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Heterotic string field theory and new relations extending L_∞ algebra

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Abstract. Based on the Wess-Zumino-Witten-like formulation, a gauge invariant action for heterotic string field theory is constructed at the sixth and eighth order of the Ramond field Ψ . A key relation is a kind of extension of the L_∞ algebra including another type of string products called the gauge products. Some general structure of a complete action is also discussed.

1. Introduction

A gauge invariant action for the Neveu-Schwarz (NS) sector, representing space-time bosons, of the open superstring field theory was constructed based on an ingenious formulation utilizing the large Hilbert space, which is now called Wess-Zumino-Witten-like (WZW-like) formulation [1]. This formulation was then cleverly extended to the NS sector of the heterotic string field theory [2] and also the NS-NS sector of the type II superstring field theory [3].¹ Despite its elegant mathematical structure it was difficult to extend this formulation to incorporate the Ramond sector, which represents space-time fermions, and to construct a complete gauge invariant action for a long time.

We have recently overcome this difficulty and succeeded to construct a complete gauge invariant action based on the WZW-like formulation (WZW-like action) for the open superstring field theory in [6]. Then it is quite natural to attempt to extend this success to the heterotic and the type II superstring field theory. In the previous paper we first considered an extension to the heterotic string field theory. We attempted to incorporate the Ramond sector perturbatively using the fermion expansion, and constructed a gauge invariant action at the quadratic and quartic order in powers of the Ramond field Ψ [7]. In this note we push this attempt further and construct a gauge invariant action at the sixth and eighth order of Ψ .

2. Action for the NS sector

The action constructed in [2] is based on the WZW-like formulation utilizing the large Hilbert space. The large Hilbert space $\mathcal{H}_{large}$ is an extension of the Hilbert space of the covariantly first-quantized NSR superstring whose superconformal ghost sector is constructed as a Fock space of ghost fields $(\phi(z), \eta(z), \xi(z))$ related to the conventional super conformal ghost $(\beta(z), \gamma(z))$ through the bosonization:

$$\beta(z) = \partial\xi(z)e^{-\phi(z)}, \quad \gamma(z) = e^{\phi(z)}\eta(z). \quad (1)$$

¹ See also [4], [5].



The NS sector of heterotic string field theory is described by a dynamical field $\tilde{V}$ which is Grassmann odd and has the ghost number 1 and picture number 0: $(g, p) = (1, 0)$. Since heterotic string is a closed string, $\tilde{V}$ satisfies the closed string constraints:

$$b_0^- \tilde{V} = L_0^- \tilde{V} = 0. \quad (2)$$

The WZW-like action for a heterotic string field theory was given by

$$S_{NS} = - \int_0^1 dt \langle \tilde{B}_t(t), QG(t) \rangle, \quad (3)$$

where $\langle, \rangle$ is the BPZ inner product for the closed string in the large Hilbert space:

$$\langle A, B \rangle = \langle A | c_0^- | B \rangle. \quad (4)$$

The operator Q is the BRST charge and $G = G(\tilde{V})$ is a functional of $\tilde{V}$ satisfying

$$QG(\tilde{V}) + \sum_{n=2}^{\infty} \frac{1}{n!} \underbrace{[G(\tilde{V}), \dots, G(\tilde{V})]_n} = 0. \quad (5)$$

Its extension $G(t) = G(\tilde{V}(t))$ is obtained by an one-parameter extension $\tilde{V}(t)$, $t \in [0, 1]$, with the conditions $\tilde{V}(1) = V$ and $\tilde{V}(0) = 0$. Another $\tilde{B}_t(t) = \tilde{B}_t(\tilde{V}(t))$ is also a functional satisfying a relation

$$\partial_t G(t) = Q\tilde{B}_t(t) + \sum_{m=1}^{\infty} \frac{1}{m!} \underbrace{[G(\tilde{V}), \dots, G(\tilde{V})]_m, \tilde{B}_t(t)}. \quad (6)$$

The square bracket denote the string products which are graded symmetric and map n string fields $\{\Phi_1, \dots, \Phi_n\}$ to a string field $[\Phi_1, \dots, \Phi_n]$. For later use, it is useful to regard them as linear maps $L_n^{(0)}$ from the symmetrized tensor product of $\mathcal{H}_{large}$ to $\mathcal{H}_{large}$:

$$L_n^{(0)} : (\mathcal{H}_{large})^{\wedge n} \rightarrow \mathcal{H}_{large}, \quad (7)$$

where (0) in the superscript denotes the picture number carried by the map, and further to consider corresponding coderivations $L_n^{(0)}$ acting on a symmetrized tensor algebra $\mathcal{SH}$ [8]. The BRST charge Q can naturally be identified with a one-string product $L_1^{(0)} = Q$. These string products satisfy the L_∞ relation, which is written as

$$[L_B(s), L_B(s)] = 0, \quad (8)$$

by defining a coderivation $L_B(s) = \sum_{n=0}^{\infty} s^n L_{n+1}^{(0)}$.

By suitably inserting operators with non-zero picture number, these *bosonic* string products can be extended to those with non-zero picture number as

$$L^{[n]}(t) = \sum_{m=0}^{\infty} t^m L_{m+n+1}^{(m)}, \quad (9)$$

where the label $[n]$ is called the picture number deficit. By introducing another string products

$$\lambda^{[n]}(t) = \sum_{m=0}^{\infty} t^m \lambda_{m+n+2}^{(m+1)}, \quad (10)$$

called gauge products, they can be obtained by solving two equations

$$\partial_t \mathbf{L}^{[n]}(t) = \sum_{m=0}^n [\mathbf{L}^{[n-m]}(t), \boldsymbol{\lambda}^{[m]}(t)], \quad (11)$$

$$(n+1)\mathbf{L}^{[n+1]}(t) = [\boldsymbol{\eta}, \boldsymbol{\lambda}^{[n]}(t)], \quad (12)$$

iteratively with a initial condition $\mathbf{L}^n(0) = \mathbf{L}_{n+1}^{(0)}$, [8].

3. Known results incorporating R sector

The Ramond sector is described by a dynamical field Ψ which is Grassmann even and has ghost and picture number $(g, p) = (2, -1/2)$. It also satisfies the closed string constraints:

$$b_0^- \Psi = L_0^- \Psi = 0. \quad (13)$$

In addition, Ψ must be further constrained by the conditions

$$\eta \Psi = 0, \quad XY \Psi = \Psi, \quad (14)$$

where X and Y are defined by

$$X = -\delta(\beta_0)G_0 + \delta'(\beta_0)b_0, \quad Y = 2c_0^+ \delta'(\gamma_0). \quad (15)$$

These operators X and Y only appear as those acting on the states in the small Hilbert space with picture number $-3/2$ and $-1/2$, on which they are the (inverse) picture changing operators in the sense that they satisfy

$$XYX = X, \quad YXY = Y. \quad (16)$$

It is important to note that X is BRST exact in the large Hilbert space: By introducing some operator Ξ , it is written in the form of $X = \{Q, \Xi\}$ using the BRST charge Q . An explicit form of Ξ is given in [7].² The kinetic term of the Ramond string is given by

$$S_{kin} = -\frac{1}{2} \langle \xi \Psi, YQ\Psi \rangle. \quad (17)$$

3.1. dual description of NS action

Incorporating interactions with the Ramond sector, it is useful to introduce a dual description [10] [7], which is written by using another L_∞ exchanging the role of $\mathbf{Q}$ and $\boldsymbol{\eta}$:

$$\mathbf{L}^\eta(t) = \sum_{n=0}^{\infty} t^n \mathbf{L}_{n+1}^\eta, \quad (18)$$

with $\mathbf{L}_1^\eta = \boldsymbol{\eta}$ satisfies $[\mathbf{L}^\eta(t), \mathbf{L}^\eta(t)] = 0$. This dual L_∞ is known to be obtained by similarity transformation of $\boldsymbol{\eta}$ as

$$\mathbf{L}^\eta(t) = \hat{\mathbf{G}}(t) \boldsymbol{\eta} \hat{\mathbf{G}}(t)^{-1}, \quad (19)$$

where $\hat{\mathbf{G}}(t)$ is a invertible cohomomorphism defined by

$$\hat{\mathbf{G}}(t) = \vec{\mathcal{P}} \exp \left(\int_0^t dt' \boldsymbol{\lambda}^{[0]}(t') \right). \quad (20)$$

² The rigorous definition of Ξ in the large Hilbert space was first given in [9] for the open superstring.

Here the symbol $\vec{\mathcal{P}}$ denotes the path ordered product from left to right. It is useful to introduce again a notation:

$$L_n^\eta(\Phi_1, \dots, \Phi_n) = [\Phi_1, \dots, \Phi_n]^\eta. \quad (21)$$

Then the L_∞ relation can be written as

$$\sum_\sigma \sum_{k=1}^n \frac{1}{k!(n-k)!} (-1)^{\epsilon(\sigma)} [[\Phi_{\sigma(1)}, \dots, \Phi_{\sigma(k)}]^\eta, \Phi_{\sigma(k+1)}, \dots, \Phi_{\sigma(n)}]^\eta = 0, \quad (22)$$

where $\Phi = V$ or Ψ ,³ and $|\Phi| = 1$ for $\Phi = V$ and $|\Phi| = 0$ for $\Phi = \Psi$. The BRST charge Q acts on these products as a derivation:

$$Q[\Phi_1, \dots, \Phi_n]^\eta + \sum_{k=1}^n (-1)^{|\Phi_1| + \dots + |\Phi_{k-1}|} [\Phi_1, \dots, Q\Phi_k, \dots, \Phi_n]^\eta = 0. \quad (23)$$

This BRST exactness requires the existence of another important string products $(\Phi_1, \dots, \Phi_n)^{[1]}$, dual gauge products with one picture number deficit, related to the dual string products as

$$[\Phi_1, \dots, \Phi_n]^\eta = Q(\Phi_1, \dots, \Phi_n)^{[1]} - \sum_{k=1}^n (-1)^{|\Phi_1| + \dots + |\Phi_{k-1}|} (\Phi_1, \dots, Q\Phi_k, \dots, \Phi_n)^{[1]}. \quad (24)$$

These products are cyclic with respect to the BPZ inner product in $\mathcal{H}_{large}$ with appropriate sign factors:

$$\langle \Phi_1, [\Phi_2, \dots, \Phi_{n+1}]^\eta \rangle = (-1)^{|V_1| + \dots + |V_n|} \langle [\Phi_1, \dots, \Phi_n]^\eta, \Phi_{n+1} \rangle, \quad (25)$$

$$\langle \Phi_1, (\Phi_2, \dots, \Phi_{n+1})^{[1]} \rangle = -(-1)^{|V_2| + \dots + |V_n|} \langle (\Phi_1, \dots, \Phi_n)^{[1]}, \Phi_{n+1} \rangle. \quad (26)$$

The NS action in the dual description is given by⁴

$$S^{(0)} = \int_0^1 dt \langle B_t(t), QG_\eta(t) \rangle, \quad (27)$$

where $G_\eta(t) = G_\eta(V(t))$ and $B_t(t) = B_t(V(t))$ are analogs of $G(t)$ and $\tilde{B}_t(t)$ in the previous subsection, and satisfy

$$\eta G_\eta(V) + \sum_{n=2}^{\infty} \frac{1}{n!} \underbrace{[G_\eta(V), \dots, G_\eta(V)]^\eta}_n = 0, \quad (28)$$

$$\partial_t G_\eta(V(t)) = D_\eta B_t(V(t)). \quad (29)$$

The shifted one string product $D_\eta \Phi = [\Phi]_{G_\eta}^\eta$ is one of the shifted string products which are defined for general n by

$$[\Phi_1, \dots, \Phi_n]_{G_\eta}^\eta = \sum_{m=0}^{\infty} \frac{1}{m!} \underbrace{[G_\eta, \dots, G_\eta]}_m, \Phi_1, \dots, \Phi_n]^\eta, \quad (30)$$

³ Here V and $\tilde{V}$ in the previous subsection should be distinguished, which are related through the cohomomorphism (20). New dynamical field V also satisfies the closed string constraint (2).

⁴ This is a dual form of that first given in [2]. See [7].

and satisfy the same L_∞ relation as (22). The action $S^{(0)}$ is invariant under the gauge transformation

$$B_\delta(V) = Q\Lambda + D_\eta\Omega. \quad (31)$$

Here B_δ is a functional of V and δV satisfying

$$\delta G_\eta(V) = D_\eta B_\delta(V), \quad (32)$$

and can be solved by δV if you want.

3.2. Fermion expansion

Since the action of the heterotic string field theory is nonpolynomial not only for NS string field V but also Ramond string field Ψ , we cannot yet write down a complete action including the Ramond sector. For now we can only construct a part of the action by expanding it in powers of the Ramond string field Ψ :

$$S = \sum_{n=0}^{\infty} S^{(2n)}. \quad (33)$$

An arbitrary variation of the action is then expanded as

$$\delta S = - \sum_{n=1}^{\infty} \langle \xi \delta \Psi, Y E^{(2n-1)} \rangle + \sum_{n=0}^{\infty} \langle B_\delta, E^{(2n)} \rangle, \quad (34)$$

which derives the equations of motion: $\sum_{n=0}^{\infty} E^{(2n)} = 0$ and $\sum_{n=1}^{\infty} E^{(2n-1)} = 0$. If we similarly expand the gauge transformation as

$$B_\delta = \sum_{n=0}^{\infty} B_\delta^{(2n)}, \quad \delta \Psi = \sum_{n=1}^{\infty} \delta^{(2n-1)} \Psi, \quad (35)$$

the gauge invariance of the action at order $\mathcal{O}(\Psi^{2n})$ requires

$$0 = - \sum_{k=1}^n \langle \xi \delta^{(2n-2k+1)} \Psi, Y E^{(2k-1)} \rangle + \sum_{k=0}^n \langle B_\delta^{(2n-2k)}, E^{(2k)} \rangle. \quad (36)$$

By solving these equations, we can simultaneously obtain the action and the gauge transformation order by order in the number of the Ramond field.

3.3. Known results

In [7] we solved the equation (36) required by gauge invariance for the quadratic and quartic order in Ψ .

The action at the quadratic order in Ψ has a form which is formally the same as that of the open superstring field theory [6]:

$$S^{(2)} = - \frac{1}{2} \langle \xi \Psi, Y Q \Psi \rangle + \frac{1}{2} \int_0^1 dt \langle B_t(t), [F(t) \Psi, F(t) \Psi]_{G_\eta(t)}^\eta \rangle. \quad (37)$$

Here the linear map F has also the same form as that of the open superstring case,

$$F = (1 + \Xi(D_\eta - \eta))^{-1}, \quad (38)$$

although D_η is now the shifted one string product (30) for the heterotic string field theory. Gauge transformation of this order is simultaneously determined as

$$B_\delta^{(2)} = \frac{1}{2}[F\Psi, F\Psi, \Lambda]_{G_\eta}^\eta - [F\Psi, F\Xi[F\Psi, \Lambda]_{G_\eta}^\eta]_{G_\eta}^\eta - [F\Psi, F\Xi\lambda]_{G_\eta}^\eta, \quad (39)$$

$$\delta^{(1)}\Psi = -X_\eta F\Xi D_\eta[F\Psi, \Lambda] + Q\lambda + X_\eta F\lambda. \quad (40)$$

Using gauge products (24) with one picture number deficit, the action at the quartic order in Ψ , which is the first nontrivial order beyond the open superstring field theory, was given by

$$S^{(4)} = \frac{1}{4!}\langle F\Psi, (F\Psi, F\Psi, F\Psi)_{G_\eta}^{[1]} \rangle, \quad (41)$$

where

$$(F\Psi, F\Psi, F\Psi)_{G_\eta}^{[1]} = \sum_{n=0}^{\infty} \frac{1}{n!} (\underbrace{G_\eta, \dots, G_\eta}_n, F\Psi, F\Psi, F\Psi)^{[1]}. \quad (42)$$

The gauge transformation $B_\delta^{(4)}$ and $\delta^{(3)}\Psi$ was also obtained in [7]. We do not give its explicit expression here since it is long and complicated. However we confirmed that its implicit form is given by

$$D_\eta B_\delta^{(2n)} = \left((D_\eta \Lambda) \frac{\delta}{\delta G_\eta} - \lambda \frac{\delta}{\delta \Psi} \right) E^{(2n)} - [E^{(2n)}, \Lambda]_{G_\eta}^\eta, \quad (43)$$

$$\delta \Psi^{(2n-1)} = \left((D_\eta \Lambda) \frac{\delta}{\delta G_\eta} - \lambda \frac{\delta}{\delta \Psi} \right) E^{(2n-1)}, \quad (44)$$

up to this order $\mathcal{O}(\Psi^4)$.

4. New results

In this note we further continue the analysis using the fermion expansion.

In order for the gauge invariance condition (36) at $\mathcal{O}(\Psi^6)$ ($n=3$) to have a solution, we find that the relation

$$\begin{aligned} & [F\Psi, (F\Psi^4)_{G_\eta}^{[1]}]_{G_\eta}^\eta - 2(F\Psi^3, [F\Psi^2]_{G_\eta}^\eta)_{G_\eta}^{[1]} \\ & + [F\Psi^2, (F\Psi^3)_{G_\eta}^{[1]}]_{G_\eta}^\eta - (F\Psi^2, [F\Psi^3]_{G_\eta}^\eta)_{G_\eta}^{[1]} = Q(*) + D_\eta(*) + \dots, \end{aligned} \quad (45)$$

must be satisfied, where dots in the right hand side represent the terms involving QG_η or $QF\Psi$. In order to show that this relation actually holds, it is necessary to introduce a sequence of new dual products,

$$\tilde{L}^{[n]}(t) = -\hat{G}(t)L^{[n]}(t)\hat{G}^{-1}(t), \quad \tilde{\lambda}^{[n]}(t) = -\hat{G}(t)\lambda^{[n]}(t)\hat{G}^{-1}(t), \quad (46)$$

which have n picture number deficit, and can be expanded as

$$\tilde{L}^{[n]}(t) = \sum_{m=0}^{\infty} t^m \tilde{L}_{m+n+1}^{(m)}, \quad \tilde{\lambda}^{[n]}(t) = \sum_{m=0}^{\infty} t^m \tilde{\lambda}_{m+n+2}^{(m+1)}. \quad (47)$$

Using relations

$$\partial_t \hat{G}(t) = \hat{G}(t)\lambda^{[0]}(t), \quad \partial_t \hat{G}(t)^{-1} = -\lambda^{[0]}(t)\hat{G}(t)^{-1}, \quad (48)$$

and the fact that

$$\mathbf{L}^{[0]}(t) = \hat{\mathbf{G}}(t)^{-1} \mathbf{Q} \hat{\mathbf{G}}(t), \quad (49)$$

we can show that

$$\partial_t \mathbf{L}^\eta(t) = [\mathbf{L}^\eta(t), \tilde{\boldsymbol{\lambda}}^{[0]}(t)] = -\tilde{\mathbf{L}}^{[1]}(t), \quad \partial_t^2 \mathbf{L}^\eta(t) = -[\mathbf{Q}, \tilde{\boldsymbol{\lambda}}^{[1]}(t)], \quad (50)$$

that is, the string products $(\dots)^{[1]}$ introduced above is nothing but the gauge products $\tilde{\boldsymbol{\lambda}}^{[1]}(t)$:

$$(\Phi_1, \dots, \Phi_n)^{[1]} = \frac{1}{(n-1)(n-2)} \tilde{\lambda}_n^{[1]}(\Phi_1, \dots, \Phi_n). \quad (51)$$

For deriving the relations necessary for the higher order, we need a little calculation. For the analogs of the equations (11) and (12) we have

$$\partial_t \tilde{\mathbf{L}}^{[n]}(t) = - \sum_{m=1}^n [\tilde{\mathbf{L}}^{[n-m]}(t), \tilde{\boldsymbol{\lambda}}^{[m]}(t)], \quad (52)$$

$$(n+1) \tilde{\mathbf{L}}^{[n+1]}(t) = [\mathbf{L}^\eta(t), \tilde{\boldsymbol{\lambda}}^{[n]}(t)]. \quad (53)$$

From these equations (52) and (53), and (50), we can obtain a key relation:

$$\begin{aligned} \partial_t [\mathbf{L}^\eta(t), \tilde{\boldsymbol{\lambda}}^{[n]}(t)] + (n+1) [\partial_t \mathbf{L}^\eta(t), \tilde{\boldsymbol{\lambda}}^{[n]}(t)] \\ + \sum_{m=1}^{n-1} \frac{n+1}{n-m+1} [[\mathbf{L}^\eta(t), \tilde{\boldsymbol{\lambda}}^{[n-m]}(t)], \tilde{\boldsymbol{\lambda}}^{[m]}(t)] = (n+1) [\mathbf{Q}, \tilde{\boldsymbol{\lambda}}^{[n+1]}(t)], \end{aligned} \quad (54)$$

for $n \geq 1$. In particular, we need the relation for $n = 1$ at $\mathcal{O}(\Psi^6)$:

$$\partial_t [\mathbf{L}^\eta(t), \tilde{\boldsymbol{\lambda}}^{[1]}(t)] + 2 [\partial_t \mathbf{L}^\eta(t), \tilde{\boldsymbol{\lambda}}^{[1]}(t)] = 2 [\mathbf{Q}, \tilde{\boldsymbol{\lambda}}^{[2]}(t)]. \quad (55)$$

After some calculation we can obtain

$$[\mathbf{L}_1^\eta, \boldsymbol{\rho}_{n+4}^{[1]}] + [\mathbf{L}_2^\eta, \boldsymbol{\rho}_{n+3}^{[1]}] + \frac{1}{2} \sum_{m=1}^n [\mathbf{L}_{m+2}^\eta, \boldsymbol{\rho}_{n-m+3}^{[1]}] = [\mathbf{Q}, \boldsymbol{\rho}_{n+4}^{[2]}], \quad (56)$$

where

$$\boldsymbol{\rho}_{n+3}^{[1]} = \frac{n!}{(n+2)!} \tilde{\boldsymbol{\lambda}}_{n+3}^{[1]} \quad (57)$$

$$\boldsymbol{\rho}_{n+4}^{[2]} = \frac{2! n!}{(n+3)!} \left(\tilde{\boldsymbol{\lambda}}_{n+4}^{[2]} + \frac{1}{8} \sum_{m=0}^{n-1} (f_n(m) - f_n(n-m-1)) [\boldsymbol{\rho}_{n-m+2}^{[1]}, \boldsymbol{\rho}_{m+3}^{[1]}] \right), \quad (58)$$

with

$$f_n(m) = (n+2m+5)(n-m)(n-m+1). \quad (59)$$

If we define the shifted products as

$$\mathbf{L}_{G:n}^\eta(\Phi_1, \dots, \Phi_n) = \pi_1 \mathbf{L}^\eta(e^{\wedge G_\eta} \wedge \Phi_1 \wedge \dots \wedge \Phi_n), \quad (60)$$

$$\rho_{G:n}^{[p]}(\Phi_1, \dots, \Phi_n) = \pi_1 \boldsymbol{\rho}^{[p]}(e^{\wedge G_\eta} \wedge \Phi_1 \wedge \dots \wedge \Phi_n), \quad (61)$$

we can further derive a relation

$$\sum_{m=0}^{[\frac{n}{2}]+1} [\mathbf{L}_{G:m+1}^\eta, \boldsymbol{\rho}_{G:n-m+4}^{[1]}] + \delta_{n,2l+1} \frac{1}{2} [\boldsymbol{\rho}_{G:l+3}^{[1]}, \mathbf{L}_{G:l+3}^\eta] = [\mathbf{Q}, \hat{\boldsymbol{\rho}}_{G:n+4}^{[2]}] - (QG_\eta) \frac{\delta}{\delta G_\eta} \hat{\boldsymbol{\rho}}_{G:n+4}^{[2]}, \quad (62)$$

with

$$\hat{\boldsymbol{\rho}}_{G:n+4}^{[2]} = \boldsymbol{\rho}_{G:n+4}^{[2]} - \frac{1}{2} (\boldsymbol{\rho}_{G:n+5}^{[1]} \boldsymbol{\rho}_{G:0}^{[1]})_{n+4} - \frac{1}{2} \sum_{l=1}^{[\frac{n}{2}]+2} [\boldsymbol{\rho}_{G:n-l+5}^{[1]}, \boldsymbol{\rho}_{G:l}^{[1]}]. \quad (63)$$

The Gauss' symbol $[\frac{n}{2}]$ is the greatest integer that is less than or equal to $\frac{n}{2}$. Acting this (with $n = 1$) on $(F\Psi)^{\wedge 5}$, and use the fact that $D_\eta F\Psi = 0$, we find the expected relation,

$$\begin{aligned} & [F\Psi, (F\Psi^4)_{G_\eta}^{[1]}]_{G_\eta}^\eta - 2(F\Psi^3, [F\Psi^2]_{G_\eta}^\eta)_{G_\eta}^{[1]} + [F\Psi^2, (F\Psi^3)_{G_\eta}^{[1]}]_{G_\eta}^\eta - (F\Psi^2, [F\Psi^3]_{G_\eta}^\eta)_{G_\eta}^{[1]} \\ &= \frac{1}{5} \left(Q(F\Psi^5)_{G_\eta}^{[2]} - D_\eta(F\Psi^5)_{G_\eta}^{[2]} - (QG_\eta, F\Psi^5)_{G_\eta}^{[2]} - 5(F\Psi^4, QF\Psi) \right), \end{aligned} \quad (64)$$

with the notation:

$$(\Phi_1, \dots, \Phi_n)_{G_\eta}^{[2]} \equiv \hat{\boldsymbol{\rho}}_{G:n}^{[2]}(\Phi_1 \wedge \dots \wedge \Phi_n). \quad (65)$$

Due to this relation we can solve the gauge invariance condition (36) and find that the action at this order is given by

$$S^{(6)} = -\frac{\kappa^4}{6!} \langle F\Psi, (F\Psi^5)_{G_\eta}^{[2]} \rangle + \frac{1}{2!3!3!} \langle F\Psi, (F\Psi^2, \Delta(F\Psi^3)_{G_\eta}^{[1]})_{G_\eta}^{[1]} \rangle, \quad (66)$$

with $\Delta = (F\Xi D_\eta - D_\eta F\Xi)/2$. The gauge transformation at this order, $B_\delta^{(6)}$ and $\delta^{(5)}\Psi$, is also obtained simultaneously, and is confirmed to be the form of (44).

We can repeat a similar analysis at $\mathcal{O}(\Psi^8)$. From the key equation with $n = 2$, we can obtain

$$\begin{aligned} & [\mathbf{L}_1^\eta, \boldsymbol{\rho}_{n+5}^{[2]}] + [\mathbf{L}_2^\eta, \boldsymbol{\rho}_{n+4}^{[2]}] + \frac{1}{2} \sum_{m=1}^{n+1} [[\mathbf{L}_1^\eta, \boldsymbol{\rho}_{n-m+4}^{[1]}], \boldsymbol{\rho}_{m+2}^{[1]}] + \frac{1}{2} \sum_{m=1}^n [[\mathbf{L}_2^\eta, \boldsymbol{\rho}_{n-m+3}^{[1]}], \boldsymbol{\rho}_{m+2}^{[1]}] \\ &+ \frac{1}{3} \sum_{m=1}^{n-1} \sum_{l=1}^{n-m} [[\mathbf{L}_{l+2}^\eta, \boldsymbol{\rho}_{n-m-l+3}^{[1]}], \boldsymbol{\rho}_{m+2}^{[1]}] = [\mathbf{Q}, \boldsymbol{\rho}_{n+5}^{[3]}], \end{aligned} \quad (67)$$

with

$$\begin{aligned} \boldsymbol{\rho}_{n+5}^{[3]} &= \frac{3!n!}{(n+4)!} \left(\tilde{\lambda}_{n+5}^{[3]} + \frac{1}{3!} \sum_{m=1}^n f_n(m) [\boldsymbol{\rho}_{n-m+4}^{[2]}, \boldsymbol{\rho}_{m+2}^{[1]}] \right. \\ &\quad \left. + \frac{1}{72} \sum_{m=1}^{n-1} \sum_{l=1}^{n-m} F_n(m, l) [[\boldsymbol{\rho}_{m+2}^{[1]}, \boldsymbol{\rho}_{l+2}^{[1]}], \boldsymbol{\rho}_{n-m-l+3}^{[1]}] \right), \end{aligned} \quad (68)$$

where

$$f_n(m) = (n+3m+4)(n-m+1)(n-m+2)(n-m+3), \quad (69)$$

$$\begin{aligned} F_n(m, l) &= -f_n(m) + f_n(l) \\ &\quad - 2(m-l)(n+3(n-m-l+1)+4)((m+l+1)(m+l+2)+2ml) \\ &= -F_n(l, m). \end{aligned} \quad (70)$$

After long and tedious calculation, we can find a relation for the shifted products for even n , which is enough for our purpose here:

$$\begin{aligned}
& [L_{G:1}^\eta, \hat{\rho}_{G:n+5}^{[2]}] + [L_{G:2}^\eta, \hat{\rho}_{G:n+4}^{[2]}] \\
& + \sum_{l=0}^{\frac{n}{2}-1} [[L_{G:1}^\eta, \rho_{G:n-l+3}^{[1]}], \rho_{G:l+3}^{[1]}] + \frac{1}{2} [[L_{G:1}^\eta, \rho_{G:\frac{n}{2}+3}^{[1]}], \rho_{G:\frac{n}{2}+3}^{[1]}] \\
& + \sum_{l=0}^{\frac{n}{2}-1} [[L_{G:2}^\eta, \rho_{G:n-l+2}^{[1]}], \rho_{G:l+3}^{[1]}] + \frac{1}{3} \sum_{l=0}^{n-2} \sum_{m=0}^{n-l-2} [[L_{G:n-m-l+1}^\eta, \rho_{G:m+3}^{[1]}], \rho_{G:l+3}^{[1]}] \\
& = [Q, \hat{\rho}_{G:n+5}^{[3]}] - (QG_\eta) \frac{\delta}{\delta G_\eta} \rho_{G:n+5}^{[3]}, \tag{71}
\end{aligned}$$

with

$$\begin{aligned}
\hat{\rho}_{G:n+5}^{[3]} &= \rho_{G:n+5}^{[3]} - (\rho_{G:n+6}^{[2]} \rho_{G:0}^{[1]})_{n+5} - [\rho_{G:n+5}^{[2]}, \rho_{G:1}^{[1]}] - [\rho_{G:n+4}^{[2]}, \rho_{G:2}^{[1]}] \\
&+ \frac{1}{6} (\rho_{G:n+7}^{[1]} \rho_{G:0}^{[1]} \rho_{G:0}^{[1]})_{n+5} + \frac{1}{6} (\rho_{G:n+6}^{[1]} (\rho_{G:1}^{[1]} \rho_{G:0}^{[1]})_0)_{n+5} + \frac{1}{6} \sum_{l=0}^{n+2} [\rho_{G:n-l+5}^{[1]}, (\rho_{G:l+2}^{[1]} \rho_{G:0}^{[1]})_{l+1}] \\
&- \frac{1}{3} [\rho_{G:2}^{[1]}, (\rho_{G:n+5}^{[1]} \rho_{G:0}^{[1]})_{n+4}] - \frac{1}{3} [\rho_{G:1}^{[1]}, (\rho_{G:n+6}^{[1]} \rho_{G:0}^{[1]})_{n+5}] \\
&+ \frac{1}{6} \sum_{l=0}^1 \sum_{m=0}^1 [[\rho_{G:n-l-m+5}^{[1]}, \rho_{G:m+1}^{[1]}], \rho_{G:l+1}^{[1]}] - \frac{1}{6} \sum_{l=2}^{n+1} \sum_{m=0}^1 [[\rho_{G:n-l-m+5}^{[1]}, \rho_{G:m+1}^{[1]}], \rho_{G:l+1}^{[1]}] \\
&- \frac{1}{6} [[\rho_{G:3}^{[1]}, \rho_{G:1}^{[1]}], \rho_{G:n+3}^{[1]}]. \tag{72}
\end{aligned}$$

Due to the relations (71) and (62), we can solve the gauge invariance condition (36), and have found that an action at $\mathcal{O}(\Psi^8)$ is given by

$$\begin{aligned}
S^{(8)} &= \frac{\kappa^6}{8!} \langle F\Psi, (F\Psi^7)_{G_\eta}^{[3]} \rangle \\
&+ \frac{\kappa^6}{3! \cdot 5!} \langle F\Psi, (F\Psi^2, D_\eta F \Xi (F\Psi^5)_{G_\eta}^{[2]})_{G_\eta}^{[1]} \rangle \\
&+ \frac{\kappa^6}{2 \cdot 2! \cdot 3! \cdot 3!} \langle F\Psi, (F\Psi^2, \Delta(F\Psi^2, \Delta(F\Psi^3)_{G_\eta}^{[1]})_{G_\eta}^{[1]})_{G_\eta}^{[1]} \rangle \\
&- \frac{\kappa^6}{2! \cdot 3! \cdot 3! \cdot 4!} \langle F\Psi, (F\Psi^2, (F\Psi^2, (F\Psi^3)_{G_\eta}^{[1]})_{G_\eta}^{[1]})_{G_\eta}^{[1]} \rangle. \tag{73}
\end{aligned}$$

We have also obtained the gauge transformation $B_\delta^{(8)}$ and $\delta^{(7)}\Psi$, and confirmed that it has the form of (44).

5. Discussion

In this note we push forward the previous attempt to construct a complete WZW-like action for the heterotic string field theory by means of the fermion expansion. We have constructed $S^{(6)}$ and $S^{(8)}$, the action at $\mathcal{O}(\Psi^6)$ and $\mathcal{O}(\Psi^8)$, respectively. The gauge transformations, $B_\delta^{(6)}$, $\delta^{(5)}\Psi$ and $B_\delta^{(8)}$, $\delta^{(7)}\Psi$ at these order have simultaneously been obtained and confirmed to be the form of (44).

The important remaining task is to give a complete action and gauge transformation in a closed form. It is still difficult task but we can now speculate that the main part of the action at $\mathcal{O}(\Psi^{2n})$ may be given by

$$S^{(2n)} = \frac{(-1)^n}{(2n)!} \langle F\Psi, \underbrace{(F\Psi, \dots, F\Psi)_{G_\eta}^{[n-1]}}_{2n-1} \rangle + \dots. \quad (74)$$

The dots represent the terms with less moduli integral, which include more than two nested string products. However this is only a formal expression because the products $(\dots)_{G_\eta}^{[n-1]}$ is not yet defined. Although its explicit expression, like (63) and (72), must be determined so that a higher order analog of the relations (62) and (71) is satisfied, for now we can only derive it by brute force calculation. It is necessary to understand more deeply how the relations (56), (67), (62), and (71) are derived from the key relation (54).

Finally we comment that we have recently succeeded to construct a complete WZW-like action and gauge transformation in a closed form with a different approach [11]. It is interesting to consider the relationship of two approaches and use it as a clue to complete the attempt examined in this note.

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