

Thom polynomials and around

By

Toru Ohmoto*

This is a quick review on Enumerative Geometry from the viewpoint of Singularity Theory. We shall feature *Thom polynomial*, the simplest universal cohomological obstruction for appearance of singular points of prescribed type in any given generic maps. The first main problem is to *compute* Thom polynomials for given singularity types. This had been thought for a bit long time as a technically difficult problem in algebraic geometry or representation theory, because traditional approaches require to construct an appropriate resolution of the orbit closure. Recently, however, a new effective method for computation has been brought from topology - classifying spaces, equivariant cohomology and cobordism theory. We work on complex analytic singularities, and H_* stands for Borel-Moore homology.

§ 1. Enumerating Singularities: Examples

Example 1.1. (Riemann-Hurwitz formula)

Let $f : M \rightarrow N$ be a branched cover of a nonsingular complete curve; Set $\mu(f, x) = e_x - 1$, where e_x is the branch index at $x \in M$. The *classical Riemann-Hurwitz formula* is rewritten as follows:

$$\begin{aligned}
 \sum \mu(f, x) &= \deg f \cdot \chi(N) - \chi(M) \\
 (1.1) \quad &= c_1(TN) \cap f_*[M] - c_1(TM) \cap [M] \\
 &= (c_1(f^*TN) - c_1(TM)) \cap [M] \\
 &= c_1(f^*TN - TM) \cap [M].
 \end{aligned}$$

If f has only A_1 -singularities (i.e., $\mu(f, x) = 1$, namely it is locally expressed by $z \mapsto z^2$), the formula counts the number of A_1 -points: Actually in this case (equidimensional case), the Thom polynomial $tp(A_1)$ for A_1 -singularity type is just the monomial $c_1 = c_1(f^*TN - TM)$.

Received February 1, 2008. Accepted July 28, 2008.

2000 Mathematics Subject Classification(s):

*Department of Mathematics, Hokkaido University, Sapporo 060-0810, Japan

Example 1.2. (Thom-Porteous formula)

Let $E \rightarrow M$ and $F \rightarrow M$ be vector bundles of rank m and n , respectively, with the same smooth base M . Let $\varphi : E \rightarrow F$ be a vector bundle morphism, that is a section of the vector bundle $\text{Hom}(E, F) \rightarrow M$, which may also be regarded as a family of ‘equivalent classes’ of linear maps $\mathbb{C}^m \rightarrow \mathbb{C}^n$ parametrized by M . We are herewith interested in the loci corresponding to ‘singular linear maps’, that is, the *degeneracy loci*

$$\Sigma^k(\varphi) = \{ x \in M, \dim \ker \varphi_x = k \}.$$

Below we denote $k = \dim \ker \varphi_x$ and $l = \dim \text{coker } \varphi_x = n - m + k$. The *Giambelli-Thom-Porteous formula* [30] states that for suitably generic φ ¹, the dual to the closure of $\Sigma^k(\varphi)$ is expressed by

$$(1.2) \quad \text{Dual}[\overline{\Sigma^k(\varphi)}] = \Delta_{l^k}(c(F - E)) = \overbrace{\begin{vmatrix} c_l & c_{l+1} & \cdots \\ c_{l-1} & c_l & \cdots \\ \vdots & \vdots & \ddots \end{vmatrix}}^k$$

where c_i is the i -th component of $c(F - E) = \frac{1 + c_1(F) + \cdots}{1 + c_1(E) + \cdots}$. The Schur function of type $l^k = (l, \dots, l)$ in the Chern classes c_i is the Thom polynomial $tp(\Sigma^k)$ for the singularity type Σ^k .

For maps $f : M^m \rightarrow N^n$, we denote by $\Sigma^k(f)$ the degeneracy loci of the differential $df : TM \rightarrow f^*TN$. Note that $tp(A_1) = tp(\Sigma^1) = c_1$ in Example 1.1.

Example 1.3. (Singularities of maps)

As an example of classification of map-germs $(\mathbb{C}^m, 0) \rightarrow (\mathbb{C}^n, 0)$ (*mono-singularities*), let us see the case of $m = n + 1$ and some generic types:

$$(1.3) \quad \begin{aligned} A_1 &: (x, y, v) \mapsto (x^2 + y^2, v), \\ A_2 &: (x, y, u, v) \mapsto (x^3 + ux + y^2, u, v), \\ A_3 &: (x, y, u_1, u_2, v) \mapsto (x^4 + u_1x^2 + u_2x + y^2, u_1, u_2, v), \end{aligned}$$

where v stands for parameters of the trivial unfolding of some dimension.

For a given generic map $f : M^{n+1} \rightarrow N^n$ between manifolds, we denote by $A_k(f)$ the set of points x of M at which the germ of f is of type A_k , that is, the germ f at x is written in the normal form in suitable local coordinates of source and target. Then $A_k(f)$ is a locally closed submanifold of codimension $k + 1$ in M . We denote by $A_1^2(f)$

¹This means that the section φ is transverse to the variety $\Sigma^s(E, F)$ (consisting of all linear maps with $\dim \ker = s$) for all $s \geq k$ in the total space of $\text{Hom}(E, F)$. One can replace the LHS of (1.2) by a certain *localized class* ([10], [16], [19]) that makes the formula hold even for non-generic φ (cf. the Milnor number μ in Example 1.1).

the transverse self-intersection locus of $f(A_1(f))$ in N (i.e., the locus corresponding to the family of fibers having exactly two nodes), which has codimension 2 in N . One can consider many other combinations like as $A_1^k, A_1 A_2, \dots$, called *multi-singularities*.

The closures $\overline{A_k(f)}$ in M and $\overline{A_1^2(f)}$ in N become possibly singular closed subvarieties, and their Poincaré dual are universally expressed by *Thom polynomials*

$$(1.4) \quad \begin{aligned} tp(A_1) &= c_1^2 - c_2, & tp(A_2) &= 2c_1(c_1^2 - c_2), & \in H^*(M) \\ tp(A_3) &= 5c_1^4 - 4c_1^2 c_2 - c_1 c_3 \\ n(A_1^2) &= (s_2 - s_{01})^2 - s_{0001} + 8s_{001}s_{01} - 7s_3 \in H^*(N) \end{aligned}$$

where c_i is the i -th component of $c(f) = c(f^*TN - TM)$ as same as before, and $s_{i_1 i_2 \dots}$ are the *Landweber-Novikov classes* $s_I := f_*(c^I(f)) = f_*(c_1^{i_1}(f) \cdots c_k^{i_k}(f))$.

The *existence theorem* of such universal polynomials $tp(\eta)$ in variables c_i for *mono-singularities* η goes back to René Thom (e.g. [33] in 1950's), while the *existence theorem* of universal polynomials $n(\eta_{ml})$ in variables s_I for *multi-singularities* η_{ml} is rather quite new, that is due to M. Kazarian ([14] '03)². Indeed it is quite recent that many computational results have been obtained (see [31], [14], [15] etc) by the method which will be explained in §4 below.

§ 2. Classification of Singularities

- 1st year Linear Algebra:

$G = GL(m) \times GL(n)$ acts on $V = \text{Hom}(\mathbb{C}^m, \mathbb{C}^n)$ by $(A, B).H = BHA^{-1}$. Orbits are determined by kernel dimension k (or rank), denoted by Σ^k .

- Classification of map/function-germs:

Roughly, the classification of map-germs is the study of 'jet space representations' of automorphism groups of coordinate changes. Basic references are e.g. [22], [1], [20].

Let $\mathcal{V} = \{f : \mathbb{C}^m, 0 \rightarrow \mathbb{C}^n, 0\} \simeq (\mathfrak{m}_m)^n$, the space of map-germs preserving the origin, where $\mathfrak{m}_m$ is the maximal ideal of the ring $\mathcal{O}_{\mathbb{C}^m, 0}$ of regular function-germs. There are two basic equivalences (group action on $\mathcal{V}$), called *$\mathcal{A}$ and $\mathcal{K}$ -equivalences*:

$$\begin{aligned} \mathcal{A} &= \text{Aut}(\mathbb{C}^m, 0) \times \text{Aut}(\mathbb{C}^n, 0) \text{ acts on } \mathcal{V} \text{ by } (\phi, \tau).f = \tau \circ f \circ \phi^{-1}, \\ \mathcal{K} &= \text{Aut}(\mathbb{C}^m, 0) \times \{B : \mathbb{C}^m, 0 \rightarrow GL(n)\} \text{ by } ((\phi, B).f)(x) = B(x)f(\phi^{-1}(x)). \end{aligned}$$

$\mathcal{A}$ -equivalent map-germs are also $\mathcal{K}$ -equivalent [22]: the $\mathcal{K}$ -equivalence is indeed the classification of variety-germs (=ideals) $f = 0$ under automorphisms of the source.

²Note that $n(A_1^2)$ has the form that the square of $f_*(tp(A_1))$ plus some '*residual polynomial*'. The residual terms are much related to the geometry of the strata adjacent to the A_1 -locus.

The infinitesimal structure of orbits looks as follows (in case of $\mathcal{K}$; we omit $\mathcal{A}$): For $f \in \mathcal{V}$, let $\theta(f)$ denote the $\mathcal{O}_{\mathbb{C}^m}$ -module consisting of all germs of sections of $f^*T\mathbb{C}^n$ at $0 \in \mathbb{C}^m$, then the space of infinitesimal deformations of f in $\mathcal{V}$ is identified with $\mathfrak{m}_m\theta(f)$. Denote the differential by $tf(\mathbf{v}) := df(\mathbf{v})$ for vector-field-germs $\mathbf{v} \in \theta(id_m)$, id_m being the identity of $\mathbb{C}^m$. The *tangent space* to the orbit $\mathcal{K}.f$, which consists of infinitesimal deformations of the form $\frac{d}{dt}(\phi_t, B_t).f|_{t=0}$, $(\phi_t, B_t) \in \mathcal{K}$, is the $\mathcal{O}_{\mathbb{C}^m}$ -submodule given by

$$T\mathcal{K}.f = tf(\mathfrak{m}_m\theta(id_m)) + f^*\mathfrak{m}_n\theta(f) \subset \mathfrak{m}_m\theta(f) = T_f\mathcal{V}.$$

All groups and spaces described above are infinite dimensional, but our classification does essentially deal with some finite dimensional representatives: For $f \in \mathcal{V}$, the k -jet $j^kf(0)$ is defined to be the class of f modulo $\mathfrak{m}_m^k\mathcal{V}$, i.e., the truncated Taylor expansion of f at 0 up to order k . Denote the k -jet space by $J^k\mathcal{V} = \mathcal{V}/\mathfrak{m}_m^k\mathcal{V}$. A key notion is *finite determinacy* for $G = \mathcal{A}$ or $\mathcal{K}$, that is, $f \in \mathcal{V}$ is said to be k - G -determined if any $g \in \mathcal{V}$ with $j^kg(0) = j^kf(0)$ is G -equivalent to f (e.g. the Morse singularity (A_1) is 2-determined for both equivalences). Our classification process goes to list up k -determined germs from low order k to higher. It is known that

- (a) f is finitely G -determined³, (i.e. k -determined for some k) $\iff$ the orbit $G.f$ in $\mathcal{V}$ has finite codimension, i.e., $\dim_{\mathbb{C}} T_f\mathcal{V}/TG.f < \infty$, [22], [1], [20];
- (b) For finitely determined map-germs f , its stabilizer subgroup has maximal compact (reductive) subgroups conjugate to some subgroup of $GL(m) \times GL(n)$ ($= J^1\mathcal{A} = J^1\mathcal{K}$), see [35].

Example 2.1. (Simple Curve Singularities: Example 1.3)

A G -orbit η is called to be *simple* (G -simple) if there are only finitely many G -orbits which intersect with a sufficiently small neighborhood of η .

In the classification of functions $\mathbb{C}^2, 0 \rightarrow \mathbb{C}, 0$ and ICIS curve-germs $\mathbb{C}^{n+1}, 0 \rightarrow \mathbb{C}^n, 0$, the list of low codimensional $\mathcal{K}$ -orbits are shown as follows (those are simple $\mathcal{K}$ -orbits):

1-jet/codim	0	2	3	4	5	6	7	...
Σ^1	A_0							...
Σ^2	$A_1 \ A_2 \ A_3 \ A_4 \ A_5 \ A_6 \ \cdots$							
	$D_4 \ D_5 \ D_6 \ \cdots$							
	$E_6 \ \cdots$							
Σ^3	$S_5 \ S_6 \ \cdots$							

³In particular, in case of $m \geq n$, f is finitely $\mathcal{K}$ -determined $\iff f = 0$ defines an *isolated* complete intersection singularity (ICIS) at 0.

Normal forms of those singularities are as follows:

$$A_\mu : x^{\mu+1} + y^2 \ (\mu \geq 1), \quad D_\mu : x^2y + y^{\mu-1} \ (\mu \geq 4), \quad E_6 : x^3 + y^4, \dots, \\ S_\mu : (x^2 + y^2 + z^{\mu-3}, yz), \dots.$$

For example, look at the case of $A_3 : (x, y) \mapsto z = f(x, y) = x^4 + y^2$. Set $\mathbf{e} = \frac{\partial}{\partial z}$ and then $\theta(f) = \mathcal{O}_{x,y} \mathbf{e}$. The normal vector space $T_f \mathcal{V} / T\mathcal{K}.f$ has a basis $\{x^3 \mathbf{e}, y \mathbf{e}, x^2 \mathbf{e}, x \mathbf{e}\}$, thus the codimension is 4. The first two vectors are parallel to $tf(\frac{\partial}{\partial x})$ and $tf(\frac{\partial}{\partial y})$. The last two vectors define a deformation of f by $f(x, y) + u_1 x^2 + u_2 x$ with parameters u_1, u_2 , which gives a *miniversal unfolding* of A_3 -singularity (see e.g. [22], [1] for the detail). The obtained map-germ is actually the third one without trivial parameters v in (1.3). Note that the source space (x, y, u_1, u_2) of the miniversal unfolding of f is identified with the normal to the $\mathcal{K}$ -orbit of f in $\mathcal{V}$.

§ 3. Classifying Spaces

Let G be a complex Lie group acting on a complex space V . In principle, any family φ of G -orbits in V parametrized by M is regarded as a certain morphism from M to the Borel construction (or the classifying stack $[V/G]$), called a *classifying morphism*⁴,

$$\bar{\varphi} : M \longrightarrow B_G V := EG \times_G V$$

where $EG \rightarrow BG$ is the universal G -principal bundle and $EG \times_G V = (EG \times V)/G$ is the total space of the associated bundle with fiber V over BG . Any two classifying morphisms corresponding to the same family φ are isomorphic in a categorical sense. The G -equivariant cohomology of V (in the sense of Borel) is defined by $H_G^*(V) := H^*(B_G V)$. If V is an G -affine space, then $H_G^*(V) \simeq H_G^*(pt) \simeq H^*(BG)$.

Definition 3.1. For a G -orbit η in a G -affine space V , the *Thom polynomial* of η is the G -characteristic class which represents the equivariant fundamental class of the orbit closure $\bar{\eta}$ via the equivariant Poincaré dual⁵:

$$tp(\eta) := \text{Dual}^G[\bar{\eta}]_G \in H_G^*(V) = H^*(BG).$$

Intuitively, $tp(\eta)$ is the dual to the ‘finite codimensional subvariety’ $B_G \bar{\eta} = EG \times_G \bar{\eta}$ in $B_G V$. An alternative description is given in the next section.

⁴For the construction in algebraic context, see [34].

⁵The G -versions of the fundamental class and the Poincaré dual are defined by the inductive limits using a finite dimensional approximation of the Borel construction (In algebraic context, see [34] and [4]).

Example 3.2. Let $G = GL(m) \times GL(n)$ and $V = \text{Hom}(\mathbb{C}^m, \mathbb{C}^n)$. Note that $BG = BGL(m) \times BGL(n) = \text{Grass}(m, \infty) \times \text{Grass}(n, \infty)$, which is the classifying space for vector bundles $\text{Hom}(E, F) \rightarrow M$. By the following diagram, the *classifying space for vector bundle morphisms* $\varphi : E \rightarrow F$ should be the total space $B_G V$ (the classifying morphism is given by $\bar{\varphi} = \bar{\rho} \circ \varphi$):

$$\begin{array}{ccccc}
 & & B_G(V \times V) & & \\
 & \nearrow \bar{\varphi}' & \Delta \uparrow & & \\
 \text{Hom}(E, F) & \xrightarrow{\bar{\rho}} & B_G V & = \bigsqcup B_G \Sigma^k & \\
 \varphi \uparrow & \nearrow \bar{\varphi} & \downarrow & & \\
 \bigsqcup \Sigma^k(\varphi) = M & \xrightarrow{\rho} & BG & &
 \end{array}$$

The diagonal embedding $V \rightarrow V \times V$ defines the *universal section* Δ . By the tautological construction, φ is induced as the pullback of Δ via $\bar{\varphi}$.

$B_G V$ is stratified via $B_G \Sigma^k = EG \times_G \Sigma^k$ ($\max(0, m - n) \leq k \leq m$), and the degeneracy locus $\Sigma^k(\varphi)$ is obtained as the intersection of the classifying morphism $\bar{\varphi}$ with the stratum $B_G \Sigma^k$. We say φ is generic if $\bar{\varphi}$ is transverse to those strata in $B_G V$. For generic φ it holds that $\text{Dual}[\overline{\Sigma^k(\varphi)}] = \bar{\varphi}^* \text{Dual}[\overline{B_G \Sigma^k}] = \rho^* tp(\Sigma^k)$.

Remark. The case of mono-germ classification is similar as Example 3.2; We just take the jet space $J^k \mathcal{V}$ and the group $J^k \mathcal{K}$ (or $J^k \mathcal{A}$) for some order k . However, the case of multi-singularities of maps is much harder (Kazarian [14]): it requires the classifying space of complex cobordism (or probably a glued space of classifying stacks).

§ 4. Actual Computations

We briefly review a new approach due to R. Rimányi [31], which is effective for computation (also see Kazarian [13], [15], Fehér-Rimányi [7]).

Let η be a G -orbit in a G -affine space V . Look at the exact sequence

$$\rightarrow H_G^*(V, V - \bar{\eta}) \xrightarrow{j} H_G^*(V) \xrightarrow{\alpha} H_G^*(V - \bar{\eta}) \rightarrow .$$

In $H_G^*(V)$, $\ker \alpha (= \text{Im } j)$ is an ideal. If its non-trivial lowest cohomological degree part, say $H_G^{2s}(V) \cap \ker \alpha$, is isomorphic to $\mathbb{Z}$, we define $tp(\eta)$ to be the generator of the homogeneous part so that $tp(\eta)$ restricted to the orbit η itself equals the Euler class of the normal bundle ν_η of η . The last condition fixes the ambiguity in the choice of the generator, and this definition of $tp(\eta)$ is compatible⁶ with Definition 3.1. Note that if $\bar{\eta}$

⁶In fact it holds that $H_G^i(V, V - \bar{\eta}) \simeq H_{\dim V - i}^G(\bar{\eta})$ (the Alexander duality, cf [4]), thus if $\bar{\eta}$ is of (complex) codimension s and irreducible, $H_G^i(V, V - \bar{\eta})$ is trivial for $i < 2s$ and is isomorphic to $\mathbb{Z}$ for $i = 2s$, the generator of which corresponds to the G -fundamental class of $\bar{\eta}$, cf. (equivariant version of) §19.1 in [10].

is smooth, $tp(\eta)$ is nothing but the j -image of the Thom class; but in general our $\bar{\eta}$ is singular.

If V admits a certainly nice G -invariant stratification, one can describe $tp(\eta)$ as a solution of some linear equations on equivariant cohomology classes:

Assume that V consists of finitely many orbits ξ , and also that their normal Euler classes are non-zero: $c_{top}(\nu_\xi) \neq 0$ for each ξ . Let G_ξ be the stabilizer subgroup of ξ , and $\iota_\xi : G_\xi \subset G$ the inclusion. Then it turns out that

$$\oplus \iota_\xi^* : H_G^*(V) = H^*(BG) \longrightarrow \oplus_\xi H^*(BG_\xi)$$

is an isomorphism. In particular it holds that

$$(4.1) \quad \iota_\xi^* tp(\eta) = 0 \quad (\text{for } \xi \not\subset \bar{\eta}), \quad \iota_\eta^* tp(\eta) = c_{top}(\nu_\eta)$$

and the system of these equations has a unique solution.

This gives an effective method for computing the correct form of $tp(\eta)$ by solving equations (4.1), which works well⁷ at least for *simple orbits* of (mono)singularities [31], [7], [15]. Also for multi-singularities, this method is used to determine the residual polynomials appearing in $n(\eta_{ml})$ [14], [15].

Example 4.1. (Computing Tp for degeneracy loci (1.2))

For instance, let us see a very quick proof of the Gambielli-Thom-Porteous formula $tp(\Sigma^k) = \Delta_{l^k}(c)$ in Example 1.2, that should be compared with proofs using the embedded resolution of $\bar{\Sigma}^k$, see [30], [16], [19] (also [10] Chap. 14; [32], [5]). Let $G = GL(m) \times GL(n)$, $V = \text{Hom}(\mathbb{C}^m, \mathbb{C}^n)$ (the space of complex $n \times m$ matrices) and $\eta = \Sigma^k$. We shall determine the universal form of $tp(\Sigma^k)$. This case, it is enough to use only the last equation in (4.1). Put $l = n - m + k$.

Take a representative h in Σ^k in a standard way, then the tangent space of the orbit at the point h consists of all matrices in the following form:

$$h = \left[\begin{array}{c|c} I_{m-k} & O \\ \hline O & O \end{array} \right] \in \Sigma^k, \quad T_h \Sigma^k = \left\{ \left[\begin{array}{c|c} * & * \\ \hline * & O \end{array} \right] \right\} \subset T_h V = V.$$

Note that the normal space is isomorphic to $\text{Hom}(\ker h, \text{coker } h)$. The stabilizer group of $h \in \Sigma^k$, denoted by G_k , consists of pairs of square matrices

$$\left(\left[\begin{array}{c|c} P & O \\ \hline * & A \end{array} \right], \left[\begin{array}{c|c} P & * \\ \hline O & B \end{array} \right] \right), \quad (A, B, P) \in GL(k) \times GL(l) \times GL(m-k).$$

⁷In general the space V contains continuous families of orbits (moduli of orbits). In that case, by dimensional reason, this method can determine $tp(\eta)$ for all the orbits η whose codimension is less than the codimension of moduli strata.

Let ρ_1, ρ_2, ρ_3 be the representations of G_k on $\ker h$, $\operatorname{coker} h$, $\operatorname{Im} h$, respectively, and put $c(\rho_1) = \prod_{i=1}^k (1 + a_i)$ and $c(\rho_2) = \prod_{j=1}^l (1 + b_j)$. Then using (4.1) we have

$$\iota_k^* tp(\Sigma^k) = c_{top}(\nu_k) = c_{kl}(\rho_1^* \otimes \rho_2) = \prod (b_j - a_i) = \Delta_{l^k}(c(\rho_2 - \rho_1))$$

(the last expression of the resultant is classical).

Denote by c'_i and c''_j universal Chern classes for $GL(m)$ and $GL(n)$, respectively, and then $H^*(BG) = \mathbb{Z}[c'_1, \dots, c'_m, c''_1, \dots, c''_n]$. The representation of G_k on the source and the target are $\lambda_1 = \rho_1 \oplus \rho_3$ and $\lambda_2 = \rho_2 \oplus \rho_3$. Thus $\iota_k^* : H^*(BG) \rightarrow H^*(BG_k)$ is determined by $c' \mapsto c(\lambda_1)$ and $c'' \mapsto c(\lambda_2)$, and in particular, it sends

$$1 + c_1 + c_2 + \dots := \frac{1 + c'_1 + \dots + c''_n}{1 + c'_1 + \dots + c'_m} \mapsto \frac{c(\lambda_2)}{c(\lambda_1)} = 1 + c_1(\rho_2 - \rho_1) + \dots$$

Hence $\iota_k^* \Delta_{l^k}(c) = \Delta_{l^k}(c(\rho_2 - \rho_1))$. Finally it is easily checked that ι_k^* is *injective* for degree $\leq kl$, thus it follows that $tp(\Sigma^k) = \Delta_{l^k}(c)$. $\square$

Example 4.2. (Computing Tp for simple ICI singularities)

Let ξ be a $\mathcal{K}$ -orbit listed in Example (2.1) of ICIS curve-germs $\mathbb{C}^{n+1}, 0 \rightarrow \mathbb{C}^n, 0$. It is finitely determined, so the maximal reductive stabilizer group G_ξ can be taken in $J^1\mathcal{K} = GL(n+1) \times GL(n)$ as mentioned in §2 (b). Furthermore the normal form is quasi-homogeneous, thus we may assume⁸ $G_\xi \simeq \mathbb{C}^*$. Let α denote the canonical 1-dimensional representation of $\mathbb{C}^*$ and put $a = c_1(\alpha) \in H^2(B\mathbb{C}^*)$.

It is easy to write down in terms of weights of the normal forms the induced homomorphism $\iota_\xi^* : H^*(BK) = H^*(BJ^1\mathcal{K}) \rightarrow H^*(BG_\xi)$ and the normal Euler class $c_{top}(\nu_\xi) \in H^*(BG_\xi)$. For example, let ξ be $A_3 : (x, y) \mapsto x^4 + y^2$. The representations of $G_{A_3} = \mathbb{C}^*$ on the source and target are $\lambda_1 = \alpha \oplus \alpha^2$ and $\lambda_2 = \alpha^4$. In particular $\iota_{A_3}^*$ sends

$$1 + c_1 + c_2 + \dots := \frac{c''}{c'} \mapsto \frac{c(\lambda_2)}{c(\lambda_1)} = \frac{1 + 4a}{(1 + a)(1 + 2a)} = 1 + a - 5a^2 + \dots$$

The group G_{A_3} acts also on the normal to the orbit in $\mathcal{V}$, such as $\tilde{\lambda}_1 = \alpha \oplus \alpha^2 \oplus \alpha^2 \oplus \alpha^3$ on parameters (w_1, w_2, u_1, u_2) where one expresses the normal vectors by $w_1 x^3 \mathbf{e} + w_2 y \mathbf{e} + u_1 x^2 \mathbf{e} + u_2 x \mathbf{e}$ as mentioned at the end of §2 (or equivalently, $\tilde{\lambda}_1 = \lambda_1 \oplus \alpha^2 \oplus \alpha^3$ on the source space (x, y, u_1, u_2) of the miniversal unfolding of A_3 in (1.3), since G_{A_3} is also the stabilizer of the unfolding). Thus $c_{top}(\nu_{A_3}) = c_{top}(\tilde{\lambda}_1) = 12a^4$.

By using these data and (4.1), Thom polynomials $tp(\eta)$ for simple ICIS can be computed. Let $\eta = D_5$, for instance. Since it has codimension 6, we may set $tp(D_5) = \sum a_I c^I$ of degree 6 with *unknown coefficients* a_I , thanks to the existence theorem of tp . The equations (4.1) are $\iota_\xi^* tp(D_5) = 0$ for $\xi = A_0, \dots, A_5, S_5$ and $\iota_{D_5}^* tp(D_5) = c_6(\nu_{D_5})$.

⁸An exceptional case is $A_1 : (x, y) \mapsto xy$, which has two obvious symmetries $\mathbb{C}^* \times \mathbb{C}^*$.

These form a system of linear equations of unknowns a_I (e.g., in case $\xi = A_3$, substitute $c_1 = a, c_2 = -5a^2, \dots$ in $tp(D_5) = \sum a_I c^I$, then the resulting coefficient of a^6 must be zero). Solving the linear equations provides the correct answer:

$$\begin{aligned} tp(D_5) &= 4c_1^6 - 2c_1^4c_2 - 18c_1^3c_3 - 6c_1^2c_4 + 12c_1^2c_2^2 \\ &\quad + 2c_1c_5 + 12c_1c_2c_3 - 4c_2^3 - 4c_3^2 + 4c_2c_4 \\ &= 24\Delta_{42} + 12\Delta_{411} + 24\Delta_{33} + 64\Delta_{321} + 26\Delta_{3111} \\ &\quad + 24\Delta_{222} + 42\Delta_{2211} + 4\Delta_{111111}. \end{aligned}$$

The last expansion is with respect to the Schur function basis, and it is observed that all the coefficients are non negative. In fact the *positivity* holds in general, see Pragacz-Weber [29].

We remark that it must be hard to find the precise forms of $tp(A_4), tp(A_5), tp(D_5)$ etc by the traditional approach using embedded resolutions.

Example 4.3. (Schubert varieties and Tp)

Let $m \leq n$ and $G = GL(m) \times GL(n)$, $V = \text{Hom}(\mathbb{C}^m, \mathbb{C}^n)$. Note that each linear map in the G -orbit Σ^{m-r} determines a r -dimensional subspace in $\mathbb{C}^n$, and that any linear maps having the same image can be translated each other by the source changes $GL(m)$. Now fix a complete flag in the target linear space $\mathbb{C}^n$ by using coordinates and put $G^\Delta := GL(m) \times \{ \text{lower triangular matrices in } GL(n) \}$, the subgroup of G which preserves the flag. Then the G -orbit Σ^{m-r} breaks into several G^Δ -orbits, which enjoy the one-to-one correspondence

$$G^\Delta\text{-orbits in } \Sigma^{m-r} \xrightarrow{1:1} \text{Schubert cells in } \text{Grass}(r, \mathbb{C}^n).$$

We may write $H_{G^\Delta}^*(V) = \mathbb{Z}[c'_1, \dots, c'_m, b_1, \dots, b_n]$, where b_i are Chern roots (corresponding to the maximal torus of $GL(n)$). In [9] it is shown that the Thom polynomial of a G^Δ -orbit (in $H_{G^\Delta}^*(V)$) coincides with a *double Schur polynomial* in the sense of [16], and that the specialization under all $b_j = 0$ gives a *Schur polynomial* $\Delta_\lambda(c')$. This picture should be related to equivariant Schubert classes in $H_T^*(\text{Grass}(r, \mathbb{C}^n))$ and their localization [18] (also [24]; see [11], [12] for other classical groups).

§ 5. Current interests

Finally I comment about a few topics in current interests working in progress.

1. *Computing Tp for moduli strata of K-orbits:* As seen above, to compute tp for simple $\mathcal{K}$ -orbits (up to certain codimension) is basically possible. The next objects are moduli (continuous families) of $\mathcal{K}$ -orbits, such as unimodular singularities, Thom-Boardmann strata $\Sigma^{i,j,k,\dots}$ etc, for definition, see [1], [23]. Although $tp(\Sigma^{i,j})$ have

already been studied in [32] by the desingularization method (for tp of some $\Sigma^{i,j,k}$, see [5]), the restriction method approach to $tp(\Sigma^{i,j})$ dealt in [6] gives a new insight.

2. *Generating functions of Tp* : It is natural to ask what each coefficient arising in $tp(\eta)$ (with respect to monomial basis or Schur function basis) does mean, or ask whether there is a universal rule for the appearance of such numbers. On one hand, there are several ‘stems’ in classifications of mono-singularities $\mathbb{C}^m, 0 \rightarrow \mathbb{C}^{m+\ell}, 0$ such as $A_\mu, D_\mu, \dots$ or multi-singularities A_1^s ($s = 1, 2, \dots$) etc. So it is reasonable to think of generating functions of tp ’s for such series of singularities where μ, ℓ or s is regarded as a parameter, see [3], [9], [15].
3. *Equivariant Chern classes for singularities (higher degree generalization of Tp)*: In general, singular varieties (such as orbit closures $\bar{\eta}$) admit several variants of ‘Chern (homology) classes’, and usually the top term of such classes is the fundamental class of the variety. The most useful functorial Chern class theory is *the Chern-Schwartz-MacPherson class* [21], and the G -equivariant version of CSM classes has been established in [25]. Applying this theory to the orbit closure $\bar{\eta}$ in a G -affine space V , we obtain a non-homogeneous series in $H^*(BG)$ whose lowest degree homogeneous term is just $tp(\eta)$, since $tp(\eta)$ corresponds to the equivariant fundamental class of the orbit closure. Such a ‘total class version’ of tp is a meaningful class in $\ker \alpha$ mentioned in §3 (i.e., the ideal of G -characteristic classes having supports on $\bar{\eta}$), which has information about the combinatorial structure of adjacencies of G -orbits and several invariants, cf. [25], [26], [27]. On one hand there are some examples of computational aspects: Chern-Schwartz-MacPherson classes of degeneracy loci $\overline{\Sigma^k}$ and of Schubert varieties have already been studied in [28] and [2], respectively.

The author would like to thank the referee for his useful comments. The author was partially supported by Grant-in-Aid for Scientific Research (No.17340013), JSPS.

References

- [1] V. I. Arnold, S. M. Gusein-Zade, and A. N. Varchenko, *Singularities of Differential Maps, Vol.I*, Monographs in Math. 82, Birkhäuser (1985).
- [2] P. Aluffi and L. Mihalcea, Chern classes of Schubert cells and varieties, to appear in *Jour. Algebraic Geometry*, math.AG/0607752.
- [3] G. Bérczi and A. Szenes, Thom polynomials of Morin singularities, math.AT/0608285.
- [4] D. Edidin and W. Graham, Equivariant intersection theory, *Invent. Math.* 131 (1998), 595–634.
- [5] J. Damon, *Thom Polynomials for Contact Class Singularities*, Dissertation, Harvard University (1972).

- [6] B. Kömüves and L. Fehér, On second order Thom-Boardman singularities, *Fund. Math.* 191 (2006), 249–264.
- [7] L. Fehér and R. Rimányi, Calculation of Thom polynomials and other cohomological obstructions for group actions, Real and Complex Singularities (Sao Carlos, 2002), *Contemp. Math.* 354, Amer. Math. Soc., (2004), 69–93.
- [8] L. Fehér and R. Rimányi, Schur and Schubert polynomials as Thom polynomials – cohomology of moduli spaces, *Cent. Eur. J. Math.* 1 (2003), 418–434.
- [9] L. Fehér and R. Rimányi, On the structure of Thom polynomials of singularities, *Bull. London Math. Soc.* 39 (2007), 541–549.
- [10] W. Fulton, *Intersection Theory*, Springer-Verlag (1984)
- [11] T. Ikeda, Schubert classes in the equivariant cohomology of the Lagrangian Grassmannian, *Adv. Math.* 215 (2007), 1–23.
- [12] T. Ikeda and H. Naruse, Excited Young diagrams and equivariant Schubert calculus, arXiv:math/0703637.
- [13] M. E. Kazarian, Characteristic Classes of Singularity Theory, Arnold-Gelfand Math. Seminars “*Geometry and Singularity Theory*”, Birkhäuser (1997), 325–340.
- [14] M. E. Kazarian, Multisingularities, cobordisms and enumerative geometry, *Russian Math. Survey* 58:4 (2003), 665–724 (*Uspekhi Mat. nauk* 58, 29–88).
- [15] M. E. Kazarian, Thom polynomials, Proc. symposium. “*Singularity Theory and its application*” (Sapporo, 2003), *Adv. Stud. Pure Math.* vol. 43 (2006), 85–136.
- [16] G. Kempf and D. Laksov, The determinantal formula of Schubert calculus, *Acta math.* 132 (1974), 153–162.
- [17] S. L. Kleiman, The enumerative theory of singularities, Proc. Nordic Summer School, “*Real and complex singularities*” (Oslo, 1976), P. Holm ed., Sijthoff and Noordhoff, (1977), 297–396.
- [18] A. Knutson and T. Tao, Puzzles and (equivariant) cohomology of Grassmannians, *Duke Math. Jour.* 119 (2) (2003), 221–260.
- [19] A. Lascoux, Puissance extérieures, déterminants et cycles de Schubert, *Bulletin Soc. Math. France* 102 (1974), 161–179.
- [20] E. Looijenga, *Isolated Singular Points on Complete Intersections*, London Math. Soc. Lect. Notes Series 77, Cambridge Univ. Press (1984).
- [21] R. MacPherson, Chern classes for singular algebraic varieties. *Annals of Math.* 100 (1974) 421–432.
- [22] J. Mather, Stability of C^∞ mappings, III, *Publ. Math. IHES* 35 (1968) 127–156.
- [23] J. Mather, On Thom-Boardman singularities, *Dynamical systems* (ed. Peixoto), Acad. Press, (1973), 195–232.
- [24] L. Mihalcea, Giambelli formulae for the equivariant quantum cohomology of the Grassmannian, to appear in *Trans. Amer. Math. Soc.* math.CO/0506335.
- [25] T. Ohmoto, Equivariant Chern classes of singular algebraic varieties with group actions, *Math. Proc. Cambridge Phil. Soc.* 140 (2006), 115–134.
- [26] T. Ohmoto, Chern classes and Thom polynomials, Proc. Trieste Singularity Summer School and Workshop (ICTP, 2005), World Scientific (2007), 464–482.
- [27] T. Ohmoto, Enumerative theory of singularities and equivariant Chern classes, to appear in *Suugaku* 60 (2008) (in Japanese)
- [28] A. Parusiński and P. Pragacz, Chern-Schwartz-MacPherson classes and the Euler characteristic of degeneracy loci and special divisors, *Jour. Amer. Math. Soc.* 8 (1995), no. 4, 793–817.

- [29] P. Pragacz and A. Weber, Positivity of Schur function expansion of Thom polynomials, *Fund. Math.* 195 (2007), 85–95 (arXiv: math.AG/0605308).
- [30] I. R. Porteous, Simple singularities of maps, *Proceedings of Liverpool Singularities I*, Springer Lecture Notes Math. 192 (1971), 286–307.
- [31] R. Rimányi, Thom polynomials, symmetries and incidences of singularities, *Invent. Math.* 143 (2001), 499–521.
- [32] F. Ronga, Le calcul des classes duals singularités de Boardman d’ordre deux, *Comment. Math. Helv.* 47 (1972), 15–35.
- [33] R. Thom, Les singularités des applications différentiables, *Ann. Inst. Fourier* 6 (1955–56), 43–87.
- [34] B. Totaro, The Chow Ring of a Classifying Space, *Proc. Symposia in Pure Math.* Amer. Math. Soc. 67 (1999), 249–281.
- [35] C. T. C. Wall, A second note on symmetry of singularities, *Bull. London Math. Soc.* 12 (1980), 347–354.