数理解析研究所講究録
第 1670 巻 2009 年 77-83



# Spaces of maps from the closed Riemann surface into the 2-sphere

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## 1 Introduction.

For connected spaces $X$ and $Y$, let $\mathrm{Map}(X,Y)$ (resp. $\mathrm{Map}^*(X,Y)$) denote the space consisting of all continuous (resp. based continuous) maps $f:X\to Y$ with compact-open topology. Let $T_g$ denote the closed Riemann surface of genus $g$. Then for each integer $d\in\mathbb{Z}=\pi_0(\mathrm{Map}(T_g,S^2))$ we denote by $\mathrm{Map}_d(T_g,S^2)$ (resp. by $\mathrm{Map}_d^*(T_g,S^2)$) the corresponding path-component of $\mathrm{Map}(T_g,S^2)$ (resp. $\mathrm{Map}^*(T_g,S^2)$) consisting of all maps (resp. of base-point preserving maps) $f:T_g\to S^2$ of degree $d$. Similarly, we denote by $\mathrm{Hol}_d(T_g,S^2)$ the subspace of $\mathrm{Map}_d(T_g,S^2)$ of all holomorphic maps $f:T_g\to S^2$ of degree $d$, and by $\mathrm{Hol}_d^*(T_g,S^2)$ the corresponding subspace of $\mathrm{Map}_d^*(T_g,S^2)$ of all base-point preserving holomorphic maps of degree $d$. Note that $\mathrm{Hol}_d(T_g,S^2)=\emptyset$ if $d<0$ and that any holomorphic map $f:T_g\to S^2$ of degree zero is a constant map. So in this paper we always assume that $d\geq 1$ and recall the following results.

**Theorem 1.1** (L. Larmore and E. Thomas, [7]). (i) *If $g=0$, $T_0=S^2$ and there are isomorphisms*

$$\pi_1(\mathrm{Map}_d(S^2,S^2))\cong\mathbb{Z}/2d,\quad \pi_1(\mathrm{Map}_d^*(S^2,S^2))\cong\mathbb{Z}.$$

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(ii) *If $g \geq 1$, there are isomorphisms*

$$\begin{cases} \pi_1(\mathrm{Map}_d(T_g, S^2)) \cong \langle \alpha, e_j \ (1 \leq j \leq 2g) | [e_k, e_{k+g}] = \alpha^2, \alpha^{2d} = 1 \rangle, \\ \pi_1(\mathrm{Map}_d^*(T_g, S^2)) \cong \langle \alpha, e_j \ (1 \leq j \leq 2g) | [e_k, e_{k+g}] = \alpha^2 \rangle, \end{cases}$$

*where $k = 1, 2, \cdots, g$, and $[x, y] = xyx^{-1}y^{-1}$.* □

**Theorem 1.2** (G. Segal, [9]). *The inclusion maps*

$$\begin{cases} i_d : \mathrm{Hol}_d^*(T_g, S^2) \to \mathrm{Map}_d^*(T_g, S^2) \\ j_d : \mathrm{Hol}_d(T_g, S^2) \to \mathrm{Map}_d(T_g, S^2) \end{cases}$$

*are homotopy equivalences up to dimension $d$ if $g = 0$, and they are homology equivalences up to dimension $D(d; g) = d - 2g$ if $g \geq 1$.* □

**Theorem 1.3** (S. Kallel, [6]). *If $d > 2g$ and $g \geq 1$, the inclusion maps $i_d$ and $j_d$ induce isomorphisms*

$$\begin{cases} i_{d*} : \pi_1(\mathrm{Hol}_d^*(T_g, S^2)) \xrightarrow{\cong} \pi_1(\mathrm{Map}_d^*(T_g, S^2)), \\ j_{d*} : \pi_1(\mathrm{Hol}_d(T_g, S^2)) \xrightarrow{\cong} \pi_1(\mathrm{Map}_d(T_g, S^2)). \end{cases} \quad \square$$

*Remark.* A map $f : X \to Y$ is called *a homotopy (resp. homology) equivalence up to dimension $D$* if $f_* : \pi_k(X) \to \pi_k(Y)$ (resp. $f_* : H_k(X, \mathbb{Z}) \to H_k(Y, \mathbb{Z})$) is an isomorphism for any $k < D$ and epimorphism for $k = D$.

We expect that the inclusions $i_d$ and $j_d$ will be homotopy equivalences up to dimension $D(d; g)$ for $g \geq 1$. For example, Theorem 1.3 supports that this might be true, and it seems valuable to investigate the homotopy types of the universal coverings of $\mathrm{Hol}_d^*(T_g, S^2)$ and $\mathrm{Hol}_d(T_g, S^2)$.

The main purpose of this note is to announce the recent work of the author given in [13], in which we shall study the homotopy types of universal coverings of the above spaces. Let $\widetilde{X}$ denote the universal covering of a connected space $X$. Then we can state our results as follows.

**Theorem 1.4** ([13]). *If $d \geq 1$, there is a homotopy equivalence*

$$\tilde{\Phi}_d : S^3 \times \widetilde{\mathrm{Hol}_d^*}(T_g, S^2) \xrightarrow{\simeq} \widetilde{\mathrm{Hol}_d}(T_g, S^2).$$

**Corollary 1.5** (([1], [8], [12])). *If $g = 0$ and $d \geq 1$, there is a homotopy equivalence $\tilde{\Phi}_d : S^3 \times \widetilde{\mathrm{Hol}_d^*}(S^2, S^2) \xrightarrow{\simeq} \widetilde{\mathrm{Hol}_d}(S^2, S^2)$.* □

Let $i : \mathrm{Hol}_d^*(T_g, S^2) \to \mathrm{Hol}_d(T_g, S^2)$ be an inclusion map and let $ev : \mathrm{Hol}_d(T_g, S^2) \to S^2$ denote the evaluation map given by $ev(f) = f(t_0)$, where $t_0 \in T_g$ is the base-point of $T_g$. Then it is known that there is a evaluation fibration sequence (e.g. [6])

$$\mathrm{Hol}_d^*(T_g, S^2) \underset{\subset}{\xrightarrow{i}} \mathrm{Hol}_d(T_g, S^2) \xrightarrow{ev} S^2.$$

**Corollary 1.6** ([13]). *If $k \geq 2$ and $d \geq 1$, the above sequence induces a split short exact sequence*

$$0 \to \pi_k(\mathrm{Hol}_d^*(T_g, S^2)) \xrightarrow{i_*} \pi_k(\mathrm{Hol}_d(T_g, S^2)) \xrightarrow{ev_*} \pi_k(S^2) \to 0.$$

**Theorem 1.7** ([13]). *Let $d \geq 1$ be an integer.*

(i) *There is a homotopy equivalence*

$$\tilde{\Phi} : S^3 \times \widetilde{\mathrm{Map}}^*(T_g, S^3) \xrightarrow{\simeq} \widetilde{\mathrm{Map}}_d(T_g, S^2)$$

*and there is a fibration sequence (up to homotopy equivalence)*

$$\Omega^2 S^3\langle 3\rangle \to \widetilde{\mathrm{Map}}^*(T_g, S^3) \to (\Omega S^3)^{2g},$$

*where $S^3\langle 3\rangle$ denotes the 3-connective covering of $S^3$.*

(ii) *For any $k \geq 2$, there is an isomorphism*

$$\pi_k(\widetilde{\mathrm{Map}}^*(T_g, S^3)) \cong \pi_{k+2}(S^3) \oplus \pi_{k+1}(S^3)^{\oplus 2g}.$$

## 2 The idea of the proofs.

Let $X$, $Y$ and $Z$ be connected spaces and let $f : X \to Y$ be a map. Define the map $f^\# : \mathrm{Map}(Y, Z) \to \mathrm{Map}(X, Z)$ by $f^\#(g) = g \circ f$ for $g \in \mathrm{Map}(Y, Z)$. Similarly, we define the map $f^\# : \mathrm{Map}^*(Y, Z) \to \mathrm{Map}^*(X, Z)$ by the restriction. It is well known that there is a cofiber sequence

$$S^1 \xrightarrow{\varphi_g} \vee^{2g} S^1 \xrightarrow{i'} T_g \xrightarrow{q_g} S^2 \xrightarrow{\Sigma\varphi_g} \vee^{2g} S^2 \tag{1}$$

where $\pi_1(\vee^{2g} S^1)$ is the free group on $2g$ generators $\{a_j, b_j : 1 \leq j \leq g\}$ and $\varphi_g = [a_1, b_1][a_2, b_2] \cdots [a_g, b_g] \in \pi_1(\vee^{2g} S^1)$.

**Lemma 2.1.** $q^{\#}_{g\,*} : \pi_k(\Omega^2_d S^2) \to \pi_k(\mathrm{Map}^*_d(T_g, S^2))$ *is a monomorphism for any* $k \geq 1$.

*Proof.* By using an easy diagram chasing we can show the assertion. $\square$

Consider the commutative diagram of evaluation fibration sequences

$$\begin{array}{ccccc} \mathrm{Map}^*_d(S^2,S^2) & \xrightarrow[\subset]{i_T} & \mathrm{Map}_d(S^2,S^2) & \xrightarrow{ev_S} & S^2 \\ {\scriptstyle q^{\#}_g}\downarrow & & {\scriptstyle q^{\#}_g}\downarrow & & \| \\ \mathrm{Map}^*_d(T_g,S^2) & \xrightarrow[\subset]{i_S} & \mathrm{Map}_d(T_g,S^2) & \xrightarrow{ev_T} & S^2 \end{array} \tag{2}$$

**Lemma 2.2.** $ev_{X*} : \pi_2(\mathrm{Map}_d(X, S^2)) \to \pi_2(S^2)$ *is trivial for* $X \in \{S^2, T_g\}$.

*Proof.* If $X = S^2$, we can easily show the assertion by using Whitehead's Theorem [11] concerning the boundary operator of the homotopy exact sequence induced from the evaluation fibartion. By using this with the diagram chasing of (2), we can also prove the assertion for $X = T_g$. $\square$

**Lemma 2.3.** *The induced homomorphisms*

$$\begin{cases} i_{T*} : \pi_2(\mathrm{Map}^*_d(T_g, S^2)) \to \pi_2(\mathrm{Map}_d(T_g, S^2)) \\ i_* : \pi_2(\mathrm{Hol}^*_d(T_g, S^2)) \to \pi_2(\mathrm{Hol}_d(T_g, S^2)) \end{cases}$$

*are epimorphisms.*

*Proof.* The proof follows from the diagram chasing and Lemma 2.2. $\square$

If we identify $S^2 = \mathbb{C}\mathrm{P}^1$, the group $SU(2)$ acts on $S^2$ by the right matrix multiplication. By using this right matrix multiplication, define the map

$$\Phi : SU(2) \times \mathrm{Map}^*_d(T_g, S^2) \to \mathrm{Map}_d(T_g, S^2) \tag{3}$$

by $(\Phi(A, f))(t) = f(t) \cdot A$ for $(A, f, t) \in SU(2) \times \mathrm{Map}^*_d(T_g, S^2) \times T_g$.

Since $\Phi(SU(2) \times \mathrm{Hol}^*_d(T_g, S^2)) \subset \mathrm{Hol}_d(T_g, S^2)$, we can define the map

$$\Phi_d : SU(2) \times \mathrm{Hol}^*_d(T_g, S^2) \to \mathrm{Hol}_d(T_g, S^2) \tag{4}$$

by the restriction $\Phi_d = \Phi | SU(2) \times \mathrm{Hol}^*_d(T_g, S^2)$.

**Theorem 2.4** ([13]). $\Phi_{d*} : \pi_k(SU(2)\times \mathrm{Hol}^*_d(T_g, S^2)) \xrightarrow{\cong} \pi_k(\mathrm{Hol}_d(T_g, S^2))$ *is an isomorphism for any* $k \geq 2$. □

*Proof.* The detail is omitted and see [13] in detail. □

Now we can prove Theorem 1.4 by using Theorem 2.4.

*Proof of Theorem 1.4.* Let

$$\begin{cases} \pi : \widetilde{\mathrm{Hol}}_d(T_g, S^2) \to \mathrm{Hol}_d(T_g, S^2) \\ \pi' : \widetilde{\mathrm{Hol}}^*_d(T_g, S^2) \to \mathrm{Hol}^*_d(T_g, S^2) \end{cases}$$

denote projection maps of the universal coverings. If we identify $SU(2) = S^3$, it is easy see that the universal covering of $SU(2) \times \mathrm{Hol}^*_d(T_g, S^2)$ is given by $1 \times \pi' : S^3 \times \widetilde{\mathrm{Hol}}^*_d(T_g, S^2) \to SU(2) \times \mathrm{Hol}^*_d(T_g, S^2)$. Then because $\pi$ is a projection of the universal covering, there is a lifting $\tilde{\Phi}_d : S^3 \times \widetilde{\mathrm{Hol}}^*_d(T_g, S^2) \to \widetilde{\mathrm{Hol}}_d(T_g, S^2)$ such that the following diagram is commutative.

$$\begin{array}{ccc} S^3 \times \widetilde{\mathrm{Hol}}^*_d(T_g, S^2) & \xrightarrow{\tilde{\Phi}_d} & \widetilde{\mathrm{Hol}}_d(T_g, S^2) \\ {\scriptstyle 1\times\pi'}\downarrow & & \downarrow{\scriptstyle \pi} \\ SU(2) \times \mathrm{Hol}^*_d(T_g, S^2) & \xrightarrow{\Phi_d} & \mathrm{Hol}_d(T_g, S^2) \end{array}$$

Since $\pi_k(\Phi_d)$ is an isomorphism for any $k \geq 2$, an easy diagram chasing shows that $\pi_k(\tilde{\Phi}_d)$ is also an isomorphism for any $k \geq 2$. Because $S^3 \times \widetilde{\mathrm{Hol}}^*_d(T_g, S^2)$ and $\widetilde{\mathrm{Hol}}_d(T_g, S^2)$ are simply connected, $\tilde{\Phi}_d$ is a homotopy equivalence. □

By using the completely similar way as above, we can see that there is a fibration sequence (up to homotopy equivalence)

$$S^1 \longrightarrow SU(2) \times \mathrm{Map}^*_d(T_g, S^2) \xrightarrow{\Phi} \mathrm{Map}_d(T_g, S^2). \tag{5}$$

**Theorem 2.5** ([13]). $\Phi_* : \pi_k(SU(2)\times \mathrm{Map}^*_d(T_g, S^2)) \xrightarrow{\cong} \pi_k(\mathrm{Map}_d(T_g, S^2))$ *is an isomorphism for any* $k \geq 2$.

*Proof.* The proof is analogous to that of Theorem 2.4. □

**Corollary 2.6** ([13]). *There is a homotopy equivalence*

$$\widetilde{\mathrm{Map}}_d(T_g, S^2) \simeq S^3 \times \widetilde{\mathrm{Map}}_0^*(T_g, S^2). \quad \square$$

**Lemma 2.7.** *There is a homotopy fibration sequence*

$$\Omega^2 S^3\langle 3\rangle \to \widetilde{\mathrm{Map}}^*(T_g, S^3) \to (\Omega S^3)^{2g}.$$

*Proof.* This can be proved by using tedious diagram chasing and the detail is omitted. $\square$

*Proof of Theorem 1.7.* (i) By using Corollary 2.6 and Lemma 2.7, to prove (i) it is sufficient to show that there is a homotopy equivalence

$$\widetilde{\mathrm{Map}}^*(T_g, S^3) \simeq \widetilde{\mathrm{Map}}_0^*(T_g, S^2). \tag{6}$$

However, by using a similar manner as that of the previous Theorem we can prove the assertion (i) and the detail is omitted.

(ii) Assume that $k \geq 2$. Since there is a homotopy equivalence $\Sigma^k T_g \simeq \vee^{2g} S^{1+k} \vee S^{2+k}$ and $S^3$ is a Lie group, $\mathrm{Map}^*(T_g, S^3)$ is an H-space. Hence, the assertion (ii) easily follows. $\square$

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