any element $v \in \mathbb{Z}^3$ is equivalent to (x, y, y) for some $x \leq y$.*

Lemma 3.4. *Let $v \in \mathbb{Z}^{2k-1}$ be an element of the form*

$$v = (\underbrace{x, \dots, x}_{2p-1}, \underbrace{y, \dots, y}_{2q}, \underbrace{z, \dots, z}_{2r})$$

for some $x, y, z, p, q, r \in \mathbb{Z}$ with $p, q, r \geq 1$ and $p + q + r = k$. If $x < y < z$ holds, then v is equivalent to

$$(\underbrace{x, \dots, x}_{2p-1}, \underbrace{y', \dots, y'}_{2q'}, \underbrace{z', \dots, z'}_{2r'})$$

for some $y', z', q', r' \in \mathbb{Z}$ with $x \leq y' \leq z' < z$ and $\{q', r'\} = \{q, r\}$.

Lemma 3.5. *Let $v \in \mathbb{Z}^{2k-1}$ be an element of the form*

$$v = (\underbrace{x, \dots, x}_{2p-1}, \underbrace{y, \dots, y}_{2q}, \underbrace{z, \dots, z}_{2r})$$

for some $x, y, z, p, q, r \in \mathbb{Z}$ with $p, q, r \geq 1$ and $p + q + r = k$. If $x < y < z$ holds, then v is equivalent to

$$(\underbrace{x, \dots, x}_{2p'-1}, \underbrace{y', \dots, y'}_{2k-2p'})$$

for some $y', p' \in \mathbb{Z}$ with $x < y'$ and $p' \in \{p, p + q, p + r\}$.

Lemma 3.6. Any element $v \in \mathbb{Z}^{2k-1}$ is equivalent to

$$\underbrace{(x, \dots, x)}_{2p-1}, \underbrace{(y, \dots, y)}_{2k-2p}$$

for some $x, y, p \in \mathbb{Z}$ with $x < y$ and $1 \leq p \leq k$.

By Lemma 2.1(i) and (ii), any element $v = (a_1, \dots, a_{2k-1}) \in \mathbb{Z}^{2k-1}$ satisfies

$$a_1 \equiv \dots \equiv a_{2k-1} \equiv \Delta(v) \pmod{d(v)}.$$

Therefore we have $a_i \equiv \Delta(v)$ or $\Delta(v) + d(v) \pmod{2d(v)}$. Moreover, the number of a_i 's congruent to $\Delta(v)$ modulo $2d(v)$ is odd.

Proposition 3.7. Let v be an element in $\mathbb{Z}^{2k-1}$. Set $\Delta = \Delta(v)$, $d = d(v)$ and

$$M(v)_{2d} = \underbrace{\{\Delta, \dots, \Delta\}}_{2p-1}, \underbrace{\{\Delta + d, \dots, \Delta + d\}}_{2k-2p} \}_{2d}$$

for some $p \in \mathbb{Z}$ with $1 \leq p \leq k$. Then we have

$$v \sim \underbrace{(\Delta, \dots, \Delta)}_{2p-1}, \underbrace{(\Delta + d, \dots, \Delta + d)}_{2k-2p}.$$

4 The case $m \geq 4$ even

Throughout this section we consider the equivalence relation $\sim$ on $\mathbb{Z}^m$ for $m = 2k \geq 4$. The following theorem provides a classification of $\mathbb{Z}^{2k}$ under $\sim$.

Theorem 4.1. For two elements $v, w \in \mathbb{Z}^{2k}$, the following are equivalent.

- (i) $v \sim w$.
- (ii) $\Delta(v) = \Delta(w)$, $d(v) = d(w)$ and $M(v)_{2d(v)} = M(w)_{2d(w)}$.

For example, $v = (-4, 11, 8, -1)$ and $w = (5, 5, 2, 8) \in \mathbb{Z}^4$ satisfy

$$\Delta(v) = \Delta(w) = -6, \quad d(v) = d(w) = 3 \quad \text{and} \quad M(v)_6 = M(w)_6 = \{2, 2, 5, 5\}_6.$$

Therefore we have $v \sim w$ by Theorem 4.1.

To prove Theorem 4.1, we prepare Lemma 4.2 and Proposition 4.3.

Lemma 4.2. Let $v \in \mathbb{Z}^{2k}$ be a nontrivial element of the form

$$v = \underbrace{(x, \dots, x)}_p, \underbrace{(x + \lambda d, x + d, \dots, x + d)}_{2k-p-1}$$

for some $x, \lambda, d, p \in \mathbb{Z}$ with $d > 0$ and $1 \leq p \leq 2k - 2$. Then we have

$$v \sim v + 2nd \cdot \mathbf{1}$$

for any $n \in \mathbb{Z}$.

Proposition 4.3. *Let v be a nontrivial element in $\mathbb{Z}^{2k}$. Set $d = d(v)$ and*

$$M(v)_{2d} = \underbrace{\{r, \dots, r\}}_p, \underbrace{\{r + d, \dots, r + d\}}_{2k-p} \}_{2d}$$

for some $r, p \in \mathbb{Z}$ with $0 \leq r < d$ and $1 \leq p \leq 2k - 1$.

(i) *If $p \geq 2$ holds, then there is an even integer λ such that*

$$v \sim \underbrace{(r, \dots, r)}_{p-1}, r + \lambda d, \underbrace{(r + d, \dots, r + d)}_{2k-p}.$$

(ii) *If $2k - p \geq 2$ holds, then there is an odd integer λ such that*

$$v \sim \underbrace{(r, \dots, r)}_p, r + \lambda d, \underbrace{(r + d, \dots, r + d)}_{2k-p-1}.$$

5 Tangles

In this section we define two coarser equivalence relations $\widetilde{\mathcal{T}}_0$ and $\widetilde{\mathcal{T}}$ on $\mathbb{Z}^m$ than $\sim$, and characterize them. Throughout this section we assume $k \geq 2$ and $m \geq 3$.

5.1 Tangles

By a k -string tangle T , we mean a tangle diagram of k strings possibly with some loops. A map $C : \{\text{arcs of } T\} \rightarrow \mathbb{Z}$ is called a $\mathbb{Z}$ -coloring of T if the equation $a + c = 2b$ holds at each crossing of T , where a and c are the integers assigned to the under-arcs at the crossing, and b is the integer assigned to the over-arc. The integer assigned to an arc is called the *color* of the arc. Let $a_1, \dots, a_{2k}$ be the colors of the $2k$ endpoints of T listed in counterclockwise order. Then we say that the endpoints receive $v = (a_1, \dots, a_{2k}) \in \mathbb{Z}^{2k}$. For example, we consider a 3-string tangle T with a single loop admitting a $\mathbb{Z}$ -coloring as shown in Figure 5.1. Then the 6 endpoints of T receive $(1, 5, 19, 20, 6, 1) \in \mathbb{Z}^6$.

Proposition 5.1. *For an element $v \in \mathbb{Z}^{2k}$, the following are equivalent.*

- (i) *There is a k -string tangle T without loops admitting a $\mathbb{Z}$ -coloring such that the $2k$ endpoints of T receive v .*
- (ii) *There is a k -string tangle T' possibly with some loops admitting a $\mathbb{Z}$ -coloring such that the $2k$ endpoints of T' receive v .*
- (iii) $\Delta(v) = 0$.

5.2 (m, m) -tangles without loops

By an (m, m) -tangle, we mean an m -string tangle diagram possibly with some loops such that each string connects between top and bottom points. We denote by $\mathcal{T}_m$ the set of (m, m) -tangles, and by $\mathcal{T}_{m,0} \subset \mathcal{T}_m$ that of (m, m) -tangles without loops.

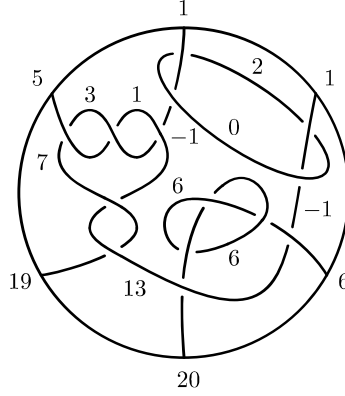


Figure 5.1: A 3-string tangle with a single loop admitting a $\mathbb{Z}$ -coloring

For two elements $v, w \in \mathbb{Z}^m$, we write $v \overset{\mathcal{T}_0}{\sim} w$ if there is an (m, m) -tangle without loops admitting a $\mathbb{Z}$ -coloring such that the top and bottom points receive v and w , respectively. Figure 5.2 shows an example of a $(3, 3)$ -tangle without loops admitting a $\mathbb{Z}$ -coloring which gives $(-1, 2, 2) \overset{\mathcal{T}_0}{\sim} (0, 3, 2)$. We remark that $(-1, 2, 2) \not\sim (0, 3, 2)$ by Lemma 2.2(ii); in fact, we have

$$d((-1, 2, 2)) = 3 \neq 1 = d((0, 3, 2)).$$

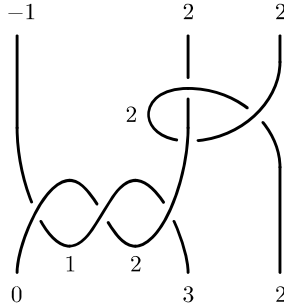


Figure 5.2: A $(3, 3)$ -tangle without loops admitting a $\mathbb{Z}$ -coloring

The following theorem provides a classification of $\mathbb{Z}^m$ under $\overset{\mathcal{T}_0}{\sim}$.

Theorem 5.2. *For two elements $v, w \in \mathbb{Z}^m$, the following are equivalent.*

- (i) $v \overset{\mathcal{T}_0}{\sim} w$.
- (ii) $\Delta(v) = \Delta(w)$, $d_2(v) = d_2(w)$ and $M(v)_{2d_2(v)} = M(w)_{2d_2(w)}$.

To prove this theorem, we prepare Lemmas 5.3–5.5, Propositions 5.6 and 5.7.

Lemma 5.3. *Let $T \in \mathcal{T}_{m,0}$ be an (m, m) -tangle without loops admitting a $\mathbb{Z}$ -coloring such that the top m points of T receive $v = (a_1, \dots, a_m) \in \mathbb{Z}^m$, and*

$$a_i = c_{i1}, c_{i2}, \dots, c_{in(i)}$$

the colors of all the arcs on each i th string of T from top to bottom ($i = 1, \dots, m$). If $c_{ij} - a_1$ is divisible by an integer $N \geq 1$ for any i, j , then we have the following.

- (i) $c_{ij} - a_i$ is divisible by $2N$ for any i, j .
- (ii) If $a_i - a_1$ is divisible by $2N$ for any i , then so is $c_{ij} - a_1$ for any i, j .

For $T \in \mathcal{T}_{m,0}$, we denote by $\pi_T \in \mathcal{S}_m$ the permutation on $\{1, \dots, m\}$ associated with T ; that is, each i th string of T connects the i th top point to the $\pi_T(i)$ th bottom point ($i = 1, \dots, m$). For $v = (a_1, \dots, a_m) \in \mathbb{Z}^m$, we define an integer $d_2(v) \geq 0$ as follows: If v is nontrivial with $d(v) = 2^s t$ for $s \geq 0$ and t odd, then we set $d_2(v) = 2^s$. If v is trivial, that is, $d(v) = 0$, then we set $d_2(v) = 0$.

Lemma 5.4. *Let $T \in \mathcal{T}_{m,0}$ be an (m, m) -tangle without loops admitting a $\mathbb{Z}$ -coloring such that the top and bottom points receive $v = (a_1, \dots, a_m)$ and $w = (b_1, \dots, b_m) \in \mathbb{Z}^m$, respectively. Then we have the following.*

- (i) $\Delta(v) = \Delta(w)$.
- (ii) $d_2(v) = d_2(w)$.
- (iii) $b_{\pi_T(i)} \equiv a_i \pmod{2d_2(v)}$ for any $i = 1, \dots, m$.
- (iv) $M(v)_{2d_2(v)} = M(w)_{2d_2(w)}$.

Lemma 5.5. *For any integers $x, s, t \in \mathbb{Z}$ with $s \geq 0$ and t odd, we have*

$$(x, x + 2^s t) \stackrel{\mathcal{T}_0}{\sim} (x + 2^s, x + 2^s(t + 1)).$$

For any element $v \in \mathbb{Z}^m$, we have

$$a_1 \equiv \dots \equiv a_m \pmod{d_2(v)}$$

by Lemma 2.1(i) and the fact that $d_2(v)$ divides $d(v)$. In the case where m is odd, it follows from Lemma 2.1(ii) that

$$a_i \equiv \Delta(v) \text{ or } \Delta(v) + d_2(v) \pmod{2d_2(v)},$$

and the number of a_i 's congruent to $\Delta(v)$ modulo $2d_2(v)$ is odd.

Proposition 5.6. *Let v be a nontrivial element in $\mathbb{Z}^{2k-1}$. Set $\Delta = \Delta(v)$, $d_2 = d_2(v)$ and*

$$M(v)_{2d_2} = \underbrace{\{\Delta, \dots, \Delta\}}_{2p-1}, \underbrace{\{\Delta + d_2, \dots, \Delta + d_2\}}_{2k-2p} \}_{2d_2}$$

for some $p \in \mathbb{Z}$ with $1 \leq p \leq k - 1$. Then we have

$$v \stackrel{\mathcal{T}_0}{\sim} \underbrace{(\Delta, \dots, \Delta)}_{2p-1}, \underbrace{(\Delta + d_2, \dots, \Delta + d_2)}_{2k-2p}.$$

We consider the case where $m = 2k$ is even. For a nontrivial element $v = (a_1, \dots, a_{2k}) \in \mathbb{Z}^{2k}$, there is an integer r with $0 \leq r < d_2(v)$ such that $a_i \equiv r \pmod{d_2(v)}$ for any $i = 1, \dots, 2k$. Then it holds that

$$a_i \equiv r \text{ or } r + d_2(v) \pmod{2d_2(v)}.$$

Proposition 5.7. *Let v be a nontrivial element in $\mathbb{Z}^{2k}$. Set $d_2 = d_2(v)$ and*

$$M(v)_{2d_2} = \{\underbrace{r, \dots, r}_p, \underbrace{r + d_2, \dots, r + d_2}_{2k-p}\}_{2d_2}$$

for some $r, p \in \mathbb{Z}$ with $0 \leq r < d_2$ and $1 \leq p \leq 2k - 1$.

(i) *If $p \geq 2$ holds, then there is an even integer λ such that*

$$v \stackrel{\mathcal{T}_0}{\sim} (\underbrace{r, \dots, r}_{p-1}, r + \lambda d_2, \underbrace{r + d_2, \dots, r + d_2}_{2k-p}).$$

(ii) *If $2k - p \geq 2$ holds, then there is an odd integer λ such that*

$$v \stackrel{\mathcal{T}_0}{\sim} (\underbrace{r, \dots, r}_p, r + \lambda d_2, \underbrace{r + d_2, \dots, r + d_2}_{2k-p-1}).$$

5.3 (m, m) -tangles possibly with loops

For two elements $v, w \in \mathbb{Z}^m$, we write $v \stackrel{\mathcal{T}}{\sim} w$ if there is an (m, m) -tangle possibly with some loops admitting a $\mathbb{Z}$ -coloring such that the top and bottom points receive v and w , respectively. We remark that $v \stackrel{\mathcal{T}_0}{\sim} w$ implies $v \stackrel{\mathcal{T}}{\sim} w$ by definition.

Figure 5.3 shows an example of a $(3, 3)$ -tangle with a single loop admitting a $\mathbb{Z}$ -coloring which gives $(6, 10, 4) \stackrel{\mathcal{T}}{\sim} (0, 0, 0)$. We remark that $(6, 10, 4) \not\stackrel{\mathcal{T}_0}{\sim} (0, 0, 0)$ by Lemma 5.4(ii); in fact, we have

$$d_2((6, 10, 4)) = 2 \neq 0 = d_2((0, 0, 0)).$$

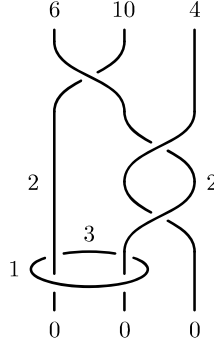


Figure 5.3: A $(3, 3)$ -tangle with a single loop admitting a $\mathbb{Z}$ -coloring

The following theorem provides a classification of $\mathbb{Z}^m$ under $\stackrel{\mathcal{T}}{\sim}$.

Theorem 5.8. *For two elements $v, w \in \mathbb{Z}^m$, the following are equivalent.*

(i) $v \stackrel{\mathcal{T}}{\sim} w$.

(ii) $\Delta(v) = \Delta(w)$ and $M(v)_2 = M(w)_2$.

To prove this theorem, we prepare Lemmas 5.9–5.11 and Proposition 5.12.

Lemma 5.9. *Let $T \in \mathcal{T}_m$ be an (m, m) -tangle possibly with some loops admitting a $\mathbb{Z}$ -coloring such that the top m points of T receive $v = (a_1, \dots, a_m) \in \mathbb{Z}^m$, and*

$$a_i = c_{i1}, c_{i2}, \dots, c_{in(i)}$$

the colors of all the arcs on each i th string of T from top to bottom ($i = 1, \dots, m$). Then $c_{ij} \equiv a_i \pmod{2}$ for any i, j .

Lemma 5.10. *If two elements $v, w \in \mathbb{Z}^m$ satisfy $v \stackrel{\mathcal{T}}{\sim} w$, then we have the following.*

(i) $\Delta(v) = \Delta(w)$.

(ii) $M(v)_2 = M(w)_2$.

Lemma 5.11. *If two integers $x, y \in \mathbb{Z}$ satisfy $x \equiv y \pmod{2}$, then we have*

$$\underbrace{(x, \dots, x)}_{2p} \stackrel{\mathcal{T}}{\sim} \underbrace{(y, \dots, y)}_{2p}.$$

Proposition 5.12. *Let v be an element in $\mathbb{Z}^m$. Set*

$$M(v)_2 = \{\underbrace{0, \dots, 0}_p, \underbrace{1, \dots, 1}_{m-p}\}_2$$

for some $p \in \mathbb{Z}$ with $0 \leq p \leq m$.

(i) *If $p \geq 1$ holds, then there is an even integer λ such that*

$$v \stackrel{\mathcal{T}}{\sim} (\underbrace{0, \dots, 0}_{p-1}, \lambda, \underbrace{1, \dots, 1}_{m-p}).$$

(ii) *If $m - p \geq 1$ holds, then there is an odd integer λ such that*

$$v \stackrel{\mathcal{T}}{\sim} (\underbrace{0, \dots, 0}_p, \lambda, \underbrace{1, \dots, 1}_{m-p-1}).$$

6 Pure braids and string links

In this section we define three finer equivalence relations $\stackrel{\mathcal{P}}{\sim}$, $\stackrel{\mathcal{L}_0}{\sim}$ and $\stackrel{\mathcal{L}}{\sim}$ on $\mathbb{Z}^m$ than $\sim$, $\stackrel{\mathcal{T}_0}{\sim}$ and $\stackrel{\mathcal{T}}{\sim}$, respectively, and characterize them. Throughout this section we assume $m \geq 3$.

6.1 Pure braids

A *pure m -braid* is an m -braid $\beta \in \mathcal{B}_m$ such that the permutation $\pi_\beta \in \mathcal{S}_m$ associated with β is the identity e ; that is, the i th top point connects to the i th bottom point by a string of β for any $i = 1, \dots, m$. Let $\mathcal{P}_m$ denote the pure m -braid group, which is the subgroup of $\mathcal{B}_m$ consisting of pure m -braids. We denote by $\stackrel{\mathcal{P}}{\sim}$ the equivalence relation on $\mathbb{Z}^m$ induced from the Hurwitz action of $\mathcal{P}_m \subset \mathcal{B}_m$.

For an element $v = (a_1, \dots, a_m) \in \mathbb{Z}^m$, we set $\vec{M}(v) = (a_1, \dots, a_m)$ as an ordered set of $\mathbb{Z}$. The following theorem provides a classification of $\mathbb{Z}^m$ under $\stackrel{\mathcal{P}}{\sim}$.

Theorem 6.1. *For two elements $v, w \in \mathbb{Z}^m$, the following are equivalent.*

- (i) $v \stackrel{\mathcal{P}}{\sim} w$.
- (ii) $\Delta(v) = \Delta(w)$, $d(v) = d(w)$ and $\vec{M}(v)_{2d(v)} = \vec{M}(w)_{2d(w)}$.

For example, we consider $v = (1, -5, 4)$, $w = (10, 7, 7)$ and $u = (7, 7, 10) \in \mathbb{Z}^3$. They satisfy

$$\begin{aligned} \Delta(v) &= \Delta(w) = \Delta(u) = 10, \quad d(v) = d(w) = d(u) = 3, \\ M(v)_6 &= M(w)_6 = M(u)_6 = \{1, 1, 4\}_6 \text{ and} \\ \vec{M}(v)_6 &= \vec{M}(u)_6 = (1, 1, 4)_6 \neq (4, 1, 1)_6 = \vec{M}(w)_6. \end{aligned}$$

Then we have

$$v \sim w, \quad v \not\stackrel{\mathcal{P}}{\sim} w \text{ and } v \stackrel{\mathcal{P}}{\sim} u$$

by Theorem 6.1. See Figure 6.1.

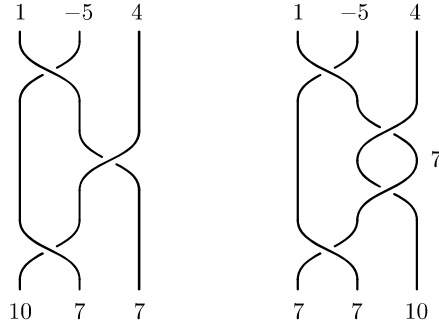


Figure 6.1: $(1, -5, 4) \sim (10, 7, 7)$ and $(1, -5, 4) \stackrel{\mathcal{P}}{\sim} (7, 7, 10)$

To prove Theorem 6.1, we prepare Lemma 6.2.

For an integer i with $1 \leq i \leq m$, we consider two subgroups of $\mathcal{S}_m$ defined by

$$\begin{aligned} \mathcal{S}_{1,i} &= \{\pi \in \mathcal{S}_m \mid \pi(j) = j \text{ for } i+1 \leq j \leq m\} \text{ and} \\ \mathcal{S}_{i,m} &= \{\pi \in \mathcal{S}_m \mid \pi(j) = j \text{ for } 1 \leq j \leq i-1\}. \end{aligned}$$

Lemma 6.2. *Let $v \in \mathbb{Z}^m$ be an element of the form*

$$v = (\underbrace{x, \dots, x}_p, x + \lambda d, \underbrace{x + d, \dots, x + d}_{m-p-1}).$$

for some $x, \lambda, d, p \in \mathbb{Z}$ with $d > 0$ and $1 \leq p \leq m - 2$.

- (i) *If λ is even, then for any $\pi \in \mathcal{S}_{1,p+1}$ and $\pi' \in \mathcal{S}_{p+2,m}$, there is an m -braid $\beta \in \mathcal{B}_m$ such that $v \cdot \beta = v$ and $\pi_\beta = \pi\pi'$.*
- (ii) *If λ is odd, then for any $\pi \in \mathcal{S}_{1,p}$ and $\pi' \in \mathcal{S}_{p+1,m}$, there is an m -braid $\beta \in \mathcal{B}_m$ such that $v \cdot \beta = v$ and $\pi_\beta = \pi\pi'$.*

For example, we consider the element $v = (0, 0, 2, 1, 1, 1) \in \mathbb{Z}^6$ as in Lemma 6.2 for $x = 0$, $\lambda = 2$, $d = 1$ and $p = 2$. For the permutations $\pi = (2\ 3) \in \mathcal{S}_{1,3}$ and $\pi' = e \in \mathcal{S}_{4,6}$, the 6-braid $\beta = (\sigma_3\sigma_2^{-1})\sigma_3(\sigma_2\sigma_3^{-1}) \in \mathcal{B}_6$ satisfies $v \cdot \beta = v$ and $\pi_\beta = \pi\pi' = (2\ 3)$. See Figure 6.2.

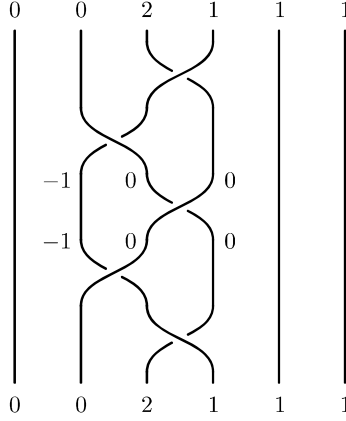


Figure 6.2: The 6-braid $\beta = \sigma_3\sigma_2^{-1}\sigma_3\sigma_2\sigma_3^{-1}$ with $\pi_\beta = (2\ 3)$

6.2 String links

An m -string link is an (m, m) -tangle T possibly with some loops such that the permutation $\pi_T \in \mathcal{S}_m$ associated with T is the identity; that is, the i th top point connects to the i th bottom point by a string for any $i = 1, \dots, m$. We denote by $\mathcal{L}_m$ the set of m -string links, and by $\mathcal{L}_{m,0} \subset \mathcal{L}_m$ that of m -string links without loops.

For two elements $v, w \in \mathbb{Z}^m$, we write $v \stackrel{\mathcal{L}_0}{\sim} w$ if there is an m -string link without loops admitting a $\mathbb{Z}$ -coloring such that the top and bottom points receive v and w , respectively. By Lemmas 5.4, 6.2, Propositions 5.6 and 5.7, we have a classification of $\mathbb{Z}^m$ under $\stackrel{\mathcal{L}_0}{\sim}$ as follows.

Theorem 6.3. *For two elements $v, w \in \mathbb{Z}^m$, the following are equivalent.*

- (i) $v \stackrel{\mathcal{L}_0}{\sim} w$.

(ii) $\Delta(v) = \Delta(w)$, $d_2(v) = d_2(w)$ and $\vec{M}(v)_{2d_2(v)} = \vec{M}(w)_{2d_2(w)}$.

For two elements $v, w \in \mathbb{Z}^m$, we write $v \stackrel{\mathcal{L}}{\sim} w$ if there is an m -string link possibly with some loops admitting a $\mathbb{Z}$ -coloring such that the top and bottom points receive v and w , respectively. By Lemmas 5.9, 5.10 and Proposition 5.12, we have a classification of $\mathbb{Z}^m$ under $\stackrel{\mathcal{L}}{\sim}$ as follows.

Theorem 6.4. *For two elements $v, w \in \mathbb{Z}^m$, the following are equivalent.*

- (i) $v \stackrel{\mathcal{L}}{\sim} w$.
- (ii) $\Delta(v) = \Delta(w)$ and $\vec{M}(v)_2 = \vec{M}(w)_2$.

7 Virtual versions

In this section we define six equivalence relations $\stackrel{v}{\sim}$, $\stackrel{v\mathcal{T}_0}{\sim}$, $\stackrel{v\mathcal{T}}{\sim}$, $\stackrel{v\mathcal{P}}{\sim}$, $\stackrel{v\mathcal{L}_0}{\sim}$ and $\stackrel{v\mathcal{L}}{\sim}$ on $\mathbb{Z}^m$ as virtual versions of $\sim$, $\mathcal{T}_0$, $\mathcal{T}$, $\mathcal{P}$, $\mathcal{L}_0$ and $\mathcal{L}$, respectively, and characterize them for $m \geq 3$.

Let $\mathcal{VB}_m$ ($m \geq 2$) be the *virtual m -braid group* with the standard generators

$$\sigma_1, \dots, \sigma_{m-1} \text{ and } \tau_1, \dots, \tau_{m-1}$$

such that σ_i corresponds to a classical crossing (see the left of Figure 2.1 again), and τ_i corresponds to a virtual crossing between the i th and $(i+1)$ st strings (cf. [2]).

The set $\mathbb{Z}^m$ has the Hurwitz action of $\mathcal{VB}_m$ such that $v \cdot \sigma_i$ is defined as the same as in Section 2 and

$$v \cdot \tau_i = (a_1, \dots, a_{i-1}, a_{i+1}, a_i, a_{i+2}, \dots, a_m),$$

where $v = (a_1, \dots, a_m) \in \mathbb{Z}^m$. See the left of Figure 7.1. We denote by $v \stackrel{v}{\sim} w$ if there is a virtual m -braid $\beta \in \mathcal{VB}_m$ with $v \cdot \beta = w$. The right figure shows that

$$(1, -5, 4) \cdot (\tau_1 \sigma_2 \sigma_1^{-1} \tau_2) = (-2, 1, 1) \in \mathbb{Z}^3,$$

and hence $(1, -5, 4) \stackrel{v}{\sim} (-2, 1, 1)$.

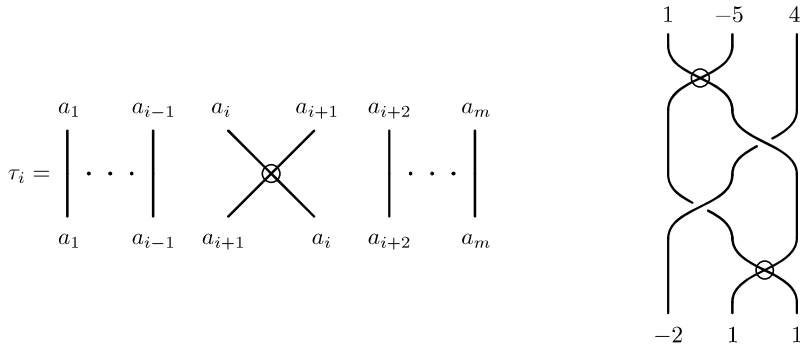


Figure 7.1: The Hurwitz action of $\mathcal{VB}_m$ on $\mathbb{Z}^m$

Theorem 7.1. *For two elements $v, w \in \mathbb{Z}^m$, the following are equivalent.*

- (i) $v \stackrel{\sim}{\sim} w$.
- (ii) $d(v) = d(w)$ and $M(v)_{2d(v)} = M(w)_{2d(w)}$.

For two elements $v, w \in \mathbb{Z}^m$, we write $v \stackrel{v\mathcal{T}_0}{\sim} w$ (or $v \stackrel{v\mathcal{T}}{\sim} w$) if there is a virtual (m, m) -tangle without loops (or possibly with some loops) admitting a $\mathbb{Z}$ -coloring such that the top and bottom points receive v and w , respectively.

Theorem 7.2. *For two elements $v, w \in \mathbb{Z}^m$, the following are equivalent.*

- (i) $v \stackrel{v\mathcal{T}_0}{\sim} w$.
- (ii) $d_2(v) = d_2(w)$ and $M(v)_{2d_2(v)} = M(w)_{2d_2(w)}$.

Theorem 7.3. *For two elements $v, w \in \mathbb{Z}^m$, the following are equivalent.*

- (i) $v \stackrel{v\mathcal{T}}{\sim} w$.
- (ii) $M(v)_2 = M(w)_2$.

The *virtual pure m -braid group* is defined by

$$\mathcal{VP}_m = \{\beta \in \mathcal{VB}_m \mid \pi_\beta = e\}.$$

Then $\mathcal{VP}_m$ acts on $\mathbb{Z}^m$ as well as $\mathcal{VB}_m$ does. For two elements $v, w \in \mathbb{Z}^m$, we write $v \stackrel{v\mathcal{P}}{\sim} w$ if there is a virtual pure m -braid $\beta \in \mathcal{VP}_m$ with $v \cdot \beta = w$.

Theorem 7.4. *For two elements $v, w \in \mathbb{Z}^m$, the following are equivalent.*

- (i) $v \stackrel{v\mathcal{P}}{\sim} w$.
- (ii) $d(v) = d(w)$ and $\vec{M}(v)_{2d(v)} = \vec{M}(w)_{2d(w)}$.

For two elements $v, w \in \mathbb{Z}^m$, we write $v \stackrel{v\mathcal{L}_0}{\sim} w$ (or $v \stackrel{v\mathcal{L}}{\sim} w$) if there is a virtual m -string link without loops (or possibly with some loops) admitting a $\mathbb{Z}$ -coloring such that the top and bottom points receive v and w , respectively.

Theorem 7.5. *For two elements $v, w \in \mathbb{Z}^m$, the following are equivalent.*

- (i) $v \stackrel{v\mathcal{L}_0}{\sim} w$.
- (ii) $d_2(v) = d_2(w)$ and $\vec{M}(v)_{2d_2(v)} = \vec{M}(w)_{2d_2(w)}$.

Theorem 7.6. *For two elements $v, w \in \mathbb{Z}^m$, the following are equivalent.*

- (i) $v \stackrel{v\mathcal{L}}{\sim} w$.
- (ii) $\vec{M}(v)_2 = \vec{M}(w)_2$.

Proposition 7.7. *For an element $v \in \mathbb{Z}^{2k}$, the following are equivalent.*

- (i) *There is a virtual k -string tangle without loops T admitting a $\mathbb{Z}$ -coloring such that the $2k$ endpoints of T receive v .*
- (ii) $\Delta(v) \equiv 0 \pmod{2d_2(v)}$.

Proposition 7.8. *For an element $v \in \mathbb{Z}^{2k}$, the following are equivalent.*

- (i) *There is a virtual k -string tangle possibly with some loops T admitting a $\mathbb{Z}$ -coloring such that the $2k$ endpoints of T receive v .*
- (ii) $\Delta(v) \equiv 0 \pmod{2}$.

8 The case $m = 2$

In the previous sections we provide the characterization results of the twelve equivalence relations on $\mathbb{Z}^m$ for $m \geq 3$. This section extends them to the case $m = 2$.

Theorem 8.1. *For two elements $v, w \in \mathbb{Z}^2$, we have the following.*

- (i) $v \sim w$ if and only if $\Delta(v) = \Delta(w)$ and $M(v)_{2d(v)} = M(w)_{2d(w)}$.
- (ii) $v \stackrel{\mathcal{T}_0}{\sim} w$ if and only if $\Delta(v) = \Delta(w)$ and $M(v)_{2d_2(v)} = M(w)_{2d_2(w)}$.
- (iii) $v \stackrel{\mathcal{T}}{\sim} w$ if and only if $\Delta(v) = \Delta(w)$ and $M(v)_2 = M(w)_2$.

Theorem 8.2. *For two elements $v, w \in \mathbb{Z}^2$, we have the following.*

- (i) $v \stackrel{\mathcal{P}}{\sim} w$ if and only if $\Delta(v) = \Delta(w)$ and $\vec{M}(v)_{2d(v)} = \vec{M}(w)_{2d(w)}$.
- (ii) $v \stackrel{\mathcal{L}_0}{\sim} w$ if and only if $\Delta(v) = \Delta(w)$ and $\vec{M}(v)_{2d_2(v)} = \vec{M}(w)_{2d_2(w)}$.
- (iii) $v \stackrel{\mathcal{L}}{\sim} w$ if and only if $\Delta(v) = \Delta(w)$ and $\vec{M}(v)_2 = \vec{M}(w)_2$.

Theorem 8.3. *For two elements $v, w \in \mathbb{Z}^2$, we have the following.*

- (i) $v \stackrel{\mathcal{V}}{\sim} w$ if and only if $d(v) = d(w)$ and $M(v)_{2d(v)} = M(w)_{2d(w)}$.
- (ii) $v \stackrel{v\mathcal{T}_0}{\sim} w$ if and only if $d_2(v) = d_2(w)$ and $M(v)_{2d_2(v)} = M(w)_{2d_2(w)}$.
- (iii) $v \stackrel{v\mathcal{T}}{\sim} w$ if and only if $M(v)_2 = M(w)_2$.

Theorem 8.4. *For two elements $v, w \in \mathbb{Z}^2$, we have the following.*

- (i) $v \stackrel{v\mathcal{P}}{\sim} w$ if and only if $d(v) = d(w)$ and $\vec{M}(v)_{2d(v)} = \vec{M}(w)_{2d(w)}$.
- (ii) $v \stackrel{v\mathcal{L}_0}{\sim} w$ if and only if $d_2(v) = d_2(w)$ and $\vec{M}(v)_{2d_2(v)} = \vec{M}(w)_{2d_2(w)}$.
- (iii) $v \stackrel{v\mathcal{L}}{\sim} w$ if and only if $\vec{M}(v)_2 = \vec{M}(w)_2$.

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