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Arithmetical properties of real numbers with low density of nonzero digits

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1 Introduction

Let b be an integer greater than 1. Borel [2] proved that almost all positive real numbers are normal in base-b. However, it is generally difficult to show the normality of a given positive real number ξ. In particular, the base-b expansions of algebraic irrational numbers are mysterious. In this paper, we study the normality of algebraic irrational numbers. Borel [3] conjectured that all algebraic irrational numbers are normal in each integral base-b. The conjecture is still an open problem. There is no known example of base-b and positive irrational ξ such that the normality of ξ in base-b was proven. There is also no known counterexample of Borel's conjecture. In particular, it is still not known whether the digit 1 appears infinitely many times in the decimal expansion of $\sqrt{2}$.

If Borel's conjecture is true, then all algebraic irrational numbers are simply normal in any integral base-b, namely, any letter from the alphabet $\{0, 1, \ldots, b-1\}$ appears with average frequency tending to $1/b$. Hence, it is widely believed that if a positive irrational number ξ has a low density of nonzero digits in base-b expansion, then ξ is a transcendental number.

In Section 2 we investigate the digits of the base-b expansions of algebraic irrational numbers. In particular, giving lower bounds for the numbers of nonzero digits of algebraic irrational numbers, we introduce criteria for transcendence of real numbers. In Section 3 we review β-expansions of real numbers, which gives generalizations of base-b expansions of real numbers. In Section 4 we give main results on the digits of β-expansions of algebraic numbers. Note that if β is a general real number, then the β-expansions of rational numbers are also mysterious. Using the main results, we obtain criteria for transcendence whose β-expansion has a low density of nonzero digits. In the last of Section 4 we also introduce algebraic independence of real numbers with low density of nonzero digits. In Section 5 we give a sketch of the proof of the main results. Let x be a real number. We write the integral and fractional parts of x by $\lfloor x \rfloor$ and $\{x\}$, respectively. Moreover, We use the Landau symbols o, O and the Vinogradov symbols $\gg, \ll$ with their regular meanings.

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2 Base-b expansions of algebraic irrational numbers

Let b be an integer greater than 1 and ξ a positive real number. We write the base-b expansion of ξ by

$$\xi = \lfloor \xi \rfloor + \sum_{n=1}^{\infty} t_n(b;\xi) b^{-n},$$

where $t_n(b;\xi) \in \{0, 1 \ldots, b-1\}$ for any $n \in \mathbb{Z}^+$ and $t_n(b;\xi) \leq b-2$ for infinitely many n's. For simplicity, put $t_0(b;\xi) := \lfloor \xi \rfloor$. In this section, we study the number of nonzero digits $\nu_b(\xi;R)$ and the number of digit changes $\gamma_b(\xi;R)$ defined by

$$\begin{aligned} \nu_b(\xi;R) &:= \mathrm{Card}\{n \in \mathbb{Z}^+ \mid n \leq R, t_n(b;\xi) \neq 0\}, \\ \gamma_b(\xi;R) &:= \mathrm{Card}\{n \in \mathbb{Z}^+ \mid n \leq R, t_n(b;\xi) \neq t_{n+1}(b;\xi)\}, \end{aligned}$$

respectively, where $R \geq 1$ is a real number and Card denotes the cardinality. The function $\gamma_b(\xi;R)$ was introduced by Bugeaud [5]. Observe that

$$\nu_b(\xi;R) \geq \frac{1}{2}\gamma_b(\xi;R) + O(1). \tag{2.1}$$

Various mathematicians have studied the digits of algebraic irrational numbers, using Diophantine approximation methods. We recall Ridout's theorem [14]. For any prime number l, we denote by $|\cdot|_l$ the l-adic absolute value, normalized such that $|l|_l = l^{-1}$. Let S_1 and S_2 be disjoint finite sets of prime numbers and ξ a real algebraic number. Then, for any positive real number ε, there are only finitely many rational numbers p/q with $q \geq 1$ such that

$$\left|\xi - \frac{p}{q}\right| \cdot \prod_{l \in S_1} |p|_l \cdot \prod_{l \in S_2} |q|_l < \frac{1}{q^{2+\varepsilon}}. \tag{2.2}$$

Let ξ be an algebraic irrational number. Bugeaud [5] pointed out that the Ridout's theorem implies

$$\lim_{R \to \infty} \frac{\nu_b(\xi;R)}{\log R} = \infty, \quad \lim_{R \to \infty} \frac{\gamma_b(\xi;R)}{\log R} = \infty. \tag{2.3}$$

Here, we only check the second inequality of (2.3) in order to introduce a typical example of Diophantine approximation methods. Set

$$\begin{aligned} &S_1 := \emptyset, S_2 := \{l \mid l \text{ is a prime dividing } b\}, \\ &\{n \in \mathbb{N} \mid t_n(b;\xi) \neq 0\} =: \{w_0 < w_1 < \cdots\}. \end{aligned}$$

Since ξ is irrational, w_m $(m = 0, 1, \ldots)$ is an infinite sequence. For simplicity, we put $t_{w_m}(b;\xi) =: \widetilde{t_m}$. Then we have $\widetilde{t_m} \in \{1, 2, \ldots, b-1\}$ for any $m \geq 1$ and

$$\xi = \sum_{m=0}^{\infty} \widetilde{t_m} b^{-w_m}.$$

We apply the Ridout's theorem with

$$p = b^{w_M} \sum_{m=0}^{M} \widetilde{t_m} b^{-w_m}, \ q = b^{w_M}$$

for sufficiently large M. Observe that

$$\prod_{l \in S_2} |q|_l = q^{-1} = b^{-w_M}.$$

Let ε be an arbitrary positive real number. Since (2.2) has only finitely many solutions of rational numbers, we get, for any sufficiently large M,

$$\begin{aligned} b^{-(1+\varepsilon)w_M} &= q^{-(1+\varepsilon)} \le \left| \xi - \frac{p}{q} \right| \\ &= \sum_{m=M+1}^{\infty} \widetilde{t_m} b^{-w_m} \le \sum_{h=0}^{\infty} (b-1) b^{-w_{M+1}-h} = b^{1-w_{M+1}}. \end{aligned}$$

Thus,

$$w_{M+1} \le 1 + (1+\varepsilon) w_M \le (1+2\varepsilon) w_M$$

for any sufficiently large M. Since ε is arbitrary, we deduce that

$$\lim_{M \to \infty} \frac{w_{M+1}}{w_M} = 1,$$

which implies the second equality of (2.3).

Applying a quantitative Ridout's theorem [11], Bugeaud [5] improved (2.3) as follows:

$$\gamma_b(\xi; R) \ge 3 (\log R)^{1+1/(\omega(b)+4)} (\log \log R)^{-1/4}$$

for any sufficiently large R, where $\omega(b)$ is the number of the distinct prime factors of b. In particular, (2.1) implies that

$$\nu_b(\xi; R) \ge (\log R)^{1+1/(\omega(b)+4)} (\log \log R)^{-1/4}$$

for any sufficiently large R. Improving the quantitative parametric subspace theorem by Evertse and Schlickewei [8], Bugeaud and Evertse [7] proved the following: there exists an effectively computable positive constant $C_1(\xi)$, depending only on ξ, such that if ξ is an algebraic irrational number of degree D, then

$$\gamma_b(\xi; R) \ge C_1(\xi) (\log R)^{3/2} (\log \log R)^{-1/2} \tag{2.4}$$

for any sufficiently large R. Bailey, Borwein, Crandall, and Pomerance [1] gave a new method to estimate the lower bounds for $\nu_b(\xi; N)$. We mention the method in Section 5. Let again ξ be an algebraic irrational number of degree D. They showed that if $b = 2$, then there exists an effectively computable positive constant $C_2(\xi)$, depending only on ξ, satisfying

$$\nu_2(\xi; R) \ge C_2(\xi) R^{1/D} \tag{2.5}$$

for any sufficiently large R. Modifying their method, we can generalize (2.5) for general integral base b as follows: there exists an effectively computable positive constant $C_2'(b;\xi)$, depending only on b and ξ, such that

$$\nu_b(\xi;R) \geq C_2'(b;\xi)R^{1/D} \tag{2.6}$$

for any sufficiently large R. For instance, see Theorem 8.5 in [4]. Inspired by the method in [1], the author [9, 10] improved (2.4) for certain classes of algebraic irrational ξ of degree D. Namely, if ξ satisfies certain assumptions on its minimal polynomial, then there exists an effectively computable positive constant $C_3(b;\xi)$ such that

$$\gamma_b(\xi;R) \geq C_3(b;\xi)R^{1/D}$$

for any sufficiently large R.

In the rest of this section we apply (2.6) to transcendence of real numbers with low density of nonzero digits. Let $\boldsymbol{w} = (w_m)_{m=0}^{\infty}$ be a sequence of nonnegative integers such that $w_{m+1} > w_m$ for any sufficiently large m. Put

$$f(\boldsymbol{w};z) := \sum_{m=0}^{\infty} z^{w_m}. \tag{2.7}$$

If Borel's conjecture for the normality of algebraic irrational numbers is true, then we obtain the following criteria: If $\boldsymbol{w}$ satisfies

$$\lim_{m\to\infty} \frac{w_m}{m} = \infty,$$

then $f(\boldsymbol{w};b^{-1}) = \sum_{m=0}^{\infty} b^{-w_m}$ is transcendental. However, it is unknown whether the criteria above hold. On the other hand, using (2.6), we deduce partial results. Namely, assume that $\boldsymbol{w}$ satisfies

$$\lim_{m\to\infty} \frac{w_m}{m^A} = \infty \tag{2.8}$$

for any positive real number A. Then $f(\boldsymbol{w};b^{-1})$ is transcendental. For example, put

$$\varphi_y(m) := \left\lfloor m^{(\log m)^y} \right\rfloor = \left\lfloor \exp\left((\log m)^{1+y}\right) \right\rfloor, \tag{2.9}$$

$$\mu_y(z) := \sum_{m=1}^{\infty} z^{\varphi_y(m)} \tag{2.10}$$

for a positive real number y. Then, since $\varphi_y(m)$ $(m = 1, 2, \ldots)$ satisfies (2.8), we deduce that $\mu_y(b^{-1})$ is transcendental for any integer $b \geq 2$.

3 β-expansions of real numbers

Let β be a real number greater than 1. The notion of the β-expansions of real numbers was introduced by Rényi [13]. Let $T_\beta : [0,1) \to [0,1)$ be the β-transformation defined by $T_\beta(x) := \{\beta x\}$ for $x \in [0,1)$. Let $\eta \in [0,1)$. Then the β-expansion of η is denoted by

$$\eta = \sum_{n=1}^{\infty} t_n(\beta;\xi)\beta^{-n}, \tag{3.1}$$

where $t_n(\beta;\eta) = \lfloor \beta T_\beta^{n-1}(\eta) \rfloor \in \mathbb{Z} \cap [0,\beta)$ for any $n \in \mathbb{Z}^+$. If $\beta = b$ is an integer greater than 1, then (3.1) gives the base-b expansion of η.

Recall that the base-b expansion of each rational number is ultimately periodic. Schmidt [15] studied the periodicity of the β-expansions of rational numbers. We call $\beta > 1$ Pisot number if β is an algebraic integer such that the conjugates except itself have moduli less than 1. Moreover, we say that $\beta > 1$ is a Salem number if β is an algebraic integer such that the conjugates except itself have moduli at most 1 and that at least one conjugate has modulus 1. Schmidt [15] showed that if the β-expansion of each rational number is ultimately periodic, then β is a Pisot or Salem number. Moreover, he showed that if β is a Pisot number, then the β-expansion of any rational number is ultimately periodic. However, if β is a Salem number, then the β-expansions of rational numbers are mysterious. Schmidt conjectured that if β is a Salem number, then the β-expansion of every rational number is ultimately periodic, which is still an open problem. In the next section, we investigate the β-expansions of rational numbers and more general algebraic numbers in connection with Schmidt's conjecture and Borel's conjecture.

In the rest of this section, we recall the expansion of 1. The expansion of 1 is defined by

$$1 = \sum_{n=1}^{\infty} t_n(\beta; 1-)\beta^{-n},$$

where

$$t_n(\beta; 1-) := \lim_{x \to 1-} t_n(\beta; x) \in \mathbb{Z} \cap [0,\beta).$$

The expansion of 1 has a crucial role for studying the β-expansions of real numbers. Moreover, the periodicity of the expansion of 1 is important for investigating the β-shifts. Parry [12] showed that if β is a Pisot number, then the expansion of 1 is ultimately periodic. We call that β a Parry number if the expansion of 1 is ultimately periodic. However, it is unknown whether there exists a non-Parry Salem number. We investigate the periodicity of the expansion of 1 in Section 4.

4 Main results

We now introduce main results which gives arithmetical properties of the values of power series at certain algebraic points. Let $\boldsymbol{s} = (s_n)_{n=0}^{\infty}$ be a bounded sequence of nonnegative integers. Put

$$g(\boldsymbol{s}; z) := \sum_{n=0}^{\infty} s_n z^n$$

and, for a nonnegative real number R,

$$\lambda(\boldsymbol{s}; R) := \mathrm{Card}\{n \in \mathbb{N} \mid n \le R, s_n \neq 0\}.$$

Moreover, let $\mathbf{K}$ and $\mathbf{L}$ be algebraic number fields with $\mathbf{K} \subset \mathbf{L}$. Then we denote by $[\mathbf{L} : \mathbf{K}]$ the degree of field extension.

THEOREM 4.1. *Let β be a Pisot or Salem number and ξ an algebraic number with $[\mathbb{Q}(\beta,\xi) : \mathbb{Q}(\beta)] = D$. Let $\boldsymbol{s} = (s_n)_{n=0}^{\infty}$ be a sequence of integers with $0 \le s_n \le B$ for any $n \in \mathbb{N}$, where B is a positive integer independent of n. Suppose that $s_n \ne 0$ for infinitely many n's. Then there exist effectively computable positive constants $C_4(\beta,\xi,B)$ and $C_5(\beta,\xi,B)$, depending only on β,ξ, and B, such that*

$$\lambda(\boldsymbol{s};R) \ge C_4(\beta,\xi,B)R^{1/(-1+2D)}(\log R)^{-1/(-1+2D)}$$

for any real number R with $R \ge C_5(\beta,\xi,B)$.

We apply Theorem 4.1 to the transcendence of $f(\boldsymbol{w};\beta^{-1})$, where $f(\boldsymbol{w};z)$ is defined by (2.7) and β is a Pisot or Salem number.

COROLLARY 4.2. *Let $\boldsymbol{w} = (w_m)_{m=0}^{\infty}$ be a sequence of nonnegative integers such that $v_{m+1} > v_m$ for any sufficiently large m. Suppose for any positive real number A that*

$$\lim_{m\to\infty} \frac{w_m}{m^A} = \infty.$$

Then $f(\boldsymbol{w};\beta^{-1})$ is transcendental for any Pisot or Salem number β.

Let y be a positive real number. Recall that $\varphi_y(m)$ and $\mu_y(z)$ are defined by (2.9) and (2.10), respectively. Corollary 4.2 implies that $\mu_y(\beta^{-1})$ is transcendental for any Pisot or Salem number β.

We apply Theorem 4.1 to the β-expansions of algebraic numbers $\eta \in [0,1)$. In the rest of this section, the implied constants in the symbol $\gg$ are effectively computable ones depending only on β and η. We generalize the notation in Section 2. Namely, put

$$\begin{aligned} \nu_\beta(\eta;R) &:= \mathrm{Card}\{n \in \mathbb{Z}^+ \mid n \le R,\ t_n(\beta;\eta) \ne 0\} \\ \gamma_\beta(\eta;R) &:= \mathrm{Card}\{n \in \mathbb{Z}^+ \mid n \le R,\ t_n(\beta;\eta) \ne t_{n+1}(\beta;\eta)\}. \end{aligned}$$

Then we have

$$\nu_\beta(\eta;R) \ge \frac{1}{2}\gamma_\beta(\eta;R) + O(1). \tag{4.1}$$

Let β be a Pisot or Salem number and $\eta \in [0,1)$ an algebraic number with $[\mathbb{Q}(\beta,\eta) : \mathbb{Q}(\beta)] = D$. Bugeaud [6] showed that if $t_n(\beta;\eta) \ne t_{n+1}(\beta;\eta)$ for infinitely many n's, then

$$\gamma_\beta(\eta;R) \gg (\log R)^{3/2}(\log\log R)^{-1/2}$$

for any sufficiently large R. Moreover, (4.1) implies that

$$\nu_\beta(\eta;R) \gg (\log R)^{3/2}(\log\log R)^{-1/2} \tag{4.2}$$

for any sufficiently large R. On the other hand, using Theorem 4.1, we obtain

$$\nu_\beta(\eta;R) \gg R^{1/(-1+2D)}(\log R)^{-1/(-1+2D)} \tag{4.3}$$

for every sufficiently large R, which gives improvement of (4.2).

We now consider the Schmidt conjecture on the periodicity of rational numbers. Suppose that β is a Salem number and that $\eta \in [0,1)$ is a rational number. If $t_n(\beta;\eta) \neq 0$ for infinitely many n's and if the sequence $t_n(\beta;\eta)$ $(n = 1,2,\ldots)$ is ultimately periodic, then we have $\nu_\beta(\eta;R) \gg R$. Now, (4.3) means that

$$\nu_\beta(\eta;R) \gg R(\log R)^{-1}$$

for each sufficiently large R, which gives partial results on the Schmidt's conjecture.

Theorem 4.1 is applicable to the study of the expansion of 1. It is well-known for any real number $\beta > 1$ that $t_n(\beta;1-) \neq 0$ for infinitely many n's. Let again β be a Salem number. Put

$$\nu_\beta(1-;R) := \mathrm{Card}\{n \in \mathbb{Z}^+ \mid t_n(\beta;1-) \neq 0\}.$$

If the expansion of 1 is ultimately periodic, then we have $\nu_\beta(1-;R) \gg R$. Bugeaud [6] showed that

$$\nu_\beta(1-;R) \gg (\log R)^{3/2}(\log\log R)^{-1/2}$$

for any sufficiently large R. On the other hand, using Theorem 4.1, we deduce that

$$\nu_\beta(1-;R) \gg R(\log R)^{-1}$$

for each sufficiently large R.

5 Sketch of the proof of main results

In this section we introduce ideas for the proof of the main results. For simplicity, we give a sketch of the proof of Corollary 4.2. We define $\boldsymbol{s} = (s_n)_{n=0}^{\infty}$ and ξ by

$$f(\boldsymbol{w};z) = \sum_{m=0}^{\infty} z^{w_m} =: \sum_{n=0}^{\infty} s_n z^n, \xi := f(\boldsymbol{w};\beta^{-1}),$$

respectively. Then $\boldsymbol{s}$ is bounded and $s_n \in \mathbb{N}$ for any $n \in \mathbb{N}$. We use the same notation as in Section 4. Then we have, for any positive real number ε,

$$\lambda(\boldsymbol{s};R) = o(R^\varepsilon)$$

as R tends to infinity. We show that $P(\xi) \neq 0$, where $P(X) = A_D X^D + A_{D-1}X^{D-1} + \cdots + A_0 \in \mathbb{Z}[X]$ is any non-constant polynomial with $A_D \neq 0$. In what follows, $C_6, C_7 \ldots$ and the the implied constants in the symbols $\ll, \gg$ are positive constants depending only on $\boldsymbol{w}$ and $P(X)$. For instance, $0 \leq s_n \leq C_6$ for any $n \in \mathbb{N}$. Put $\Gamma := \{n \in \mathbb{N} \mid s_n \neq 0\}$. Without loss of generality, we may assume that $0 \in \Gamma$. We now calculate ξ^k for any k with $1 \leq k \leq D$. Then we

get

$$\begin{aligned}\xi^k &= \left(\sum_{m\in\Gamma} s_m\beta^{-m}\right)^k \\ &= \sum_{m_1,\dots,m_k\in\Gamma} s_{m_1}\cdots s_{m_k}\beta^{-m_1-\cdots-m_k} \\ &= \sum_{m=0}^{\infty}\beta^{-m}\sum_{\substack{m_1,\dots,m_k\in\Gamma\\ m_1+\cdots+m_k=m}} s_{m_1}\cdots s_{m_k} \\ &=: \sum_{m=0}^{\infty}\beta^{-m}\rho(k;m),\end{aligned}$$

where

$$\rho(k;m) = \sum_{\substack{m_1,\dots,m_k\in\Gamma\\ m_1+\cdots+m_k=m}} s_{m_1}\cdots s_{m_k}.$$

Let $m\in\mathbb{N}$. Then $\rho(k;m)$ is a nonnegative integer because s_n is a nonnegative integer for any $n\in\mathbb{N}$. Set

$$k\Gamma := \{m_1+\cdots+m_k \mid m_1,\dots,m_k\in\Gamma\}.$$

Since $0\in\Gamma$, we have

$$\Gamma\subset 2\Gamma\subset\cdots\subset(D-1)\Gamma\subset D\Gamma.$$

Observe that $\rho(k;m)$ is positive if and only if $m\in kS$. Now we introduce BBP tails. Let $R\in\mathbb{N}$. Using

$$\begin{aligned}P(\xi) &= A_0+\sum_{k=1}^{D}A_k\xi^k \\ &= A_0+\sum_{k=1}^{D}A_k\sum_{m=0}^{\infty}\beta^{-m}\rho(k;m),\end{aligned}$$

we obtain

$$\begin{aligned}\beta^R P(\xi) &= A_0\beta^R+\sum_{k=1}^{D}A_k\sum_{m=0}^{\infty}\beta^{-(m-R)}\rho(k;m) \\ &= A_0\beta^R+\sum_{k=1}^{D}A_k\sum_{m=-R}^{\infty}\beta^{-m}\rho(k;m+R).\end{aligned}$$

Put

$$\begin{aligned}Y_R &:= \sum_{k=1}^{D}A_k\sum_{m=1}^{\infty}\beta^{-m}\rho(k;m+R), \\ Z_R &:= A_0\beta^R+\sum_{k=1}^{D}A_k\sum_{m=-R}^{0}\beta^{-m}\rho(k;m+R).\end{aligned}$$

Then we have

$$\beta^R P(\xi) = Y_R + Z_R. \tag{5.1}$$

Note that Z_R is an algebraic integer. In the paper [1], Y_R is called BBP tails in the case of $\beta = 2$. For the proof of $P(\xi) \neq 0$, we consider BBP tails in the case where β is a Pisot or Salem number. In what follows, we estimate Y_R and Z_R, respectively. If $\beta = 2$, then one of the key ideas for the proof of (2.5) is the following:

$$\text{If } Z_R \neq 0, \text{ then } |Z_R| \geq 1. \tag{5.2}$$

In the case where β is a general Pisot or Salem number, we get the following:

$$\text{If } Z_R \neq 0, \text{ then } |Z_R| \geq C_7 R^{-C_8}. \tag{5.3}$$

Put

$$y_N := \mathrm{Card}\{R \in \mathbb{N} \mid R \leq N,\ Y_R \neq 0\}$$

for positive integer N. Bailey, Borwein, Crandall, and Pomerance [1] estimated y_N, using the relation (5.2). However, (5.3) does not seem enough for the estimation of y_N. Hence, we calculate Y_R. Division of the interval of $[0, N)$ into subintervals is also one of the key ideas for the proof of (2.5).

In the rest of this section, we give an outline. We write the length of an interval $[x, y) \in \mathbb{R}$ by $|[x, y)| = y - x$. First, using a combinatorial method, we construct an subinterval $J \subset [0, N)$ satisfying

$$|J| \gg \frac{N}{\lambda(\boldsymbol{s}; N)^{D-1}}$$

and $Y_R > 0$ for any $R \in J$. Next, we divide J into subintervals. Consequently, we construct $I \subset J$ fulfilling

$$|I| \gg \frac{N}{\lambda(\boldsymbol{s}; N)^{-1+2D}}$$

and

$$0 < Y_R < \frac{1}{2} C_7 R^{-C_8} \tag{5.4}$$

for any R with $R \in I$. Hence, if $R \in I$, then combining (5.1), (5.3), and (5.4), we deduce that $P(\xi) \neq 0$.

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