

A localization lemma and its applications*

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Abstract

In this article, we give alternative proofs of two famous facts, the Poincaré-Hopf index theorem and the compatibility of two definitions of the degree of a divisor on a compact Riemann surface, and define a generalization of the tangential index [Br] and [Ho] and prove its index theorem by the method of the localization of the Chern class of a virtual bundle. The tangential index and its index formula was ordinary defined and proved by M. Brunella [Br] for a curve and a singular foliation on a compact complex surface and the author reproved it for a compact curve and a singular foliation on a complex surface [Ho].

1 Introduction

Let X be a C^∞ manifold of dimension m and E a C^∞ complex vector bundle of rank n . We consider the Chern class $c(E) \in H^*(X; \mathbf{C})$ of E . Note that we use the complex number field $\mathbf{C}$ as the coefficient of the cohomology groups although in fact $c(E)$ itself is in $H^*(X; \mathbf{Z})$, since we use the Chern-Weil theory for the construction of Chern classes. If E has a global section $s : X \rightarrow E$, which is not identically zero, we can make a frame, including s , of the restriction of E to the complement of the zero set of s . Therefore the top Chern class can be localized to the neighborhood of the zero set of s . This fact have many applications. In this article, we consider a simple generalization of this fact.

Let $\mathcal{V} = \{V_\alpha\}$ be an open covering of X such that the vector bundle E has a section $s_\alpha : V_\alpha \rightarrow E$ on an open set V_α , which is not a zero section. Assume that there exist non-vanishing functions $f_{\alpha\beta}$ on $V_\alpha \cap V_\beta$ such that

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$s_\beta = s_\alpha f_{\alpha\beta}$ and the system $\{f_{\alpha\beta}\}$ is a cocycle. We denote by F the line bundle defined by $\{f_{\alpha\beta}\}$. Then we consider the Chern class of the virtual bundle $E - F$. It is localized to the neighborhood of each connected component of the union of the zero set of each s_α . Then we can define the index of E by F and get its index formula.

In section 2, we consider a localization lemma and, as examples, the Poincaré-Hopf index formula and the compatibility of two definitions of the degree of a divisor on a compact Riemann surface. Although the Čech-de Rham cohomology theory and its integration theory play important roles in this article, we refer to [BT], [Leh1], [Leh2], [LS] and [Su] for the details of these theories. In section 3, a generalization of the tangential index [Br] and [Ho] are defined and we prove its index formula. This index can be considered to represent how a variety and a one dimensional singular foliation intersect, and it is a kind of indices relative not only to a singular foliation but also to a variety. The tangential index is defined by M. Brunella [Br] for a curve and a singular foliation on a compact complex surface. We generalize it for a variety and a dimension one singular foliation on X .

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2 Localization lemma

Let X be a C^∞ manifold of dimension m , E a complex vector bundle of rank n , $\mathcal{V} = \{V_\alpha\}_{\alpha \in A}$ an open covering of X and $s_\alpha : V_\alpha \rightarrow E$ a C^∞ section of E on each V_α . We can assume that E is trivial on each V_α if necessary taking a refinement of $\mathcal{V}$. Moreover we assume the following condition.

Assumption 2.1 *For any $\alpha, \beta \in A$ such that $V_\alpha \cap V_\beta \neq \emptyset$, there exists a non-vanishing C^∞ function $f_{\alpha\beta} : V_\alpha \cap V_\beta \rightarrow \mathbb{C}^*$ such that $s_\beta = s_\alpha f_{\alpha\beta}$ on $V_\alpha \cap V_\beta$ and the system $\{f_{\alpha\beta}\}$ forms a cocycle.*

We denote by F the line bundle which is defined by this cocycle $\{f_{\alpha\beta}\}$. Then there exists a bundle map $f : F \rightarrow E$ such that

- (1) $f(F_p) \subset E_p$ for all $p \in E$
- (2) there exist a subset $S \subset X$ such that $f_p : F_p \rightarrow E_p$ is injective for $p \in X - S$.

We call S the set of singularities of the line bundle F . Let $S = \coprod_{\lambda \in \Lambda} T_\lambda$ be the decomposition to connected components. We assume that each T_λ is compact. Take an open set U_λ for each λ such that $U_\lambda \supset T_\lambda$ and $U_\lambda \cap U_\mu = \emptyset$

for $\lambda \neq \mu$. Then $\mathcal{U} = \{U_0, (U_\lambda)_{\lambda \in \Lambda}\}$, where $U_0 = X - S$, is an open covering of X .

We consider the Čech-de Rham cohomology group $H^*(A^\bullet(\mathcal{U}))$ associated with this open covering $\mathcal{U}$. Note that this cohomology is isomorphic to the de Rham cohomology (see [BT]). The n -th Chern class $c_n(E - F)$ of the virtual bundle $E - F$ has a representative $(\sigma_n^0, (\sigma_n^\lambda)_\lambda, (\sigma_n^{0\lambda})_\lambda)$ in the Čech-de Rham cohomology group $H^{2n}(A^\bullet(\mathcal{U}))$ of degree $2n$, where σ_n^0 and σ_n^λ are $2n$ -closed forms which are representatives of $c_n(E - F)$ on U_0 and U_λ , respectively, in the de Rham cohomology group and $\sigma_n^{0\lambda}$ is a $(2n - 1)$ -form on $U_0 \cap U_\lambda$ such that $d\sigma_n^{0\lambda} = \sigma_n^\lambda - \sigma_n^0$. Note that we can construct σ_n^0 , σ_n^λ and $\sigma_n^{0\lambda}$ from connections of E and F , using the Chern-Weil theory, and the Čech-de Rham cohomology class represented by these forms is independent on the choice of the connections.

Lemma 2.2 (localization) *Let $j^* : H^{2n}(X, X - S; \mathbb{C}) \longrightarrow H^{2n}(X; \mathbb{C})$ be natural map. Then there exists $c \in H^{2n}(X, X - S; \mathbb{C})$ such that $j^*(c) = c_n(E - F)$.*

Proof. Since F can be considered a subbundle of E on U_0 , there exists the decomposition $F \oplus E'$ of E . The system $\{s_\alpha\}$ forms a frame of F . Let ∇_0^F be a trivial connection of F respect to the frame, $\nabla_0^{E'}$ a connection of E' on U_0 and κ the curvature matrix of the connection $\nabla_0 = \nabla_0^F \oplus \nabla_0^{E'}$ of E on U_0 . Then $\sigma_n^0 = \det \kappa = 0$. We can construct σ_n^λ and $\sigma_n^{0\lambda}$ from ∇_0 and connections of E and F on U_λ .

Therefore the representative of $c_n(E - F)$ in the Čech-de Rham cohomology is $\sigma = (0, (\sigma_n^\lambda)_\lambda, (\sigma_n^{0\lambda})_\lambda)$. This is a $2n$ -cocycle in the Čech-de Rham complex relative to $X - S$. Let τ be a $2n$ -form on X corresponding to σ . Then $c = [\tau] \in H^{2n}(X, X - S; \mathbb{C})$ and $j^*(c) = c_n(E - F)$. ■

We denote $c \in H^{2n}(X, X - S; \mathbb{C})$ by $c_n(E; F)$. This is a localization of $c_n(E - F)$.

If X is compact, there exists following commutative diagram.

$$\begin{array}{ccc} H^{2n}(X, X - S; \mathbb{C}) & \xrightarrow{A} & H_{m-2n}(S; \mathbb{C}) = \bigoplus_{\lambda \in \Lambda} H_{m-2n}(T_\lambda; \mathbb{C}) \\ j^* \downarrow & & \downarrow i_* \\ H^{2n}(X; \mathbb{C}) & \xrightarrow{\cap [X]} & H_{m-2n}(X; \mathbb{C}), \end{array}$$

where A is the Alexander duality, i the natural inclusion and $[X]$ the fundamental class of X .

Definition 2.3 We define an index $I(E, F; T_\lambda) \in H_{m-2n}(T_\lambda; \mathbb{C})$ of E by F at T_λ by

$$A(c) = (I(E, F; T_\lambda))_{\lambda \in \Lambda}.$$

Remark 2.4 We can define the index $I(E, F; T_\lambda)$ if S is compact.

From the commutativity of the above diagram, we have following theorem.

Theorem 2.5 If X is compact, we have

$$\sum_{\lambda \in \Lambda} i_* I(E, F; T_\lambda) = c_n(E - F) \frown [X].$$

In the rest of this section, we assume that X is compact and S consists only of isolated points. Since each T_λ consists of a point p_λ under this assumption, we can take a sufficiently small open neighborhood U_λ of p_λ . Then we can assume each σ_n^λ is 0 without losing generalities. Hence the localized Chern class $c_n(E; F)$ has a representative $(0, 0, (\sigma_n^{0\lambda})_\lambda)$. So it is important to write the $(2n-1)$ -form $\sigma_n^{0\lambda}$ explicitly. We have to mention the Bochner-Martinelli kernel for the purpose of writing $\sigma_n^{0\lambda}$ clearly.

Definition 2.6 We call following $(n, n-1)$ -form β_n on $\mathbb{C}^n$ the Bochner-Martinelli kernel;

$$\beta_n = C_n \sum_{i=1}^n (-1)^{i-1} \frac{\bar{z}_i d\bar{z}_1 \wedge \cdots \wedge d\bar{z}_i \wedge \cdots \wedge d\bar{z}_n \wedge dz_1 \wedge \cdots \wedge dz_n}{\|z\|^{2n}},$$

where

$$C_n = (-1)^{\frac{n(n-1)}{2}} \frac{(n-1)!}{(2\pi\sqrt{-1})^n}.$$

Remark 2.7 Let $S^{2n-1} \subset \mathbb{C}^n$ be a $(2n-1)$ -sphere centered at the origin 0. Then the Bochner-Martinelli kernel β_n is real on S^{2n-1} and a generator of the cohomology group $H^{2n-1}(S^{2n-1}; \mathbb{C})$:

$$\int_{S^{2n-1}} \beta_n = 1$$

Theorem 2.8 Assume that $p_\lambda \in V_\alpha$ for some α . Then we have

$$\sigma_n^{0\lambda} = -s_\alpha^* \beta_n,$$

To prove this theorem, the Chen-Weil theory and the integration along the fiber are needed. Here the proof is omitted.

Hereafter we assume that X is oriented. We introduce the integration on the Čech-de Rham cohomology group.

Let R_λ be a closed neighborhood of p_λ such that $R_\lambda \subset U_\lambda$ for each λ . Put $R_0 = X - \bigcup_{\lambda \in \Lambda} \text{int} R_\lambda$ and $R_{0\lambda} = R_0 \cap R_\lambda$. R_0 and R_λ are oriented as submanifolds of X for each λ and $R_{0\lambda}$ as the boundary of R_0 ; $R_{0\lambda} = \partial R_0 = -\partial R_\lambda$. We call a family $\mathcal{R} = \{R_0, (R_\lambda)_\lambda, (R_{0\lambda})_\lambda\}$ a system of honey-comb cells adapted to the open covering $\mathcal{U}$.

Then we can define the integration on the Čech-de Rham cohomology group $H^m(A^\bullet(\mathcal{U}))$ associated with $\mathcal{U}$ when X is compact. For any $\sigma = [(\sigma_0, (\sigma_\lambda)_\lambda, (\sigma_{0\lambda})_\lambda)] \in H^m(A^\bullet(\mathcal{U}))$, we define the integration by

$$\int_X \sigma = \int_{R_0} \sigma_0 + \sum_{\lambda \in \Lambda} \int_{R_\lambda} \sigma_\lambda + \sum_{\lambda \in \Lambda} \int_{R_{0\lambda}} \sigma_{0\lambda}.$$

This definition is well-defined and compatible with the integration on the de Rham cohomology group;

$$\int_X \sigma = \int_X \tau,$$

where τ is a $2n$ -form on X corresponding to σ . See [Leh1], [Leh2], [LS] and [Su] for the details and more general definitions.

Then we describe examples.

Corollary 2.9 *Let C be a compact Riemann surface, $D = \{(U_i, f_i)\}$ a Cartier divisor on C , $D' = \sum_{i=1}^n n_i p_i$ the Weil divisor corresponding to D . Then*

$$\int_C c_1([D]) = \sum_{i=1}^n n_i,$$

where $[D]$ is the line bundle associated with D .

Proof. We can assume that each point p_i in D' is contained in U_i and not contained in other U_j . There exists a coordinate z_i on each U_i such that $z_i(p_i) = 0$ and $f_i(z_i) = z_i^{n_i}$. Then $\mathcal{U} = \{U_0, (U_i)_i\}$, where $U_0 = C - \{p_1, p_2, \dots, p_n\}$ is an open covering of C . Let $\mathcal{R}$ be a system of honey-comb cell adapted to $\mathcal{U}$. Note that each f_i is a section of $[D]$ on U_i . From theorem(2.5) and (2.8), we have

$$\begin{aligned} \int_C c_1([D]) &= \sum_{i=1}^n \int_{R_{0i}} -f_i^* \beta_1 \\ &= \sum_{i=1}^n \frac{1}{2\pi\sqrt{-1}} \int_{S_{p_i}^1} \frac{df_i}{f_i} = \sum_{i=1}^n \frac{1}{2\pi\sqrt{-1}} \int_{S_{p_i}^1} \frac{n_i dz}{z} \\ &= \sum_{i=1}^n n_i, \end{aligned}$$

where $S_{p_i}^1$ is a 1-sphere in $\mathbb{C}$ centered at p_i and oriented naturally. ■

As the second example, we consider the Poincaré-Hopf index formula for a dimension one reduced singular foliation.

Definition 2.10 *A dimension one singular foliation $\mathcal{F}$ on a complex manifold X is determined by a triple $(\{V_\alpha\}, v_\alpha, e_{\alpha\beta})$ such that*

- (1) $\{V_\alpha\}$ is an open covering of X and, for each α , v_α is a holomorphic vector field on V_α ,
- (2) for each pair (α, β) , $e_{\alpha\beta}$ is a non-vanishing holomorphic function on $V_\alpha \cap V_\beta$ which satisfies the cocycle condition, $e_{\alpha\gamma} = e_{\alpha\beta}e_{\beta\gamma}$ on $V_\alpha \cap V_\beta \cap V_\gamma$,
- (3) $v_\beta = v_\alpha e_{\alpha\beta}$ on $V_\alpha \cap V_\beta$.

The cocycle $\{e_{\alpha\beta}\}$ defines a line bundle which is called the holomorphic tangent bundle of $\mathcal{F}$.

Note that this definition is adapted to the assumption (2.1) if we regard the holomorphic tangent bundle TX as a C^∞ complex vector bundle E . The singular set of a foliation is defined similarly. A dimension one singular foliation is said to be reduced if its singular set consists only of isolated points.

Corollary 2.11 (Poincaré-Hopf) *Let X be a compact complex manifold of complex dimension n , $\mathcal{F} = (\{V_\alpha\}, v_\alpha, e_{\alpha\beta})$ a reduced dimension one singular foliation and F a holomorphic tangent bundle of $\mathcal{F}$. Then we have*

$$\sum_{p \in S} PH(v, p) = \int_X c_n(TX - F),$$

where S is the singular set of $\mathcal{F}$ and $PH(v, p)$ is the Poincaré-Hopf index of v at p .

Proof. Note that the Poincaré-Hopf index $PH(v, p)$ is written as

$$\int_{S_p^{2n-1}} v^* \beta_n = \int_{R_{0\lambda}} \sigma_n^{0\lambda},$$

for some λ . Hence this formula is an obvious corollary of the theorem (2.5) and (2.8). ■

Remark 2.12 *The ordinal Poincaré-Hopf index formula is*

$$\sum_{p \in S} PH(v, p) = \chi(X),$$

where v is a vector field on X . If there exists a global vector field with only isolated zero points, the tangent bundle F is trivial and we get the classical formula, using the fact $\int_X c_n(X) = \chi(X)$, from the theorem(2.11). This formula is a special case of the Baum-Bott residue theorem [BB].

Note that some other formulas, for example, the Riemann-Hurwitz formula, can be proved in this way.

3 Tangential index

Let X be a complex manifold of dimension $n + k$, $V \subset X$ a strong locally complete intersection (SLCI) of dimension n (See [LS] for the definition of SLCI), $\mathcal{V} = \{V_\alpha\}$ an open covering of X and $V' = \text{Reg}(V) = V - \text{Sing}(V)$ a regular part of V . Since V is an SLCI, there exists a C^∞ complex vector bundle $\tilde{N}$ on a neighborhood U of V in X such that the restriction $\tilde{N}|_{V'}$ is the normal bundle $N_{V'}$ of V' .

Assumption 3.1 *There exists a bundle map $\tilde{\pi} : TX|_U \longrightarrow \tilde{N}$ such that a diagram*

$$\begin{array}{ccccccc} 0 & \longrightarrow & TV' & \longrightarrow & TX|_{V'} & \xrightarrow{\pi} & N_{V'} \longrightarrow 0 \quad (\text{exact}) \\ & & & & \downarrow i & & \downarrow i \\ & & & & TX|_U & \xrightarrow{\tilde{\pi}} & \tilde{N} \end{array}$$

is commutative.

The above assumption(3.1) is satisfied, for example, when an SLCI V is defined by a holomorphic section s of a holomorphic vector bundle E ; $V = s^{-1}(0)$. In this case, E is isomorphic to $\tilde{N}$.

Let $f_1^\alpha, f_2^\alpha, \dots, f_k^\alpha$ be defining functions of V on V_α ;

$$V \cap V_\alpha = \{f_1^\alpha = f_2^\alpha = \dots = f_k^\alpha = 0\}.$$

We can take coordinates $(x_1^\alpha, x_2^\alpha, \dots, x_{n+k}^\alpha)$ on V_α such that $x_{n+i}^\alpha = f_i^\alpha$ for $i = 1, 2, \dots, k$. Then

$$\pi \frac{\partial}{\partial x_{n+1}^\alpha}, \pi \frac{\partial}{\partial x_{n+2}^\alpha}, \dots, \pi \frac{\partial}{\partial x_{n+k}^\alpha}$$

form a frame of $N_{V'} = (TX|_{V'})/TV'$. We can assume $\tilde{N}$ is trivial on each $V_\alpha \cap U$ and there exists a frame $\{e_1^\alpha, e_2^\alpha, \dots, e_k^\alpha\}$ of $\tilde{N}$ such that

$$e_i^\alpha|_{V'} = \pi \frac{\partial}{\partial x_{n+i}^\alpha}$$

for each i . This frame is said to be associated with $\{f_1^\alpha, f_2^\alpha, \dots, f_k^\alpha\}$.

Let $\mathcal{F} = \{(V_\alpha, v_\alpha, e_{\alpha\beta})\}$ be a dimension one singular foliation and F the holomorphic tangent bundle of $\mathcal{F}$.

Assumption 3.2 *The SLCI V is not invariant by $\mathcal{F}$, i.e. $v_\alpha(f_{\alpha,i}) \notin I(V \cap V_\alpha)$, where $I(V \cap V_\alpha)$ is the ideal of holomorphic functions vanishing on $V \cap V_\alpha$ and generated by the defining functions of V on V_α .*

Take a frame $\{e_i^\alpha\}$ of $\tilde{N}$ associated with $\{f_i^\alpha\}$. Then we get

$$\tilde{\pi}(v_\alpha)|_V = \sum_{i=1}^k v_\alpha(f_{\alpha,i}) e_i^\alpha.$$

Let

$$T_\alpha = \text{Sing} V \cup \{p \in V' \cap V_\alpha \mid v_\alpha(p) \in T_p V'\}.$$

Then we have

$$\begin{aligned} \tilde{\pi}(v_\alpha)|_V &= \tilde{\pi}(v_\beta)|_V e_{\beta\alpha} \\ T_\alpha &= \{p \in V \cap V_\alpha \mid \tilde{\pi}(v_\alpha)(p) = 0\}. \end{aligned}$$

So $T = \cup_\alpha T_\alpha$ is well-defined. T is the set of tangential points of $\mathcal{F}$ and V . Let $T = \coprod_{\lambda \in \Lambda} T_\lambda$ be the decomposition to connected components and we assume that T is compact. Then there exists generalized tangential index.

Theorem 3.3 (tangential index) *There exists index*

$$I(N, F; T_\lambda) \in H_{2(n-k)}(T_\lambda; \mathbb{C})$$

of N by F at T_λ . Moreover if V is compact,

$$\sum_{\lambda \in \Lambda} i_* I(N, F; T_\lambda) = c_k(N - F) \frown [V]$$

Theorem 3.4 *If $n = k$ and T consists only of isolated points, then*

$$I(N, F; p) = \int_L (\pi(v))^* \beta_k,$$

where $p \in T$ and L is a link of V at p with a usual orientation; $L = \{f_1 = f_2 = \dots = f_k = 0, |v(f_1)|^2 + |v(f_2)|^2 + \dots + |v(f_k)|^2 = \varepsilon\}$ for a sufficiently small $\varepsilon > 0$ and $d \arg v(f_1) \wedge d \arg v(f_2) \wedge \dots \wedge d \arg v(f_k) > 0$.

These two theorems are corollaries of theorem(2.5) and (2.8), respectively. Apply these theorems to a virtual bundle $\tilde{N} - F$ and V .

This index can be considered to represent how tangent a variety and on dimensional singular foliation. If $n = k = 1$ then we have

$$I(N, F; p) = \frac{1}{2\pi\sqrt{-1}} \int_L \frac{dv(f)}{v(f)}.$$

This coincides with an intersection number $(v(f), f)_p$ at p (See [GH] Chapter 5) and the original tangential index [Br] and [Ho].

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