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Delayed feedback control and phase reduction of unstable quasi-periodic orbits

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The delayed feedback control (DFC) is applied to stabilize unstable quasi-periodic orbits (QPOs) in discrete-time systems. The feedback input is given by the difference between the current state and a time-delayed state in the DFC. However, there is an inevitable time-delay mismatch in QPOs. To evaluate the influence of the time-delay mismatch on the DFC, we propose a phase reduction method for QPOs and construct a phase response curve (PRC) from unstable QPOs directly. Using the PRC, we estimate the rotation number of QPO stabilized by the DFC. We show that the orbit of the DFC is consistent with the unstable QPO perturbed by a small state difference resulting from the time-delay mismatch, implying that the DFC can certainly stabilize the unstable QPO. © 2014 AIP Publishing LLC. [<http://dx.doi.org/10.1063/1.4896219>]

A quasi-periodic orbit (QPO) possesses the properties of both a periodic orbit and a chaotic orbit, which are almost periodic and aperiodic, respectively. Whereas the stabilization of a QPO presents a challenging problem due to its aperiodicity, its almost periodicity gives us the way to stabilize an unstable QPO because several control methods are available for periodic orbits. To stabilize unstable periodic orbits, the delayed feedback control (DFC) was proposed by Pyragas.¹ Our idea is that the DFC may be applicable to stabilize an unstable QPO because of its almost periodicity. However, even if the stabilization is achieved, there is an inevitable difference between the control orbit and the unstable QPO because of its aperiodicity. Therefore, it is not clear whether the DFC stabilizes the unstable QPO itself. In this study, we show that the orbit of the DFC is consistent with the unstable QPO by using a phase reduction method.

I. INTRODUCTION

The DFC proposed by Pyragas¹ is widely used to stabilize unstable periodic orbits even if systems are chaotic. The feedback input of the DFC is given by the difference between the current state and a time-delayed state. The delay is chosen to be equal to the period of an unstable periodic orbit. Besides the period, the DFC does not require any exact model of the unstable periodic orbit to stabilize it. The DFC is also noninvasive, since the feedback input vanishes when the system is stabilized.²

In this study, we show that the DFC is applicable to the stabilization of QPOs. Although QPOs are dynamics defined on a high-dimensional invariant torus, in general,³ we focus on the simplest case of dynamics defined on an invariant closed curve in discrete-time systems. Since the QPO is

dense in the invariant closed curve, the solution does not return to an exact state, in contrast to the periodic orbit. However, the QPO is almost periodic, i.e., if $\mathbf{x}_n$ is the state of the QPO at time n , we can choose recurrence time d for small $\epsilon > 0$ and any n as follows:

$$\|\mathbf{x}_{n+d} - \mathbf{x}_n\| < \epsilon. \quad (1)$$

Therefore, if we use recurrence time d as the feedback delay, the DFC can be applied to the QPO in the same way as the periodic case.

The problem is that the feedback input never vanishes because there is an inevitable time-delay mismatch in the QPO. There always exists a difference between the orbit of the DFC and the unstable QPO that serves as the control target. Because of this, it is not clear whether the orbit of the DFC can be understood to be derived from the unstable QPO. To this end, we evaluate the influence of the time-delay mismatch on the orbit of the DFC.

Novičenko and Pyragas have shown that the DFC with a small time-delay mismatch can be evaluated by using the phase reduction method.² They have constructed a phase response curve (PRC) of the DFC with the exact delay and evaluated the period of the orbit of the mismatch system by using it. However, a similar analysis cannot be applied to our case directly because we cannot obtain the exact delay, wherein the feedback input vanishes in the QPO. As mentioned above, the DFC is noninvasive, i.e., the orbit of the DFC coincides with the unstable orbit if the delay is exact. Therefore, we use the PRC constructed directly from the unstable QPO, which can be considered as the orbit having no time-delay mismatch.

In this study, we show a phase reduction method for QPO in discrete-time systems. We also show that a PRC can be constructed from an unstable QPO. Using the PRC, we explain that the orbit of the DFC is consistent with the unstable QPO perturbed by a small state difference resulting from the time-delay mismatch.

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II. PHASE REDUCTION OF QUASI-PERIODIC ATTRACTORS

We focus on QPOs in discrete-time systems

$$\mathbf{x}_{n+1} = \mathbf{F}(\mathbf{x}_n), \quad (2)$$

where $\mathbf{x}_n \in \mathbb{R}^M$ is the state vector and $\mathbf{F} : \mathbb{R}^M \rightarrow \mathbb{R}^M$ is the function representing the system. The rotation number is an important invariant in QPOs.⁴ If a certain phase is defined on a QPO, the rotation number is defined by the average phase difference for an iteration of $\mathbf{F}$. If the rotation number is irrational, then the QPO is topologically conjugate to the irrational rotation⁵

$$\theta_{n+1} = \theta_n + \omega, \quad (3)$$

where $\theta_n \in \mathbb{S}$ is the phase of the circle $\mathbb{S} = \mathbb{R}/\mathbb{Z}$ and $\omega \in [0, 1]$ is the rotation number. For the QPO $\tilde{\mathbf{x}}_n$, we obtain the following relation:

$$\tilde{\mathbf{x}}_n = \boldsymbol{\psi}(\theta_n), \quad (4)$$

where $\boldsymbol{\psi} : \mathbb{S} \rightarrow \mathbb{R}^M$ is the homeomorphism from the phase to the state. Since several numerical calculation methods of rotation numbers have been proposed,^{4,6,7} we can connect the QPO to the irrational rotation practically.

Let the QPO $\tilde{\mathbf{x}}_n$ be an attractor. The state $\mathbf{x}_m$ in the basin of attraction corresponds to the phase as follows: if,

$$\lim_{k \rightarrow \infty} \|\tilde{\mathbf{x}}_{n+k} - \mathbf{x}_{m+k}\| = 0, \quad (5)$$

then the phase of $\mathbf{x}_m$ is assigned to the same phase of $\tilde{\mathbf{x}}_n$, i.e., the two states have the same isochron.⁸ This implies that the phase can be defined not only in the QPO but also in its basin. Therefore, the phase reduction method is applicable to the discrete-time QPO in the same way as a continuous-time periodic orbit.

Let $\phi : \mathbb{B} \rightarrow \mathbb{S}$ be the map of the isochron for each state in the basin of attraction $\mathbb{B}$. If the orbit is restricted within the QPO (i.e., $\boldsymbol{\psi}(\mathbb{S})$), ϕ is the inverse map of the homeomorphism $\boldsymbol{\psi} : \phi = \boldsymbol{\psi}^{-1} : \boldsymbol{\psi}(\mathbb{S}) \rightarrow \mathbb{S}$. The system (2) can be represented in the phase domain by using the irrational rotation (3)

$$\phi(\mathbf{x}_{n+1}) = \phi(\mathbf{F}(\mathbf{x}_n)) = \phi(\mathbf{x}_n) + \omega. \quad (6)$$

Here, we add a small perturbation to the system

$$\mathbf{x}_{n+1} = \mathbf{F}(\mathbf{x}_n) + \epsilon \mathbf{g}_n, \quad (7)$$

where ϵ is the small parameter and $\mathbf{g}_n$ is the time-dependent perturbation. We can express the small perturbation in the phase domain by using Eq. (6)

$$\phi(\mathbf{F}(\mathbf{x}_n) + \epsilon \mathbf{g}_n) = \phi(\mathbf{x}_n) + \omega + \epsilon \left. \frac{\partial \phi}{\partial \mathbf{x}} \right|_{\mathbf{x}=\mathbf{F}(\mathbf{x}_n)} \cdot \mathbf{g}_n + O(\epsilon^2), \quad (8)$$

where we assume that the first and second derivatives of ϕ exist. Since $\boldsymbol{\psi}$ is differentiable for almost every rotation number,⁹ ϕ is differentiable within the QPO for almost every

rotation number because ϕ is the inverse map of $\boldsymbol{\psi}$ within the QPO. However, since ϕ is defined in the basin of attraction $\mathbb{B}$ including the QPO and is not ensured to be differentiable in general, the above assumption is necessary. The derivative $\partial \phi / \partial \mathbf{x}$ is called the PRC. Since the perturbed orbit lies in the neighborhood of the original orbit because of the small perturbation, the perturbed system is approximately represented by the scalar phase equation

$$\theta_{n+1} = \theta_n + \omega + \epsilon \mathbf{Z}(\mathbf{x}_{n+1}) \cdot \mathbf{g}_n, \quad (9)$$

where $\mathbf{Z}(\mathbf{x}) = \partial \phi / \partial \mathbf{x}$ is the PRC. It should be noted that we use the state of the perturbation-free system (2) as an approximation for $\mathbf{x}_{n+1}$ in Eq. (9).

In the continuous-time periodic orbit, PRCs can be estimated by the adjoint method.¹⁰ In the discrete-time QPO, we can obtain PRCs in a similar way. By differentiating Eq. (6) over $\mathbf{x}$, we construct the adjoint method in the discrete-time QPO

$$\mathbf{Z}(\mathbf{x}_n) = \mathbf{F}'(\mathbf{x}_n)^T \mathbf{Z}(\mathbf{x}_{n+1}), \quad (10)$$

where $\mathbf{F}'$ is the Jacobian matrix of $\mathbf{F}$. Since the amplitude of $\mathbf{Z}(\mathbf{x}_n)$ is not determined by Eq. (10), the initial condition is necessary. If $\mathbf{x} = \boldsymbol{\psi}(\theta)$ holds, we obtain the following relation:

$$\phi(\mathbf{x}) = \theta. \quad (11)$$

We differentiate Eq. (11) over θ . Using the chain rule, we obtain the initial condition

$$\mathbf{Z}(\mathbf{x}_N) \cdot \boldsymbol{\psi}'(\theta_N) = 1, \quad (12)$$

where N is the starting point of the iteration of Eq. (10), $\boldsymbol{\psi}'(\theta) = d\boldsymbol{\psi}/d\theta$, and $\mathbf{x}_N = \boldsymbol{\psi}(\theta_N)$. Note that this condition can be replaced by $\mathbf{Z}(\mathbf{x}_n) \cdot \boldsymbol{\psi}'(\theta_n) = 1$ for any n . However, the initial condition holds only in the QPO because $\boldsymbol{\psi}$ is defined in it. If the orbit is restricted within the QPO, the PRC is represented by the function of the phase

$$\tilde{\mathbf{Z}}(\theta) = \mathbf{Z}(\boldsymbol{\psi}(\theta)). \quad (13)$$

As an example, we consider the coupled map lattice with asymmetric connections¹¹

$$\begin{aligned} x_{n+1} &= f(x_n) + \frac{1}{2}(\beta - \delta)(f(z_n) - f(x_n)) \\ &\quad + \frac{1}{2}(\beta + \delta)(f(y_n) - f(x_n)), \\ y_{n+1} &= f(y_n) + \frac{1}{2}(\beta - \delta)(f(x_n) - f(y_n)) \\ &\quad + \frac{1}{2}(\beta + \delta)(f(z_n) - f(y_n)), \\ z_{n+1} &= f(z_n) + \frac{1}{2}(\beta - \delta)(f(y_n) - f(z_n)) \\ &\quad + \frac{1}{2}(\beta + \delta)(f(x_n) - f(z_n)), \end{aligned} \quad (14)$$

where f is the logistic map $f(x) = 1 - \alpha x^2$. There is a fixed point

$$x_n = y_n = z_n = x^*, \quad x^* = f(x^*) = \frac{-1 + \sqrt{4\alpha + 1}}{2\alpha}. \quad (15)$$

At the fixed point, the Jacobian matrix has a real eigenvalue and a pair of complex eigenvalues

$$\mu_1 = \gamma, \quad \mu_{2,3} = \gamma \frac{2 - 3\beta \pm i\sqrt{3}\delta}{2}, \quad (16)$$

where $\gamma = f'(x^*) = 1 - \sqrt{4\alpha + 1}$. Let α be the bifurcation parameter. When $|\mu_{2,3}| > 1$ by increasing α , the Neimark-Sacker bifurcation occurs and a QPO can be obtained. It is necessary for the stable QPO that $|\mu_1| < 1$. Thus, we choose the parameters $(\alpha, \beta, \delta) = (0.7, 0.02, 0.6)$ such that $|\mu_1| < 1 < |\mu_{2,3}|$. The homeomorphism ψ is estimated by the Fourier series constructed from the QPO. ψ' is also obtained by differentiating the Fourier series over θ . We use the ten-order Fourier series.

The estimated PRC is shown in Fig. 1. Although the PRC has three components $\tilde{\mathbf{Z}}(\theta) = (\tilde{Z}_x(\theta), \tilde{Z}_y(\theta), \tilde{Z}_z(\theta))^T$, we show only $\tilde{Z}_x$ since the other components have the same profile with phase shifts $\pm 1/3$. We consider the small perturbation of the parameter $\alpha = \alpha_0 + \epsilon$. We estimate the rotation number by using the phase equation (9). Since the rotation number is the average phase difference, the estimated rotation number $\tilde{\omega}$ is defined by

$$\tilde{\omega} = \omega + \frac{\epsilon}{N} \sum_{n=0}^{N-1} \mathbf{Z}(\mathbf{x}_{n+1}) \cdot \mathbf{g}_n, \quad (17)$$

where $\mathbf{x}_{n+1}$ is the state of the perturbation-free system (2).

In Fig. 2, we compare the estimated rotation number with that of the direct calculations of the coupled map lattice (14). In Eq. (17), the dependence of the estimated rotation number on the parameter ϵ is linear. This implies that Eq. (17) estimates the derivative of the dependence of the rotation number at $\epsilon = 0$. In the bifurcation of QPOs, however, since the phase locked (periodic) regions exist (Fig. 2), the dependence of the rotation number gives a devil's staircase and is non-smooth.¹² Therefore, the derivative of the dependence of the rotation number can not intrinsically be defined. We estimate the rotation number by considering the dependence of the rotation

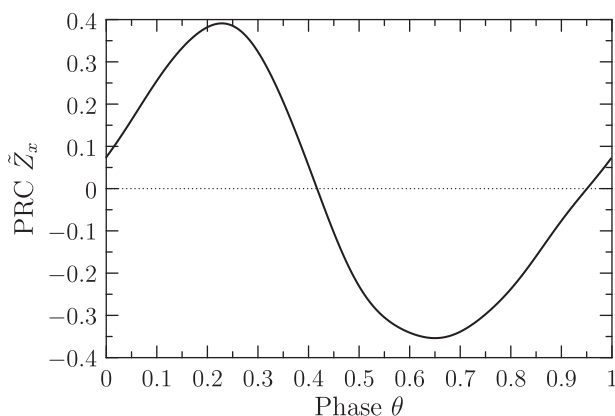


FIG. 1. PRC $\tilde{Z}_x(\theta)$ of the coupled map lattice (14). The parameters are fixed at $(\alpha, \beta, \delta) = (0.7, 0.02, 0.6)$ where the system shows the stable QPO.

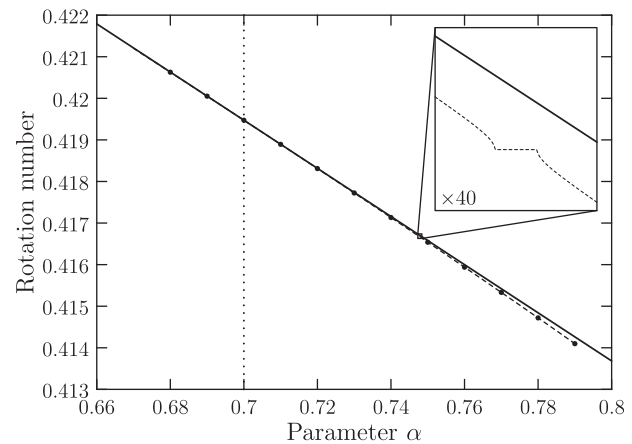


FIG. 2. Rotation number for parameter perturbation $\alpha = \alpha_0 + \epsilon$. The solid line shows the estimated rotation number by using Eq. (17). The PRC is constructed at $\alpha_0 = 0.7$ (shown by a vertical dotted line). The dashed curve with filled circles shows the rotation number obtained from the direct calculations. The partial area including period-12 points is enlarged to indicate the existence of phase locked regions. The other parameters are fixed at $(\beta, \delta) = (0.02, 0.6)$ and $N = 10^5$.

number on the parameter to be approximately smooth in the neighborhood of $\epsilon = 0$. Nevertheless, the estimated rotation number shows the correct results for small values ϵ as shown in Fig. 2. This is the different point from the phase reduction of periodic orbits in a continuous-time system, in which the change of the period given by the perturbation is smooth in general. These results suggest that the smooth approximation of the dependence of the rotation number is feasible for the discrete-time QPOs in the neighborhood of the parameter value giving a QPO.

III. STABILIZATION OF UNSTABLE QPO BY DFC

The feedback input of the DFC is given by the difference between the d -past state and the current state when the period of the unstable periodic orbit is d

$$\mathbf{x}_{n+1} = \mathbf{F}(\mathbf{x}_n) + \mathbf{K}\mathbf{u}_n, \quad (18)$$

where $\mathbf{K}$ is the $(M \times M)$ -matrix of the feedback coefficients and $\mathbf{u}_n$ is the feedback input

$$\mathbf{u}_n = \mathbf{x}_{n-d} - \mathbf{x}_n. \quad (19)$$

If the unstable periodic orbit is stabilized, the feedback input $\mathbf{u}_n$ vanishes and Eq. (18) becomes equivalent to the control-free system (2). Therefore, the DFC preserves the position of the periodic orbit and changes its stability solely.

Unfortunately, in the case of the unstable QPO, we cannot choose the delay d such that the feedback input $\mathbf{u}_n$ vanishes. However, because of the recursive property of QPO, there is a delay d in which the feedback input is always small. Our idea is that the DFC may be applicable by using such a delay in the same way as the periodic case.

In Fig. 3, we show an example of orbit stabilization by the DFC for the coupled map lattice (14). To obtain the unstable QPO, we choose the parameters $(\alpha, \beta, \delta) = (0.79, 0.02, 0.06)$ such that $|\mu_1|, |\mu_{2,3}| > 1$. If $\alpha = 0.75$, the period-doubling bifurcation of the fixed point occurs because

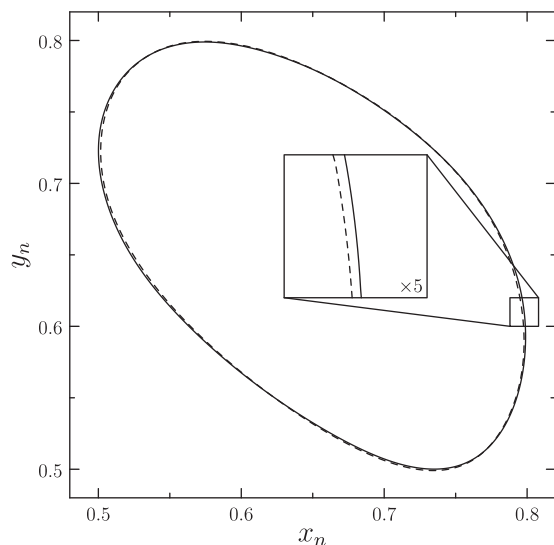


FIG. 3. Unstable QPO (solid circle) and stabilized orbit by DFC (dashed circle) of the coupled map lattice (14). The orbits are projected onto the x - y plane. The delay is $d=59$ and the feedback coefficient is $k=-0.4$. The other parameters are fixed at $(\alpha, \beta, \delta)=(0.79, 0.02, 0.06)$.

$|\mu_1|=1$. In this case, however, the Neimark-Sacker bifurcation has not occurred yet because $|\mu_{2,3}|<1$. Although the Neimark-Sacker bifurcation of the fixed point also occurs by increasing α to 0.79 (i.e., $|\mu_{2,3}|>1$), the QPO is unstable because the fixed point is unstable. In this case, the period-2 points are stable. The unstable QPO is numerically estimated by the bisection method.¹³ We numerically verify that the unstable orbit obtained by the bisection method is quasi-periodic. Using the unstable QPO, we choose the delay $d=59$ in which the average of the difference $\|x_{n-d}-x_n\|$ is sufficiently small ($\approx 3.65 \times 10^{-3}$). The feedback input is given only to x_n

$$K = \begin{pmatrix} k & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \quad (20)$$

Unfortunately, there has been no appropriate method to determine the coefficient yet. Therefore, we check the dynamics of the system with respect to the value of k exhaustively and we choose $k=-0.4$. Although the stabilized orbit by the DFC lies in the neighborhood of the unstable QPO, the difference does not disappear (Fig. 3).

IV. PHASE REDUCTION OF DFC

Recent studies have indicated that the phase reduction method is applicable to periodic orbits of time-delay systems.^{14,15} In the continuous-time case, the time-delay systems have infinite-dimensional phase space and hence the standard adjoint method cannot be directly applied. In the discrete-time case, since the system is finite-dimensional even if there is time delay, we can directly apply the adjoint method to the DFC system. However, the results in the literature^{14,15} suggest that the adjoint method can be reduced to a smaller problem in the discrete-time case.

We rewrite the delayed states in Eq. (18) as explicit states

$$x_{n+1}^{(0)} = F(x_n^{(0)}) + K(x_n^{(d)} - x_n^{(0)}), \quad (21)$$

$$x_{n+1}^{(1)} = x_n^{(0)}, x_{n+1}^{(2)} = x_n^{(1)}, \dots, x_{n+1}^{(d)} = x_n^{(d-1)}. \quad (22)$$

Let $Z^{(0)}, Z^{(1)}, \dots, Z^{(d)}$ be the PRCs corresponding to the explicit states. For simple notation, let $Z_n^{(i)} = Z^{(i)}(x_n^{(i)})$. The adjoint method of Eq. (10) is expressed as follows:

$$\begin{pmatrix} Z_n^{(0)} \\ Z_n^{(1)} \\ \vdots \\ Z_n^{(d-1)} \\ Z_n^{(d)} \end{pmatrix} = \begin{pmatrix} F'(x_n^{(0)})^T - K^T & I & 0 & \cdots & 0 \\ 0 & 0 & I & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & I \\ K^T & 0 & 0 & \cdots & 0 \end{pmatrix} \times \begin{pmatrix} Z_{n+1}^{(0)} \\ Z_{n+1}^{(1)} \\ \vdots \\ Z_{n+1}^{(d-1)} \\ Z_{n+1}^{(d)} \end{pmatrix}. \quad (23)$$

Since $Z_n^{(d)} = K^T Z_{n+1}^{(0)}$ and $Z_n^{(d-1)} = Z_{n+1}^{(d)}, Z_n^{(d-2)} = Z_{n+1}^{(d-1)}, \dots, Z_n^{(1)} = Z_{n+1}^{(2)}$, the following relation holds:

$$Z_n^{(d-i+1)} = K^T Z_{n+1}^{(i)}, \quad i = 1, 2, \dots, d. \quad (24)$$

Therefore, Eq. (23) is reduced to the following form:

$$Z(x_n) = F'(x_n)^T Z(x_{n+1}) + K^T (Z(x_{n+d+1}) - Z(x_{n+1})), \quad (25)$$

where we rewrite $x_n = x_n^{(0)}$ and $Z(x_n) = Z_n^{(0)}$ again. Note that the phase equation of the DFC is the same as Eq. (9) because the perturbation is applied only to $x_{n+1}^{(0)}$. The initial condition can be also reduced as follows:

$$Z(x_N) \cdot \psi'(\theta_N) + \sum_{i=1}^d (K^T Z(x_{N+d-i+1})) \cdot \psi'(\theta_{N-i}) = 1. \quad (26)$$

The reduced form of the initial condition (26) is more important than that of the adjoint method because it enables us to connect the PRC constructed from the unstable QPO to the PRC of the DFC, as discussed in Secs. V and VI.

V. CONSTRUCTION OF PRC FROM UNSTABLE QPO

The PRC profile of the DFC is equivalent to that of a control-free system if the feedback input u_n vanishes.¹⁴ In this case, the difference of PRCs ($Z(x_{n+d+1}) - Z(x_{n+1})$) becomes zero and the adjoint method of the DFC (25) becomes equivalent to that of the control-free system (10). Therefore, the PRC of the control-free system can be expressed by the PRC of the DFC and vice versa. In the QPO, however, we cannot obtain the exact delay such that the feedback input vanishes. Thus, we construct the PRC of the control-free system that has the unstable QPO.

In order to construct the PRC from the unstable QPO, we adopt a QR-based method used for computing Lyapunov exponents.^{16,17} To apply the QR-based method to the adjoint method, we first decompose the transposed Jacobian matrix $F'(x_N)^T$ by using the QR decomposition

$$F'(x_N)^T = Q_N R_N, \quad (27)$$

where Q_N is the orthogonal matrix and R_N is the upper triangular matrix. Then, we successively define the matrix D_n for $n = N-1, N-2, \dots, 2, 1$ by

$$D_n = F'(x_n)^T Q_{n+1}, \quad (28)$$

and decompose D_n

$$D_n = Q_n R_n, \quad (29)$$

where Q_n and R_n are orthogonal and upper triangular, respectively. Let $Z_n = Z(x_n)$ for simplicity. We consider the transformed vector Y_n of Z_n by Q_n

$$Z_n = Q_n Y_n. \quad (30)$$

Then, the adjoint method (10) can be represented by Y_n

$$Y_n = R_n Y_{n+1}, \quad (31)$$

because $R_n = Q_n^T F'(x_n)^T Q_{n+1}$ from Eqs. (28), (29), and $Q_n^T = Q_n^{-1}$. It should be noted that the products of the diagonal components of R_n give the exponents equivalent to the Lyapunov exponents λ_i ¹⁶

$$\lambda_i = \lim_{N \rightarrow \infty} \frac{1}{N} \log \prod_{n=1}^N |r_n^{(i)}| = \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N \log |r_n^{(i)}|, \quad (32)$$

where $r_n^{(i)}$ is the i -th diagonal component of R_n . It should be also noted that we obtain the exponents λ_i in decreasing order.¹⁶

We first consider the stable QPO in the case of Fig. 1. We have the three exponents: the largest exponent is $\lambda_1 = 0$ and the others are negative because the QPO is stable. In this case, only the first component $y_n^{(1)}$ of Y_n has a nonzero value and the other components $y_n^{(2)}$ and $y_n^{(3)}$ converge to zero. Therefore, the adjoint method (31) is reduced to the following scalar equation:

$$y_n^{(1)} = r_n^{(1)} y_{n+1}^{(1)}. \quad (33)$$

Equivalently, we can obtain $y_n^{(1)}$ directly by applying the initial condition (12) to each step

$$y_n^{(1)} = \frac{1}{q_n^{(1)} \cdot \psi'(\theta_n)}, \quad (34)$$

where $q_n^{(1)}$ is the first column vector of Q_n .

Novičenko and Pyragas have recently proposed the computation of PRCs via a direct method.¹⁸ Our method may be similar to theirs because they used the Floquet multipliers whereas we use the Lyapunov exponents. For the quasi-periodic and discrete-time case, the Lyapunov exponents may

be a better choice because the Floquet multipliers are mainly applicable to periodic orbits in continuous-time systems. The QR-based method can also give us the decomposable form between unstable and stable components as discussed below.

In the case of the unstable QPO of Fig. 3, the exponents consist of $(\lambda_1, \lambda_2, \lambda_3) = (+, 0, -)$. Thus, the first component $y_n^{(1)}$ of Y_n diverges in the adjoint method. However, $y_n^{(1)}$ has no influence on the other components $y_n^{(2)}$ and $y_n^{(3)}$ because R_n is upper triangular. Therefore, $y_n^{(2)}$ has a nonzero value and $y_n^{(3)}$ converges to zero in the same way as the stable case. We here assume that the DFC stabilizes the unstable QPO. Then, the first component $y_n^{(1)}$ should be zero because λ_1 becomes negative. Consequently, we can construct the adjoint method by calculating only the second component $y_n^{(2)}$.

In general, if the number of positive exponents of the unstable QPO is m , the PRC can be obtained as

$$Z_n = y_n^{(m+1)} q_n^{(m+1)}, \quad (35)$$

where

$$y_n^{(m+1)} = \frac{1}{q_n^{(m+1)} \cdot \psi'(\theta_n)}, \quad (36)$$

and $q_n^{(m+1)}$ is the $(m+1)$ -th column vector of Q_n .

VI. EVALUATION OF DFC FOR TIME-DELAY MISMATCH

As mentioned in the Introduction, we cannot obtain the exact feedback delay for the QPO such that the feedback input vanishes. Accordingly, we assume that we obtain a hypothetical delay $\hat{d}$ in the neighborhood of d such that the feedback input vanishes completely, i.e., $x_{n-\hat{d}} = x_n$. Obviously, we cannot obtain the delay $\hat{d}$ in integers. However, we can rewrite the DFC (18) in the form with the feedback input $\hat{u}_n$ and the perturbation $u_n^{(\epsilon)}$

$$x_{n+1} = F(x_n) + K\hat{u}_n + Ku_n^{(\epsilon)}, \quad (37)$$

where

$$\hat{u}_n = x_{n-\hat{d}} - x_n, \quad u_n^{(\epsilon)} = x_{n-d} - x_{n-\hat{d}}. \quad (38)$$

The feedback input $\hat{u}_n$ should vanish and $u_n^{(\epsilon)}$ is regarded as the perturbation. Since $x_{n-\hat{d}} = x_n$, we obtain the small perturbation from $u_n^{(\epsilon)}$

$$\epsilon g_n = K(x_{n-d} - x_n). \quad (39)$$

Since the PRC profile of the DFC is equivalent to that of the control-free system and the phase equation is the same for the two cases, we can evaluate the small perturbation even if we do not know the hypothetical delay $\hat{d}$ exactly.

However, the amplitude of the PRC should be normalized again for the delay $\hat{d}$.¹⁴ We obtain the PRC of the DFC $\hat{Z}_n$ with the delay $\hat{d}$ from that of the unstable QPO Z_n

$$\hat{Z}_n = Z_n / C_{\hat{d}}, \quad (40)$$

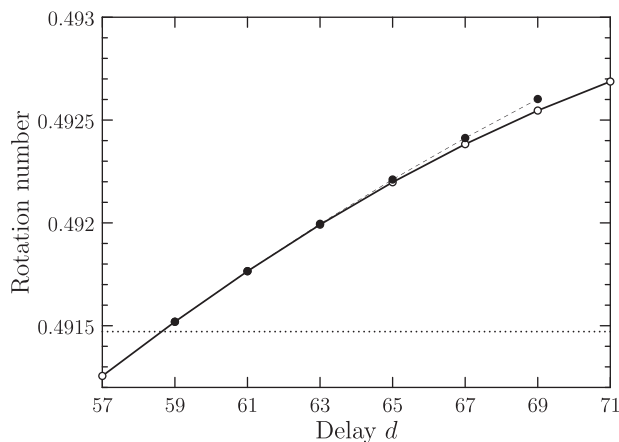


FIG. 4. Rotation number of DFC estimated from the unstable QPO for the delay d . The solid curve with open circles shows the estimated rotation number by the PRC of the unstable QPO. The dashed curve with filled circles shows the rotation number obtained from the direct calculations. The horizontal dotted line shows the rotation number of the unstable QPO. The feedback coefficient is $k = -0.4$. The other parameters are fixed at $(\alpha, \beta, \delta) = (0.79, 0.02, 0.06)$ and $N = 10^5$.

where $C_{\hat{d}}$ is the normalization factor dependent on $\hat{d}$. Since we cannot determine the hypothetical delay $\hat{d}$, we estimate $C_{\hat{d}}$ by changing the rotation number ω . We consider the modulated rotation number $\hat{\omega}$

$$\hat{\omega} = \frac{\text{nint}(d\omega)}{d}, \quad (41)$$

where nint is the nearest integer function. If we regard $\hat{\omega}$ as the rotation number of the system, the feedback input vanishes because $\theta_n = \theta_{n-d} + \text{nint}(d\omega)$ and the integer $\text{nint}(d\omega)$ is regarded as zero in $\mathbb{S}$. Note that $\hat{\omega} \approx \omega$ because $x_{n-d} \approx x_n$. Using $\hat{\omega}$ and the initial condition of the DFC (26) with the PRC $\tilde{\mathbf{Z}}$ represented in the phase domain, we estimate the normalization factor $C_{\hat{d}}$

$$C_{\hat{d}} \approx \tilde{\mathbf{Z}}(\theta_N) \cdot \psi'(\theta_N) + \sum_{i=1}^d (\mathbf{K}^T \tilde{\mathbf{Z}}(\theta_N + (d-i+1)\hat{\omega})) \times \psi'(\theta_N - i\hat{\omega}). \quad (42)$$

In Fig. 4, we compare the rotation number estimated from the unstable QPO with that obtained from the direct calculations of the DFC of the coupled map lattice (14). Since the difference between the current state and the 2-past state in the unstable QPO is small ($\approx 1.96 \times 10^{-2}$), we can obtain the small feedback input for odd delays around $d = 59$ shown in Fig. 3. However, the DFC did not stabilize the QPO for shorter delays than $d = 59$. In the neighborhood of $d = 59$, the rotation number is estimated correctly, implying that the rotation number of the orbit of the DFC can be estimated from the unstable QPO.

VII. CONCLUSIONS

We have proposed a method for constructing a PRC from unstable QPOs. Using the PRC, we have evaluated the DFC stabilizing the unstable QPO against the time-delay

mismatch. We have shown that the orbit of the DFC is consistent with the unstable QPO perturbed by the small state difference resulting from the time-delay mismatch. This fact implies that the DFC can certainly stabilize the unstable QPO despite the existence of the inevitable time-delay mismatch.

A QPO is almost periodic even if its dynamics are defined on a high-dimensional invariant torus. Furthermore, a QPO on an invariant closed curve in discrete-time systems is understood as the Poincaré section of a QPO on a 2-torus in continuous-time systems. These facts suggest that the DFC is applicable to the stabilization of unstable QPOs on the high-dimensional invariant torus in both discrete-time and continuous-time systems and hence further study is necessary.

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