

# Anti-Loewner matrices; Numerical radius and unitarity

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We review results on two topics by Hidaka and Sano; Sano and A. Uchiyama. For details, we refer [4, 5].

## 1 Anti-Loewner matrices

Let  $f$  be a positive  $C^1$  function on  $(0, \infty)$ . Let  $H^n$  be the subspace of  $\mathbb{C}^n$  consisting of all  $x = (x_1, \dots, x_n)^T \in \mathbb{C}^n$  for which  $\sum_{i=1}^n x_i = 0$ . An  $n \times n$  Hermitian matrix  $A$  is said to be *conditionally positive definite* (c.p.d. for short) if

$$\langle x, Ax \rangle \geq 0 \quad \text{for all } x \in H^n,$$

and *conditionally negative definite* (c.n.d. for short) if  $-A$  is c.p.d.

For positive numbers  $t_1, \dots, t_n$ , the matrices

$$K_f(t_1, \dots, t_n) = \left[ \frac{f(t_i) + f(t_j)}{t_i + t_j} \right]$$

have been of some interest. We call it an *anti-Loewner matrix*. Kwong showed that if  $f$  is a non-negative operator monotone function on  $(0, \infty)$  then all  $K_f$  are p.s.d. On the other hand, it is shown in [3] that if  $f$  is operator convex on  $[0, \infty)$  with  $f(0) \leq 0$ , or  $f(t) = tg(t)$  for an operator convex function  $g$  with  $f''(0) \geq 0$  then all  $K_f$  are c.n.d.

Recently, Audenaert in [2] gives a characterisation of functions  $f$  for which all  $K_f$  are p.s.d; by [2, Theorem 2.1], for a positive  $C^1$  function  $f$  on  $(0, \infty)$ , all  $K_f$  are p.s.d. if and only if  $f(\sqrt{t})\sqrt{t}$  is matrix monotone of any order  $n$ , i.e., operator monotone. Hence, such a function  $f$  is of the form

$$f(t) = \frac{\alpha}{t} + \beta t + \int_0^\infty \frac{t}{\lambda + t^2} d\nu(\lambda), \quad (1.1)$$

where  $\alpha, \beta \geq 0$  and  $\nu$  is a positive measure on  $(0, \infty)$ .

Here are our complementary results in [4]:

**Theorem 1.1.** Let  $f$  be a positive, differentiable function on  $(0, \infty)$  with  $f(0) = f'(0) = 0$  and  $t_1, t_2, \dots, t_n > 0$  given. Suppose that  $K_f(t_1, \dots, t_n, t_{n+1})$  is c.n.d. for any  $t_{n+1} > 0$ . Then  $K_{f(t)/t^2}(t_1, \dots, t_n)$  is p.s.d. Conversely, if  $K_{f(t)/t^2}(t_1, \dots, t_n)$  is p.s.d., then  $K_f(t_1, \dots, t_n)$  is c.n.d.

By Audenaert's characterisation (1.1),

**Theorem 1.2.** Let  $f$  be a positive  $C^1$  function on  $(0, \infty)$  with  $f(0) = f'(0) = 0$ . Then all  $K_f$  are c.n.d. if and only if all  $K_{f(t)/t^2}$  are p.s.d. or  $f$  is of the form

$$f(t) = \beta t^3 + \int_0^\infty \frac{t^3}{\lambda + t^2} d\nu(\lambda), \quad (1.2)$$

where  $\beta \geq 0$  and  $\nu$  is a positive measure on  $(0, \infty)$ .

In the case where  $f$  is of the form (1.2), we can consider the inverse  $f^{-1}$  of  $f$ .

**Corollary 1.3.** Let  $f$  be a positive  $C^1$  function of the form (1.2). Then  $K_{f^{-1}}$  is infinitely divisible.

**Proposition 1.4.** (1) For a function  $f$  on  $(0, \infty)$ ,  $K_f(t_1, t_2)$  are c.n.d. for all  $t_1, t_2 > 0$  if and only if  $f(t)/t$  is increasing.

(2) For a non-negative function  $f$  on  $(0, \infty)$ ,  $K_f(t_1, t_2)$  are p.s.d. for all  $t_1, t_2 > 0$  if and only if  $f(t)/t$  is decreasing and  $tf(t)$  is increasing.

**Corollary 1.5.** For  $f(t) = t^p$  ( $p \in \mathbb{R}$ ) on  $(0, \infty)$ , the following hold:

- (1)  $K_f(t_1, t_2)$  are c.n.d. for all  $t_1, t_2 > 0$  if and only if  $1 \leq p$ .
- (2)  $K_f(t_1, t_2)$  are p.s.d. for all  $t_1, t_2 > 0$  if and only if  $-1 \leq p \leq 1$ .
- (3)  $K_f(t_1, t_2, t_3)$  are c.n.d. for all  $t_1, t_2, t_3 > 0$  if and only if  $1 \leq p \leq 3$ .

## 2 Numerical radius and unitarity

Let  $\mathcal{H}$  be a Hilbert space and  $B(\mathcal{H})$  denote the set of all bounded linear operators on  $\mathcal{H}$ . Here we study the following condition: for an invertible operator  $A \in B(\mathcal{H})$ ,

$$|\langle A\xi, \xi \rangle| \leq 1, \quad |\langle A^{-1}\xi, \xi \rangle| \leq 1$$

for all unit vectors  $\xi \in \mathcal{H}$ . In this case, we show that  $A$  is unitary. It is clear that  $A$  is unitary if  $A$  is invertible,  $\|A\| \leq 1$ , and  $\|A^{-1}\| \leq 1$ . Hence, our theorem means that the operator norm can be replaced by the numerical radius; for  $A \in B(\mathcal{H})$  the numerical range  $W(A)$  and the numerical radius  $w(A)$  are defined as

$$\begin{aligned} W(A) &= \{\langle A\xi, \xi \rangle : \|\xi\| = 1\}, \\ w(A) &= \sup\{|\langle A\xi, \xi \rangle| : \|\xi\| = 1\}. \end{aligned}$$

We remark that the main result already appeared as Corollary 1 to Theorem 1 in [7] and as Theorem B in [6] with a more general result, whose proof seems to be involved.

**Theorem 2.1.** Let  $A \in B(\mathcal{H})$  be invertible. If  $w(A) \leq 1$  and  $w(A^{-1}) \leq 1$ , then  $A$  is unitary.

**Proof.** Let  $A = U|A|$  be the polar decomposition. Since  $(A^{-1})^* = (|A|^{-1}U^{-1})^* = U|A|^{-1}$ ,  $w(U|A|^{-1}) = w(A^{-1}) \leq 1$ . Let  $B := U \frac{|A| + |A|^{-1}}{2}$ . Then  $w(B) \leq 1$ , and  $|B| = \frac{|A| + |A|^{-1}}{2} \geq I$ . Applying the following lemma, we have  $|B| = I$  or  $|A| = I$ ; therefore,  $A$  is unitary. ■

**Lemma 2.2.** Let  $B \in B(\mathcal{H})$  be invertible. If  $w(B) \leq 1$  and  $|B| \geq I$ , then  $B$  is unitary.

## 参考文献

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