

Generalized Tian-Todorov theorems

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1 The classical Tian-Todorov theorem

Recall the classical Tian-Todorov theorem (see [4],[5]) about the smoothness of the moduli spaces of Calabi-Yau manifolds:

Theorem 1.1 *If X is a compact Kähler manifold with $c_1(X) = 0 \in \text{Pic}(X)$, then the Kuranishi space of deformations of complex structures on X is smooth of dimension $h^{n-1,1}(X) := \text{rk } H^{n-1,1}(X)$ where $n = \dim(X)$. Manifold X with deformed complex structure is again a Kähler manifold with $c_1(X) = 0 \in \text{Pic}(X)$. Similarly, if X is projective and $\omega \in H^2(X, \mathbb{Z})$ is an ample class, then the Kuranishi space of deformations of X which polarization ω is also smooth, of dimension $\text{rk } H_{\text{prim}}^{n-1,1}(X)$ of the primitive cohomology. Moreover, any choice of a splitting of the Hodge filtration on $H^n(X)$ (resp. of $H_{\text{prim}}^n(X)$) defines an analytic affine structure on the Kuranishi space.*

The goal of my talk is to explain that there are many generalizations of this theorem.

First, I present a sketch of a proof.

2 Smoothness via dg BV algebras

Definition 2.1 *A differential graded Batalin-Vilkovisky algebra A (a dg BV algebra for a short) over $\mathbb{C}$ is a commutative unital super-algebra endowed with two odd operators d, Δ satisfying*

- $d^2 = \Delta^2 = d\Delta + \Delta d = 0$,
- $d(1_A) = \Delta(1_A) = 0$,
- operator d is a differential operator of order ≤ 1 ,

- operator Δ is a differential operator of order ≤ 2 .

The vector space $\mathfrak{g} := \Pi A$ obtained from A by the changing of parity, carries a natural structure of Lie super-algebra:

$$[a, b] = \Delta(ab) - \Delta(a)b - (-1)^{\deg a} a\Delta(b).$$

Operators d, Δ on $\mathfrak{g}$ are odd derivations with respect to the Lie bracket.

Proposition 2.2 *Let us assume that $H^\bullet(A[[u]], d + u\Delta)$ is a free $\mathbb{C}[[u]]$ -module, where u is a formal even variable. Then the formal moduli space associated with dg Lie algebra $(\mathfrak{g}, d)$ is smooth. Any trivialization of $\mathbb{C}[[u]]$ -module $H^\bullet(A[[u]], d + u\Delta)$ gives a formal affine structure (“flat coordinates”) on the moduli space.*

The proof of the above proposition can be found e.g. in [3], (also see [1] for a slightly weaker result). The Tian-Todorov theorem follows from the Proposition, applied to

$$A_X := \Gamma(X, \Omega^{0,\bullet} \otimes_{\mathcal{O}_X} \wedge^\bullet T_X)$$

which is the algebra of $\bar{\partial}$ -forms on X with values in polyvector fields. The differential d is $\bar{\partial}$, and the operator Δ is the divergence with respect to the holomorphic volume form on X . The freeness property of the cohomology with respect to the deformed differential follows from the $\partial\bar{\partial}$ -lemma.

3 Generalizations

Instead of an individual Calabi-Yau manifold X we can consider:

1. a pair (X, D) where X is smooth projective variety (typically X is Fano), and $D \subset X$ is a divisor with normal crossing such that $[D] = -c_1(X) \in \text{Pic}(X)$,
2. a pair (X, D) where X is a Calabi-Yau manifold, $c_1(X) = 0 \in \text{Pic}(X)$, and $D \subset X$ is a divisor with normal crossings,
3. a triple $(X, (D_i)_{i \in I}, (a_i)_{i \in I})$ where X is a smooth projective variety, $(D_i)_{i \in I}$ is a finite collection of irreducible divisors whose union is a divisor with normal crossings, and $(a_i)_{i \in I}$ is a collection of rational numbers $0 < a_i < 1 \ \forall i \in I$ such that

$$\sum_{i \in I} a_i [D_i] = -c_1(X) \in \text{Pic}(X) \otimes \mathbb{Q}$$

4. a pair (X, W) where X is a smooth quasi-projective variety with $c_1(X) = 0 \in \text{Pic}(X)$ and $W : X \rightarrow \mathbb{A}^1$ is a proper map.
5. “broken Calabi-Yau variety” X , a singular projective scheme which is a reduced divisor with normal crossing in a larger smooth non-proper variety Y with $c_1(Y) = 0$, given by $X = W_Y^{-1}(0)$ where $W_Y : Y \rightarrow \mathbb{A}^1$ is a proper map.

All these examples can be merged together, i.e. one can consider broken non-compact X with a proper map to $\mathbb{A}^1$ and a fractional divisor with weights in $[0, 1] \cap \mathbb{Q}$ representing $-c_1(X)$ in $\text{Pic}(X) \otimes \mathbb{Q}$.

The proof of the classical Tian-Todorov theorem presented in the previous section, extends immediately to all cases. The dg BV algebra in cases 1,2,3 is

$$A_{X,D} := \Gamma(X, \Omega^{0,\bullet} \otimes_{\mathcal{O}_X} \wedge^\bullet T_{X,D})$$

where $T_{X,D}$ is the sheaf of holomorphic vector fields on X preserving D . The differential d is given by $\bar{\partial}$, and operator Δ is the divergence with respect to a (multi-valued) holomorphic volume form on $X \setminus D$. The contraction of these polyvector fields with the volume form gives the $\bar{\partial}$ -resolution of the sheaf of holomorphic forms on X which either have poles of first order on D (case 1), vanish on D (case 2), or take values in a local system with finite monodromy (case 3). The freeness property of cohomology follows from the theory of mixed Hodge structures.

The mirror symmetry for Calabi-Yau manifolds generalizes to some of our examples. The case 1 with smooth D is dual to the case 4, e.g. $X = \mathbb{CP}^n$ with a smooth anticanonical hypersurface $D \subset X$ of degree $n + 1$, is mirror dual to $(X^\vee, W^\vee)$ where $X^\vee$ is a partial compactification of $\mathbb{G}_m^n$ endowed with a function

$$W(x_1, \dots, x_n) = x_1 + \dots + x_n + \frac{1}{x_1 \dots x_n}.$$

Similarly, the case 2 with smooth D is dual to the case 5, e.g. the pair (X, D) where X is an elliptic curve and $D \subset X$ is a collection of k points, it mirror dual to a singular elliptic curve $X^\vee$ with double points, which is a wheel of k copies of $\mathbb{CP}^1$. One of the corollaries of the mirror symmetry is that the mapping class group of the open surface $X - D$ acts by automorphisms of $D^b(\text{Coh}(X^\vee))$ (modulo powers of the shift functor).

I do not know what are mirror partners for cases 1 and 2 with a non-smooth divisor D , and also for the case 3.

4 Non-compact Calabi-Yau manifolds

Let X be a smooth projective manifold with a section of its anti-canonical bundle which vanish with multiplicities strictly > 1 at a divisor $D \subset X$ with normal crossings. On the complement $X \setminus D$ we have a non-vanishing holomorphic volume element Ω . We can define a dg BV algebra associated with X and Ω to be a subalgebra of $A_{X,D}$ consisting of such elements for which the contraction with Ω produces a form with logarithmic poles at D . Hence we obtain again certain smooth moduli spaces. Here is one important class of examples: let $f = f(x, y)$ be polynomial defining a smooth curve in $\mathbb{C}^2$. We associate with it a non-compact 3-dimensional Calabi-Yau manifold $Y \subset \mathbb{C}^4$ given by the equation

$$uv = f(x, y).$$

One can show that Y can be represented as a complement $X \setminus D$ of the type described above. Hence we obtain a smooth moduli space. E.g. for the case of hyperelliptic curve $f(x, y) = y^2 + a_0 + a_1x + \dots + a_{2g}x^{2g} + x^{2g+2}$ the universal family is obtained by variations of coefficients $a_0, \dots, a_g$. The flat coordinates on the moduli space are associated with an appropriate splitting of the Hodge filtration, and are exactly those which appear in the matrix models, see e.g. [2].

5 Speculations about Calabi-Yau motives

The construction presented above gives many examples of variations of (mixed) Hodge structures of Calabi-Yau type over smooth bases. This leads to the following question, which I formulate for simplicity only in the pure case.

Question 5.1 *Let H be an absolutely indecomposable pure Hodge structure of weight w of algebro-geometric origin with coefficients in a number field (i.e. H is a direct summand of the cohomology space of some smooth projective variety), and such that there exists $k \in \mathbb{Z}$ such that H is of Calabi-Yau type, i.e.*

$$\mathrm{rk} H^{k, w-k} = 1, \quad H^{k', w-k'} = 0 \quad \forall k' > k.$$

Does there exist a smooth universal family of variations of H of an algebro-geometric origin, of dimension equal to $\mathrm{rk} H^{k-1, w-k+1}$?

There are many examples supporting this, e.g. one can take H to be the primitive part of the middle cohomology of hypersurface $X \subset \mathbb{CP}^{N-1}$ of degree $d|N$. The proof of the generalized Tian-Todorov theorems does not apply in this case, but still the dimension of the moduli space and of the corresponding Hodge component match. It would be wonderful if the answer to the question is positive. It means that we have nice smooth moduli stacks of Calabi-Yau motives (generalizations of Shimura varieties). With any pure Hodge structure H one can associate another Hodge structure of Calabi-Yau type (maybe decomposable), by taking the exterior power $\wedge^m H$ where $m \in \mathbb{Z}_+$ is the dimension of a term $F^l H$ of the Hodge filtration of H , i.e. $m = \text{rk } \oplus_{k' \geq l} H^{k', w-k'}$. In the case $H = H^1(C)$ where C is a smooth projective curve of genus g , the absolutely indecomposable summand H' of $\wedge^g H$ containing the one-dimensional component $\wedge^g H^{1,0}$, is a Hodge structure of Calabi-Yau type varying over an appropriate Shimura variety. One can check that the dimension of this variety always coincides with the corresponding Hodge number of H' .

References

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