

A method for computing the bifurcation set of a complex polynomial mapping II

S. Tajima

Graduate School of Science and Technology, Niigata University

K. Nabeshima

Department of Applied Mathematics, Tokyo University of Science

Abstract

Bifurcation sets of complex polynomial mappings are considered in the context of symbolic computation. Based on a result of A. Parusiński, a new effective method is proposed for computing bifurcation sets of complex polynomial mappings. The keys of our approach are the concept of local cohomology and the Grothendieck local duality on residues. The resulting method can treat the case of a family of polynomial mappings that depend on deformation parameters.

1 Introduction

The bifurcation set plays a fundamental role in the study of topology of a polynomial mapping. After the study of S. A. Broughton [5, 6], many researchers have studied bifurcation sets from several different viewpoints. e.g. [1, 2, 3, 4, 7, 8, 9, 10, 12, 13, 14, 15, 16, 18, 24, 25, 26, 27]. Many interesting and deep results were obtained. However, the determination of bifurcation sets is still quite difficult problem [16, 24, 25]. We propose in this paper a new effective method for computing them.

In Section 2, we recall the notion of a bifurcation set and a result of A. Parusiński [13]. In Section 3, we briefly recall the concept of local cohomology and the Grothendieck local duality on residues. In Section 4, we give a method for computing bifurcation sets. In Section 5, we show that the proposed method can compute parameter dependency of bifurcation sets of a family of polynomial mappings.

2 Bifurcation set and Milnor number at infinity

Let $f : \mathbb{C}^n \longrightarrow \mathbb{C}$ be a complex polynomial mapping. Then, there exists a finite set $\Gamma_f \subset \mathbb{C}$ s.t.

$$f : \mathbb{C}^n - f^{-1}(\Gamma_f) \longrightarrow \mathbb{C} - \Gamma_f$$

is a locally trivial $\mathcal{C}^\infty$ fibration. The smallest set that satisfies the condition above is called bifurcation set, or atypical set [5, 6] of the mapping f .

Let $Sing(f)$ be the singular set of f and let C_f denote the image by f of $Sing(f)$. Let B_f denote the bifurcation set of f .

Example ([5]) $f(x, y) = x^2y - x$

Let $J_f \subset K[x, y]$ be the Jacobi ideal of f . Since $J_f = \langle 2xy - 1, x^2 \rangle = \langle 1 \rangle$, we have $C_f = \emptyset$.

Now consider the fiber $f^{-1}(c)$ of $c \in \mathbb{C}$. Since $x^2y - x = x(xy - 1)$, we have $f^{-1}(c) \cong \mathbb{C}^*$ for $c \neq 0$, and $f^{-1}(0) \cong \mathbb{C} \cup \mathbb{C}^*$ for $c = 0$. Therefore $B_f = \{0\}$ which means in particular $C_f \neq B_f$.

Let $f(x)$ be a polynomial in n variables $x = (x_1, x_2, \dots, x_n)$ of degree d . Let $\tilde{f}(x, \eta)$ denote the homogenization of f , i.e. $\tilde{f}(x, \eta) = \eta^d f(\frac{x_1}{\eta}, \frac{x_2}{\eta}, \dots, \frac{x_n}{\eta})$, where η is a new variable. Now consider a homogeneous function $F_t(x, \eta)$ defined by

$$F_t(x, \eta) = \tilde{f}(x, \eta) - t\eta^d,$$

where t is regarded as a parameter.

Let A be the singular locus of $F_t(x, \eta)$ at infinity $\eta = 0$

$$A = \left\{ [x_1 : x_2 : \dots : x_n : \eta] \in \mathbb{P}^n \mid \frac{\partial F_t}{\partial x_1} = \frac{\partial F_t}{\partial x_2} = \dots = \frac{\partial F_t}{\partial x_n} = \frac{\partial F_t}{\partial \eta} = \eta = 0 \right\}$$

Lemma Let $f = f_d(x) + f_{d-1}(x) + \dots + f_0(x)$ be a polynomial of degree d , where $f_k(x)$ is a polynomial of degree k . Then, the following holds.

$$A = \left\{ [x_1 : x_2 : \dots : x_n : 0] \in \mathbb{P}^n \mid \frac{\partial f_d}{\partial x_1} = \frac{\partial f_d}{\partial x_2} = \dots = \frac{\partial f_d}{\partial x_n} = f_{d-1} = 0 \right\}.$$

Example $f(x, y) = x^2y^2 + x$

From $\tilde{f}(x, y, \eta) = x^2y^2 + x\eta^3$, $F_t(x, y, \eta) = x^2y^2 + x\eta^3 - t\eta^4$,

we have $A = \{[x : y : \eta] \in \mathbb{P}^2 \mid xy^2 = x^2y = \eta = 0\} = \{[0 : 1 : 0], [1 : 0 : 0]\}$

Example $f(x, y) = x^4 + y^4 - 4xy$

From $\tilde{f}(x, y, \eta) = x^4 + y^4 - 4xy\eta^2$, $F_t(x, y, \eta) = x^4 + y^4 + 4xy\eta^2 - t\eta^4$,

we have $A = \{[x : y : 0] \in \mathbb{P}^2 \mid x^3 = y^3 = 0\} = \emptyset$.

We assume hereafter that A is a finite set. For $A = \{a_1, a_2, \dots, a_m\}$, with $a_j \in A$, let μ_{t, a_j} denote the Milnor number of F_t at the point a_j .

In 1995, A. Parusiński obtained the following result.

Theorem ([13]) Let $t_0 \notin C_f$. Then the following are equivalent.

- (1) $t_0 \notin B_f$
- (2) There is a neighborhood $U \subset \mathbb{C}$ of t_0 s.t.
 $\mu_{t, a_j} = \mu_{t_0, a_j}, j = 1, 2, \dots, m, \forall t \in U$.

The result implies in particular that if the Milnor number μ_{t, a_j} jumps at the point t_0 , then, t_0 belongs to bifurcation set B_f .

3 Local cohomology and duality

In this section we recall some basics on local cohomology and duality. Let X be a neighborhood in $\mathbb{C}^n$ of the origin O . Let $\mathcal{O}_X$ be the sheaf on X of holomorphic functions and let $\mathcal{O}_{X, O}$ be the stalk at O of the sheaf $\mathcal{O}_X$. Let $\hat{\mathcal{O}}_{X, O}$ be the ring of formal power series at O

Let $\mathcal{H}_{\{O\}}^n(\mathcal{O}_X)$ be the sheaf of local cohomology supported at O defined as

$$\mathcal{H}_{\{O\}}^n(\mathcal{O}_X) = H^n(R\Gamma_{\{O\}}(\mathcal{O}_X)),$$

where $R\Gamma_O$ is a derived functor. Let $\mathcal{H}_{[O]}^n(\mathcal{O}_X)$ be algebraic local cohomology supported at O :

$$\mathcal{H}_{[O]}^n(\mathcal{O}_X) = \lim_{k \rightarrow \infty} \text{Ext}_{\mathcal{O}_X}^n(\mathcal{O}_X/\mathfrak{m}^k, \mathcal{O}_X),$$

where $\mathfrak{m}$ is the maximal ideal of O .

The classical theory of Fréchet Schwartz and dual Fréchet Schwartz locally convex topological vector spaces implies that the pairings

$$\mathcal{H}_{\{O\}}^n(\mathcal{O}_X) \times \mathcal{O}_{X,O} \longrightarrow \mathbb{C}, \quad \mathcal{H}_{[O]}^n(\mathcal{O}_X) \times \hat{\mathcal{O}}_{X,O} \longrightarrow \mathbb{C}$$

are perfect. Namely, $\mathcal{H}_{\{O\}}^n(\mathcal{O}_X)$ and $\mathcal{O}_{X,O}$ are mutually dual and $\mathcal{H}_{[O]}^n(\mathcal{O}_X)$ and $\hat{\mathcal{O}}_{X,O}$ are mutually dual.

Note that the pairings are defined by the Grothendieck local residues. Here we identified $\mathcal{O}_{X,O}$ with the set of germs of holomorphic n -forms $\Omega_{X,O}^n$ and $\hat{\mathcal{O}}_{X,O}$ with $\hat{\Omega}_{X,O}^n$.

Let $I \subset \mathcal{O}_X$ be an ideal s.t. $V(I) = \{O\}$ and let $\hat{I} = I\hat{\mathcal{O}}_{X,O}$

We define H_I and $H_{\hat{I}}$ by

$$H_I = \{\sigma \in \mathcal{H}_{\{O\}}^n(\mathcal{O}_X) \mid I\sigma = 0\}, \quad H_{\hat{I}} = \{\sigma \in \mathcal{H}_{[O]}^n(\mathcal{O}_X) \mid \hat{I}\sigma = 0\}.$$

Since $V(I) = \{O\}$, H_I and $H_{\hat{I}}$ are finite dimensional vector spaces and they are isomorphic: $H_I \cong H_{\hat{I}}$.

A complex analytic version of the Grothendieck local duality yields the following [19, 20].

Theorem Let $I \subset \mathcal{O}_X$ be an ideal s.t. $V(I) = \{O\}$. Then, the following pairing are non-degenerate.

$$H_I \times \mathcal{O}_{X,O}/I \longrightarrow \mathbb{C}, \quad H_{\hat{I}} \times \hat{\mathcal{O}}_{X,O}/\hat{I} \longrightarrow \mathbb{C}.$$

The non-degeneracy immediately implies the following.

Corollary $\dim_{\mathbb{C}}(\mathcal{O}_{X,O}/I) = \dim_{\mathbb{C}}(H_{\hat{I}})$ holds.

Let $g(x) \in \mathbb{C}[x] = \mathbb{C}[x_1, x_2, \dots, x_n]$ be a polynomial, s.t. $g(0) = 0$ and let $J_g = \langle \frac{\partial g}{\partial x_1}, \frac{\partial g}{\partial x_2}, \dots, \frac{\partial g}{\partial x_n} \rangle$ denote the Jacobi ideal in the polynomial ring $\mathbb{C}[x]$.

Assume that, there is a neighborhood X of the origin $O \in \mathbb{C}^n$ s.t. $V(J_g) \cap X = \{O\}$.

Let

$$H_{J_g} = \{\psi \in \mathcal{H}_{\{O\}}^n(\mathcal{O}_X) \mid p\psi = 0, \forall p \in J_g\}$$

Then, we have the following. [19, 20, 22]

Proposition Let $\mu(g)$ be the Milnor number of g at the origin. Then, the following holds.

$$\mu(g) = \dim_{\mathbb{C}}(H_{J_g}).$$

Proof Let $\mathcal{O}_{X,O}J_g$ be the Jacobi ideal of g in the ring $\mathcal{O}_{X,O}$ of convergent power series at the origin. It is easy to see that

$$\{\psi \in \mathcal{H}_{\{O\}}^n(\mathcal{O}_X) \mid p\psi = 0, \forall p \in J_g\} = \{\psi \in \mathcal{H}_{\{O\}}^n(\mathcal{O}_X) \mid h\psi = 0, \forall h \in \mathcal{O}_{X,O}J_g\}.$$

Since the Milnor number $\mu(g)$ of g at the origin is defined as $\mu(g) = \dim_{\mathbb{C}}(\mathcal{O}_{X,O}/(\mathcal{O}_{X,O}J_g))$, we have the result.

In [22, 23], an algorithm for computing a basis of the vector space H_I of local cohomology classes is described.

4 Algorithm

Let $t = (t_1, t_2, \dots, t_\ell)$ and let $I_t \subset K[t][x_1, x_2, \dots, x_n]$ be a family of ideals in the polynomial ring, where $x = (x_1, x_2, \dots, x_n)$ are variables and $t = (t_1, t_2, \dots, t_\ell)$ are regarded as parameters. Let $V(I_t)$ denote the zero loci in $\mathbb{C}^n = \{x = (x_1, x_2, \dots, x_n) \mid x_i \in \mathbb{C}, i = 1, 2, \dots, n\}$ depending on parameters $t_1, t_2, \dots, t_\ell$. We assume that there is a neighborhood $X \subset \mathbb{C}^n$ s.t. $V(I_t) \cap X = \{O\}$, the origin.

Let H_{I_t} denote the set of algebraic local cohomology classes that are annihilated by the ideal I_t , where t are regarded as parameters. Structure of H_{I_t} depends on parameters.

In [11], the authors of the present paper gave an algorithm for computing a basis of the vector spaces H_{I_t} . More precisely, the algorithm compute parameter dependency of the space H_{I_t} .

By using this algorithm **ALCohomology** the Milnor numbers μ_{t,a_j} of $F_t(x, \eta)$ at the point $a_j \in \mathbb{P}^n$ are computable.

We give some examples for illustration.

Example(S. A. Broughton [2]) $f(x, y) = x^2y - x$

$$F_t(x, y, \eta) = x^2y - x\eta^2 - t\eta^3, \quad A = \{[0 : 1 : 0]\}.$$

Let $x = uy, \eta = hy$. Then $F_t = y^3g_t(u, h)$, where $g_t(u, h) = u^2 - uh^2 - th^3$.

Local cohomology $H_{J_{g_t}} = \{\psi \mid \frac{\partial g_t}{\partial u}\psi = \frac{\partial g_t}{\partial h}\psi = 0\}$ are given as

$$(i) \quad \text{the case } t \neq 0, H_{J_{g_t}} = \text{Span}\left\{\begin{bmatrix} 1 \\ uh \end{bmatrix}, \begin{bmatrix} 1 \\ uh^2 \end{bmatrix}\right\}$$

$$(ii) \quad \text{the case } t = 0, H_{J_{g_0}} = \text{Span}\left\{\begin{bmatrix} 1 \\ uh \end{bmatrix}, \begin{bmatrix} 1 \\ uh^2 \end{bmatrix}, \begin{bmatrix} 1 \\ u^2h \end{bmatrix} + 2\begin{bmatrix} 1 \\ uh^3 \end{bmatrix}\right\}.$$

Accordingly, $\mu_t = 2$ for $t \neq 0$, and $\mu_0 = 3$ for $t = 0$. Therefore the bifurcation set B_f is equal to $\{0\}$.

Example $f(x, y) = x^2y^2 + x$

$$F_t(x, y, \eta) = x^2y^2 + x\eta^3 - t\eta^4, \quad A = \{[0 : 1 : 0], [1 : 0 : 0]\}.$$

(1) computation at $a_1 = [0 : 1 : 0]$

Let $x = uy, \eta = hy$. Then $F_t = y^4g_t(u, h)$, $g_t(u, h) = u^2 + uh^3 - th^4$

Local cohomology $H_{J_{g_t}} = \{\psi \mid \frac{\partial g_t}{\partial u}\psi = \frac{\partial g_t}{\partial h}\psi = 0\}$ are given as

$$(i) \quad \text{the case } t \neq 0, H_{J_{g_t}} = \text{Span}\left\{\begin{bmatrix} 1 \\ uh \end{bmatrix}, \begin{bmatrix} 1 \\ uh^2 \end{bmatrix}, \begin{bmatrix} 1 \\ uh^3 \end{bmatrix}\right\}$$

$$(ii) \quad \text{the case } t = 0, H_{J_{g_0}} = \text{Span}\left\{\begin{bmatrix} 1 \\ uh \end{bmatrix}, \begin{bmatrix} 1 \\ uh^2 \end{bmatrix}, \begin{bmatrix} 1 \\ uh^3 \end{bmatrix}, \begin{bmatrix} 1 \\ u^2h \end{bmatrix} + 2\begin{bmatrix} 1 \\ uh^4 \end{bmatrix}, \begin{bmatrix} 1 \\ u^2h^2 \end{bmatrix} + 2\begin{bmatrix} 1 \\ uh^5 \end{bmatrix}\right\}.$$

Accordingly, $\mu_{t,a_1} = 3$ for $t \neq 0$, and $\mu_{0,a_1} = 5$ for $t = 0$.

(2) computation at $a_2 = [1 : 0 : 0]$, We see that $\mu_{t,a_2} = 2$ for all t

Therefore, we have $B_f = \{0\}$.

Example (A. Dimca [8]) $f(x, y, z) = x^2y + y^2z + x$

$$F_t(x, y, z, \eta) = x^2y + y^2z + x\eta^2 - t\eta^3, \quad A = \{[0 : 0 : 1 : 0]\}.$$

Let $x = uz, y = vz, \eta = hz$. Then. $F_t = z^3g_t(u, v, h)$ with $g_t(u, v, h) = u^2v + v^2 + uh^2 - th^3$.

Local cohomology $H_{J_{g_t}} = \{\psi \mid \frac{\partial g_t}{\partial u}\psi = \frac{\partial g_t}{\partial v}\psi = \frac{\partial g_t}{\partial h}\psi = 0\}$ are given as

$$H_{J_{g_t}} = \text{Span}\left\{\begin{bmatrix} 1 \\ uvh \end{bmatrix}, \begin{bmatrix} 1 \\ uvh^2 \end{bmatrix}, \begin{bmatrix} 1 \\ u^2vh \end{bmatrix}, \begin{bmatrix} 1 \\ u^3vh \end{bmatrix} - \frac{1}{2}\begin{bmatrix} 1 \\ uv^2h \end{bmatrix}, \right. \\ \left. \begin{bmatrix} 1 \\ u^4vh \end{bmatrix} - \frac{1}{2}\begin{bmatrix} 1 \\ u^2v^2h \end{bmatrix} + \frac{3}{2}\begin{bmatrix} 1 \\ u^2vh^2 \end{bmatrix} + \begin{bmatrix} 1 \\ uvh^3 \end{bmatrix}\right\}$$

$H_{J_{g_t}}$ depend on t . Whereas Milnor numbers does not depend on t . Therefore, we have $B_f = \emptyset$.

Remark: In 2004, A. Bodin obtained an algorithm for computing bifurcation sets. The method proposed in this section is different from the algorithm presented in [3].

5 Bifurcation sets and deformations

The proposed method can treat a family of polynomial map depending on deformation parameters. Namely, the algorithm **ALCohomology** can compute parameter dependency of bifurcation sets. We give some examples.

Example (A. Bodin [2]) $f = (x - s^2 + 1)(x^2y + 1)$, where s is a deformation parameter.

$$\tilde{f}(x, y, \eta) = x^3y - (s^2 + 1)x^2y\eta + x\eta^3 - (s^2 + 1)\eta^4$$

$$F_t(x, y, \eta) = x^3y - (s^2 + 1)x^2y\eta + x\eta^3 - (s^2 + 1)\eta^4 - t\eta^4$$

$$A = \{[0 : 1 : 0]\}.$$

Now let $x = uy, y = y, \eta = hy$. Then

$$F_t = y^4g_t(u, h) \text{ where } g_t(u, h) = u^3 - (s^2 + 1)u^2h + uh^3 - (s^2 + 1 + t)h^4.$$

Compute bases of parametric local cohomology $H_{J_{g_t}}$ by using **ALCohomology**. We regard s and t as parameters. The output consists of four cases.

- (i) $s^2 + 1 = 0, t = 0$,
- (ii) $s^2 + 1 = 0, t \neq 0$
- (iii) $s^2 + 1 + t = 0, s^2 + 1 \neq 0, t \neq 0$,
- (iv) $(s^2 + 1)t(s^2 + 1 + t) \neq 0$

For each cases, local cohomology are given as follows.

- (i) $s^2 + 1 = 0, t = 0, \dim_K(H_{J_{g_t}}) = 7$

$$H_{J_{g_t}} = \text{Span}\left\{\begin{bmatrix} 1 \\ uh \end{bmatrix}, \begin{bmatrix} 1 \\ uh^2 \end{bmatrix}, \begin{bmatrix} 1 \\ u^2h \end{bmatrix}, \begin{bmatrix} 1 \\ uh^3 \end{bmatrix}, \begin{bmatrix} 1 \\ u^2h^2 \end{bmatrix}, \begin{bmatrix} 1 \\ uh^4 \end{bmatrix} - \frac{1}{3}\begin{bmatrix} 1 \\ u^3h \end{bmatrix}, \begin{bmatrix} 1 \\ uh^5 \end{bmatrix} - \frac{1}{3}\begin{bmatrix} 1 \\ u^3h^2 \end{bmatrix}\right\}.$$

- (ii) $s^2 + 1 = 0, t \neq 0, \dim_K(H_{J_{g_t}}) = 6$

$$H_{J_{g_t}} = \text{Span}\left\{\begin{bmatrix} 1 \\ uh \end{bmatrix}, \begin{bmatrix} 1 \\ uh^2 \end{bmatrix}, \begin{bmatrix} 1 \\ u^2h \end{bmatrix}, \begin{bmatrix} 1 \\ uh^3 \end{bmatrix}, \begin{bmatrix} 1 \\ u^2h^2 \end{bmatrix}, \begin{bmatrix} 1 \\ uh^4 \end{bmatrix} - \frac{1}{3}\begin{bmatrix} 1 \\ u^3h \end{bmatrix} + \frac{4}{3}t\begin{bmatrix} 1 \\ u^2h^3 \end{bmatrix}\right\}$$

- (iii) $s^2 + 1 + t = 0, s^2 + 1 \neq 0, t \neq 0, \dim_K(H_{J_{g_t}}) = 6$

$$H_{J_{g_t}} = \text{Span}\left\{\begin{bmatrix} 1 \\ uh \end{bmatrix}, \begin{bmatrix} 1 \\ uh^2 \end{bmatrix}, \begin{bmatrix} 1 \\ u^2h \end{bmatrix}, \begin{bmatrix} 1 \\ uh^3 \end{bmatrix}, t\begin{bmatrix} 1 \\ uh^4 \end{bmatrix}, -\frac{1}{2}\begin{bmatrix} 1 \\ u^2h^2 \end{bmatrix}, t^3\begin{bmatrix} 1 \\ uh^5 \end{bmatrix} - \frac{1}{2}t^2\begin{bmatrix} 1 \\ u^2h^3 \end{bmatrix} - \frac{9}{4}\begin{bmatrix} 1 \\ u^2h^2 \end{bmatrix}, +\frac{3}{2}t\begin{bmatrix} 1 \\ u^3h \end{bmatrix}\right\}$$

- (iv) $(s^2 + 1 + t)(s^2 + 1)t \neq 0, \dim_K(H_{J_{g_t}}) = 5$

$$H_{J_{g_t}}: \text{omitted}$$

Therefore bifurcation sets are given as follows.

$$\text{If } s^2 + 1 = 0, B_f = \{0\}$$

$$\text{If } s^2 + 1 \neq 0, B_f = \{-s^2 - 1\}$$

The following example is due to M. Tibar.

Example (M. Tibar [26]) $f(x, y, z) = x^2y + x + z^2 + sz^3$, where s is a deformation parameter.

The singular set at infinity consists of one point: $A = \{[0 : 1 : 0 : 0]\}$.

Compute bases of parametric local cohomology $H_{J_{g_t}}$ by using **ALCohomology**.

The output consists of 5 cases. Here we give the conditions and Milnor numbers.

- (i) $s = 0, t = 0, \dim_K(H_{J_{g_t}}) = 5$
- (ii) $s = 0, t \neq 0, \dim_K(H_{J_{g_t}}) = 4$
- (iii) $t = 0, s \neq 0, \dim_K(H_{J_{g_t}}) = 5$

- (iv) $27s^2t - 4 = 0$, $\dim_K(H_{J_{g_t}}) = 5$
- (v) $(27s^2t - 4)st \neq 0$, $\dim_K(H_{J_{g_t}}) = 4$

It is easy to see that bifurcation sets are given as follows.

If $s = 0$, $B_f = \{0\}$

If $s \neq 0$, $B_f = \{0, \frac{4}{27s^2}\}$

Notice that the bifurcation sets depend on deformation parameter s .

Acknowledgements

This work has been partly supported by JSPS Grant-in-Aid for Scientific Research(C)(Nos. 22K03334, 23K03076).

References

- [1] E. A. Bartolo, I. Luengo and A. Melle-Hernández, Milnor number at infinity, topology and Newton boundary of a polynomial function, *Math. Z.* **233** (2000), 679-696.
- [2] A. Bodin, Newton polygons and families of polynomials, *Manuscripta Math.* **113** (2004), 371-382.
- [3] A. Bodin, Computation of Milnor numbers and critical values at infinity, *J. Symbolic Comput.* **38** (2004), 1417-1427,
- [4] A. Bodin and M. Tibar, Topological equivalence of complex polynomials, *Adv. in Math.* **199** (2006), 136-150.
- [5] S. A. Broughton, On the topology of polynomial hypersurfaces, *Proc. Sympos. in Pure Math.* **40** (1983), 167-178.
- [6] S. A. Broughton, Milnor numbers and the topology of polynomial hypersurfaces, *Invent. Math.* **92** (1988), 217-241
- [7] P. Cassou-Noguès and A. Dimca, Topology of complex polynomials via polar curves, *Kodai Math. J.* **22** (1999), 131-139.
- [8] A. Dimca, *Singularities and Topology of Hypersurfaces*, Universitext, Springer 1992.
- [9] M. Ishikawa, The bifurcation set of a complex polynomial function of two variables and the Newton polygons of singularities at infinity, *J. Math. Soc. Japan* **54** (2002), 161-196.
- [10] V. T. Lê and M. Oka, Estimation of the number of the critical values at infinity of a polynomial function, *Publ. RIMS, Kyoto Univ.* **31** (1995), 577-598.
- [11] K. Nabeshima and S. Tajima, Algebraic local cohomology with parameters and parametric standard bases for zero-dimensional ideals, *J. of Symbolic Comput.* **82** (2017), 91-122,
- [12] A. Némethi and A. Zaharia, On the bifurcation set of a polynomial function and Newton boundary, *Publ. RIMS, Kyoto Univ.* **26** (1990), 681-689.
- [13] A. Parusiński, On the bifurcation set of complex polynomial with isolated singularities at infinity, *Compositio Math.* **97** (1995), 369-384.
- [14] A. Parusiński, A note on singularities at infinity of complex polynomials, *Banach Center Publications* **39** (1997), 131-141.
- [15] T. S. Pham, On the topology of the Newton boundary at infinity, *J. Math. Soc. Japan* **60** (2008), 1065-1081.
- [16] N. Rodrigues Ribeiro, Clasification at infinity of polynomials of degree 3 in 3 variables, *Journal of Singularities* **25** (2022), 377-392.

- [17] D. Siersma and M. Tibar, Singularities at infinity and their vanishing cycles, *Duke Math. J.* **80** (1995), 771–783.
- [18] D. Siersma and M. Tibar, Two lectures on the topology of polynomial functions and singularities at infinity, *Advanced School and Workshop on Singularities in Geometry and Topology*, (2005), The Abdus Salam International Center for Theoretical Physics.
- [19] S. Tajima, A method for computing zero-dimensional algebraic local cohomology and standard bases (in Japanese), *RIMS Kôkyûroku*, **1456**(2005), 126–132.
- [20] S. Tajima, A method for computing zero-dimensional algebraic local cohomology and standard bases II (in Japanese), *RIMS Kôkyûroku* **1568** (2007), 74–80.
- [21] S. Tajima and K. Nabeshima, A method for computing the bifurcation set of a complex polynomial mapping I (in Japanese), *RIMS Kôkyûroku* **2255** (2023), 88–95.
- [22] S. Tajima and Y. Nakamura, Annihilating ideals for an algebraic local cohomology class, *J. Symbolic Computation*, **44** (2009), 435–448.
- [23] S. Tajima, Y. Nakamura and K. Nabeshima, Standard bases and algebraic local cohomology for zero dimensional ideals, *Advanced Studies in Pure Mathematics*, **56** (2009), 341–361.
- [24] K. Takeuchi, Bifurcation values of polynomial functions and perverse sheaves, *Ann. Inst. Fourier* **70** (2020), 597–619.
- [25] S. Tanabe and A. Gündüs, On curves approaching asymptotic critical value set of a polynomial map, *Kumamoto J. Math.* **36** (2023), 17–40.
- [26] M. Tibar, *Polynomials and Vanishing Cycles*, Cambridge Univ. Press 2007.
- [27] A. Zaharia, On the bifurcation set of a polynomial function and Newton boundary, II, *Kodai Math. J.* **19** (1996), 218–233.

Shinichi Tajima

Graduate School of Science and Technology, Niigata University

8050, Ikarashi 2-no-cho, Nishi-ku Niigata, Japan

E-mail address: tajima@emeritus.niigata-u.ac.jp

新潟大学大学院自然科学研究科 田島慎一

Katsusuke Nabeshima

Department of Mathematics, Faculty of Sciences

1-3, Kagurazaka, Tokyo, Japan

E-mail address: nabeshima@rs.tus.ac.jp

東京理科大学理学部第一部応用数学科 鍋島克輔