

A p -adic analytic approach to the absolute Grothendieck conjecture

Takahiro MUROTANI (RIMS, Kyoto University)

e-mail: murotani@kurims.kyoto-u.ac.jp

Introduction

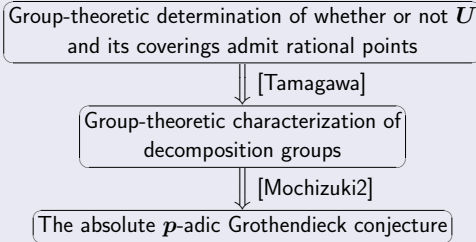
Let K be a field of characteristic 0, $\overline{K}$ an algebraic closure of K , G_K the absolute Galois group of K , U a hyperbolic curve over K and $\pi_1(U)$ the étale fundamental group of U . We have the following natural exact sequence:

$$1 \rightarrow \pi_1(U_{\overline{K}}) \rightarrow \pi_1(U) \xrightarrow{\text{pr}} G_K \rightarrow 1.$$

The relative and absolute Grothendieck conjectures ask:

$$\begin{aligned} \text{relative : } & \pi_1(U) \twoheadrightarrow G_K \overset{?}{\underset{\text{recoverable}}{\rightsquigarrow}} U, \\ \text{absolute : } & \pi_1(U) \overset{?}{\underset{\text{recoverable}}{\rightsquigarrow}} U. \end{aligned}$$

When K is a p -adic field (i.e. a finite extension of $\mathbb{Q}_p$), the absolute version of Grothendieck conjecture (the absolute p -adic Grothendieck conjecture) is unsolved. On the other hand, the absolute p -adic Grothendieck conjecture is reduced to the problem of rational points of curves as follows:



To consider this problem, we shall introduce the “ i -invariant” of **compact** analytic manifold over a p -adic field K .

Theorem-Definition [Serre]

Let q be the cardinality of the residue field of K and Y a nonempty compact analytic manifold over K . Then Y is the disjoint union of a finite number of balls and the number of (closed) balls is well determined mod $(q - 1)$. We call the number of balls $i_K(Y) \in \mathbb{Z}/(q - 1)\mathbb{Z}$ the i -invariant of Y over K . Moreover, we set $i_K(\emptyset) \equiv 0 \pmod{q - 1}$.

Let X be a **proper** hyperbolic curve over K and $X(K)$ the set of K -rational points of X . In terms of the i -invariants, the absolute p -adic Grothendieck conjecture (for hyperbolic curves **not necessarily proper**) is reduced to the following two problems:

Question

- (A) Data of i -invariants of sets of rational points of X and its étale coverings $\rightsquigarrow$ decomposition groups?
- (B) Group-theoreticity of the i -invariant of $X(K)$?

Theorem A below gives a complete affirmative answer to (A) and Theorem B below gives a partial affirmative answer to (B).

Theorem A [M]

Let K be a p -adic field, X a proper hyperbolic curve over K and q the cardinality of the residue field of K . Assume that $q \neq 2$ and let $m > 1$ be a divisor of $q - 1$. Then the following conditions are equivalent:

- (i) $X(K) \neq \emptyset$.
- (ii) $\exists X'$: a finite étale covering of X such that $X'(K) \neq \emptyset$.
- (iii) $\exists X'$: a finite étale covering of X such that $i_K(X'(K)) \not\equiv 0 \pmod{q - 1}$.
- (iv) $\exists X'$: a finite étale covering of X such that $i_K(X'(K)) \not\equiv 0 \pmod{m}$.
- (v) $\exists X'$: a finite étale covering of X such that $i_K(X'(K)) \equiv (\text{a power of } p) \pmod{q - 1}$.

Theorem B [M]

For $i = 1, 2$, let p_i be an odd prime number, K_i a p_i -adic field and X_i a proper hyperbolic curve over K_i which has log smooth reduction (i.e. has stable reduction after tame base extension). Suppose that we are given an isomorphism of profinite groups $\alpha : \pi_1(X_1) \xrightarrow{\sim} \pi_1(X_2)$. Then we have

$$i_{K_1}(X_1(K_1)) \equiv i_{K_2}(X_2(K_2)) \pmod{2}.$$

In fact, by [Mochizuki1], if we are given an isomorphism $\alpha : \pi_1(X_1) \xrightarrow{\sim} \pi_1(X_2)$, we have $p_1 = p_2$ (more precisely, we have $q_1 = q_2$ where q_i is the cardinality of the residue field of K_i). Moreover, X_1 has log smooth reduction if and only if X_2 has log smooth reduction.

Remark

If we prove Theorem B without assuming that X_i has log smooth reduction, we can prove the absolute p -adic Grothendieck conjecture for p odd.

References

- [Mochizuki1] Shinichi Mochizuki, The absolute anabelian geometry of hyperbolic curves, *Galois Theory and Modular Forms*, Kluwer Academic Publishers, 2004, pp. 77–122.
- [Mochizuki2] Shinichi Mochizuki, Topics in absolute anabelian geometry II: Decomposition groups and endomorphisms, *J. Math. Sci. Univ. Tokyo* **20** (2013), pp. 171–269.
- [M] Takahiro Murotani, A p -adic Analytic Approach to the Absolute Grothendieck Conjecture, RIMS Preprint **1892** (August 2018).
- [Serre] J.-P. Serre, *Lie Algebras and Lie Groups*, Lecture Notes in Mathematics **1500**, Springer-Verlag, (1992).
- [Tamagawa] Akio Tamagawa, The Grothendieck conjecture for affine curves, *Compositio Mathematica* **109** (1997), pp. 135–194.