

Elliptic 3-folds and Non-Kähler 3-folds

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Introduction

The purpose of this report is to study the relationship between Calabi-Yau 3-folds with elliptic fibrations and compact non-Kähler 3-folds with $K = 0$, $b_2 = 0$, $q = 0$. The non-Kähler 3-folds referred to here have firstly appeared in Friedman's paper [3]. In this paper he has shown that if there are sufficiently many (mutually disjoint) $(-1, -1)$ -curves on a Calabi-Yau 3-fold, then one can contract these curves and can deform the resulting variety to a smooth non-Kähler 3-fold with $K_2 = 0$, $b_2 = 0$, $q = 0$. For example, in the case of a (general) quintic hypersurface in $\mathbb{P}^4$, one can do this procedure for two lines on it. This phenomenon is analogous to the one for (-2) -curves on a K3 surface. In fact a (-2) -curve on a K3 surface often disappears in deformation, and this fact just says that one can contract this (-2) -curve to a point and can deform the resulting variety to a (smooth) K3 surface. By this phenomenon, we can explain the variance of the Picard number of K3 surface in deformation, and it is well-known that a general point of the moduli space of K3 surfaces corresponds to a non-projective (but Kähler) K3 surface on which there are no (-2) -curves. Taking such a non-projective surface into consideration, one has a famous theorem that two arbitrary K3 surfaces are connected by deformation. But there is a difference between Calabi-Yau 3-folds and K3 surfaces, that is, a $(-1, -1)$ -curve never disappears like a (-2) -curve in deformation. This is closely related to the fact that Calabi-Yau 3-folds have a

large repertory of topological Euler numbers. For the speculation around this area, one may refer to the paper of M. Reid [12]. The main results of this paper is the following:

Theorem A.

Let X be a Calabi-Yau 3-fold which has an elliptic fibration with a rational section. Then the bimeromorphic class of X is obtained as a semi-stable degeneration of a compact non-Kähler 3-fold with $K = 0$, $b_2 = 0$ and $q = 0$, i.e. there is a surjective proper map f of a smooth 4-dimensional variety $\mathbb{X}$ to a 1-dimensional disc Δ such that

- 1) $f^{-1}(t)$ is a compact non-Kähler 3-fold with $K = 0$, $b_2 = 0$, $q = 0$ for $t \in \Delta^*$,
- 2) $f^{-1}(0) = \sum_{i=0}^n X_i$ is a normal crossing divisor of $\mathbb{X}$, and
- 3) X_0 is bimeromorphic to X .

Here we will explain the motivation of the formulation in Theorem A. If there are sufficiently many $(-1, -1)$ -curves on X in the Friedman's sense explained above, one has a flat morphism f of a complex analytic variety $\mathbb{X}$ to a disc Δ whose central fibre is the variety obtained by contraction of these curves, and whose general fibre is a non-Kähler 3-fold with $K = 0$, $b_2 = 0$, $q = 0$. In this situation, $\mathbb{X}_0 := f^{-1}(0)$ has a number of ordinary double points, but one may assume that the total space $\mathbb{X}$ is smooth in a suitable condition (e.g. (1.1) in this report). Next blow up these points. Then the central fibre consists of a number of irreducible components, namely, the smooth variety $\tilde{\mathbb{X}}_0$ obtained by the blowing ups of the ordinary double points on $\mathbb{X}_0$ and the P^3 's corresponding to each point blown up. However this is not yet a semi-stable degeneration because the multiplicity of each P^3 is two. So taking a suitable base change, one has a semi-stable degeneration. This is a typical example of Theorem A.

The Construction of the Proof

In this report, a Calabi-Yau 3-fold means a smooth projective 3-fold with $c_2 \neq 0$, $q = 0$, K trivial. Since $c_2 \neq 0$, those 3-folds are excluded which are, up to étale covers, Abelian 3-folds or the products of $k3$ surfaces and elliptic curves. Here we will briefly review the Friedman's construction of non-Kähler 3-fold with $K = 0$, $b_2 = 0$. Assume that X is a smooth compact 3-fold with K_X trivial and that mutually disjoint $(-1, -1)$ -curves $C_1, \dots, C_n$ are given on X . Here a $(-1, -1)$ -curve means a smooth rational curve $\mathbb{P}^1$ whose normal bundle $N_{\mathbb{P}^1/X}$ is isomorphic to $\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$. Then one can contract these curves to points and one has a compact 3-fold $\bar{X}$ with ordinary double points: $\pi: X \rightarrow \bar{X}$. For simplicity we will write $P_i = \pi(C_i)$, $Z = \coprod_i P_i$ and $C = \coprod_i C_i$. We have the following commutative exact diagram:

$$\begin{array}{ccccccccc}
 0 \rightarrow H^1(\pi_* \theta_X) & \rightarrow & H^1(\theta_X) & \rightarrow & H^0(R^1 \pi_* \theta_X) & \rightarrow & H^2(\pi_* \theta_X) & \rightarrow & H^2(\theta_X) \rightarrow 0 \\
 \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 0 \rightarrow H^1(T_{\bar{X}}^0) & \rightarrow & T_{\bar{X}}^1 & \rightarrow & H^0(T_{\bar{X}}^1) & \xrightarrow{\alpha} & H^2(T_{\bar{X}}^0) & \rightarrow & T_{\bar{X}}^2 \rightarrow 0
 \end{array}$$

In the above diagram, the map α is interpreted as follows:

First we have an isomorphism $\beta: H^0(T_{\bar{X}}^1) \rightarrow H^2(T_{\bar{X}}^0)$ by using the exact sequence defined locally at each P_i :

$$0 \rightarrow T_{\bar{X}}^0 \rightarrow \theta_{\mathbb{C}^4|_{\bar{X}}} \rightarrow \theta_{\bar{X}} \rightarrow T_{\bar{X}}^1 \rightarrow 0.$$

Here we note that (X, P_i) can be embedded into $(\mathbb{C}^4, 0)$ because P_i is an ordinary double point. By the isomorphism β , α is identified with the natural map $H^2(T_{\bar{X}}^0) \rightarrow H^2(T_{\bar{X}}^0)$. In our case it is easily shown

that $\pi_* \theta_X = T_{\bar{X}}^0$. Next using the Leray spectral sequences:

$$H_Z^p(R^q \pi_* \theta_X) \Rightarrow H_C^{p+q}(\theta_X) \quad \text{and} \quad H^p(R^q \pi_* \theta_X) \Rightarrow H^{p+q}(\theta_X), \quad \text{we have}$$

$$H_Z^2(T_{\bar{X}}^0) = H_C^2(\theta_X) \quad \text{and} \quad H^2(T_{\bar{X}}^0) = H^2(\theta_X), \quad \text{which implies that the above}$$

map is identified with the following maps:

$$\begin{array}{ccc} H_C^2(\theta_X) & \rightarrow & H^2(\theta_X) \\ || & & || \\ H_C^2(\Omega_X^2) & \xrightarrow{\theta} & H^2(\Omega_X^2), \end{array}$$

where the vertical identifications come from the fact that K_X is trivial. If the map θ is surjective, then we have $T_{\bar{X}}^2 = 0$. On the other hand, $H^0(T_{\bar{X}}^1) \simeq H_Z^2(T_{\bar{X}}^0) \simeq H_C^2(\Omega_X^2)$ are isomorphic to a n -dimensional vector space $\bigoplus_{i=1}^n \mathbb{C}$, where each factor corresponds to C_i . θ is nothing but the map which associates each basis of the above vector space to the fundamental class of C_i in X . Summing up these results, we have the following fact(1.1):

(1.1) *Let X be a Calabi-Yau 3-fold and $C_1, \dots, C_n$ mutually disjoint $(-1, -1)$ -curves on X . We employ the same notation as above. Then since $H^2(\Omega_X^2) = H^4(X, \mathbb{C}) = H_2(X, \mathbb{C})$ by the Hodge decomposition and the Poincare duality, the map θ can be identified with the map $i_* : \bigoplus_{i=1}^n H_2(C_i, \mathbb{C}) \rightarrow H_2(X, \mathbb{C})$. In particular, if i_* is surjective and there is an element $(a_1, \dots, a_n) \in \text{Ker } i_*$ such that $a_i \neq 0$ for all i , then $\bar{X}$ is deformed to a smooth compact non-Kähler 3-fold with $K = 0$, $b_2 = 0$ and $q = 0$.*

A typical example of (1.1) is a general quintic hypersurface X in $\mathbb{P}^4$ and two lines on it. In this case, since $\text{Pic}(X) = \mathbb{Z}$, it is rather easy to check the conditions in (1.1). But in general it is very difficult to find the curves satisfying the condition in (1.1) even

if a Calabi-Yau 3-fold X is given explicitly. In another sense, (1.1) tell us an interesting example where the class $\mathcal{E}$ is not stable by small deformation. In fact, $\bar{X}$ is a Moishezon space, hence is in class $\mathcal{E}$. But the non-Kähler 3-fold V obtained by a small deformation of $\bar{X}$ is not in class $\mathcal{E}$. This is shown as follows. First one has $h^{0,2}(V) = 0$, because $h^{0,1}(V) = 0$ and $K_V = 0$. If V is in class $\mathcal{E}$, then it is bimeromorphic to some compact Kähler manifold Y . Since $h^{0,2}(V) = 0$, $h^{0,2}(Y) = 0$. In fact, by the desingularization theorem [4], we have a complex manifold $\tilde{V}$ which dominates both V and Y , birationally and properly. Using spectral sequences and Chow lemma [5] for $(\tilde{V}, V)$ and $(\tilde{V}, Y)$, we have the result. But $h^{0,2}(Y) = 0$ implies that Y is a projective manifold. Since the algebraic dimension of V equals to 0, this is a contradiction. So V is not in class $\mathcal{E}$. Since $\kappa(\bar{X}) = 0$, this is a counter-example to a question posed in [2].

To return from the digression, we will explain the construction of the proof. First we define a Weierstrass model.

(1.2) Definition

A Weierstrass model $W(\mathcal{L}, a, b)$ over a variety S is a closed subvariety in $P_S(\mathcal{O} \oplus \mathcal{L}^2 \oplus \mathcal{L}^3)$ defined by the equation $Y^2Z = X^3 + aXZ^2 + bZ^3$, where $\mathcal{L} \in \text{Pic}(S)$, $a \in H^0(S, \mathcal{L}^{-4})$, $b \in H^0(S, \mathcal{L}^{-6})$ and

$$Z: \mathcal{O} \rightarrow \mathcal{O} \oplus \mathcal{L}^2 \oplus \mathcal{L}^3$$

$$X: \mathcal{L}^2 \rightarrow \mathcal{O} \oplus \mathcal{L}^2 \oplus \mathcal{L}^3$$

$$Y: \mathcal{L}^3 \rightarrow \mathcal{O} \oplus \mathcal{L}^2 \oplus \mathcal{L}^3$$

are natural injections.

We denote by Σ , the section of $W(\mathcal{L}, a, b)$ over S defined by $X = Z = 0$, and denote by π the natural projection of $W(\mathcal{L}, a, b)$ to S .

Next let X be a Calabi-Yau 3-fold which has an elliptic fibration with a rational section. Then by [8] (Theorem 3.4), X is birational equivalent to a Weierstrass model $W = W(K_S, a, b)$ with only canonical singularities, where S is one of the following: P^2, Σ_i ($0 \leq i \leq 12$). This is a starting point of our proof. Since W has singularities in the case $S = \Sigma_i$ ($3 \leq i \leq 12$) if we take a and b generally, we must set up the following definition.

(1.3) Definition

Let $W = W(K_S, a, b)$ be a Weierstrass model over $S = \Sigma_i$ ($3 \leq i \leq 12$). Then W is called *general* if

- (1) W has singularities only on $F = \{ p \in W ; p \in \pi^{-1}(D_0), X = Y = 0 \}$, where D_0 is a negative section of S , and
- (2) let mD_0 and nD_0 be the fixed components of $|K_S^{-4}|$ and $|K_S^{-6}|$, respectively, then $\text{div}(a) = G + mD_0$ and $\text{div}(b) = H + nD_0$, where G (resp. H) intersects D_0 transversely and $G.D_0 \cap H.D_0 = \emptyset$. Here $G.H_0$ (resp. $H.D_0$) denotes the intersection of D_0 and G (resp. H).

We will employ Definitions (1.2), (1.3). Let $W = W(\mathcal{L}, a, b)$ be a Weierstrass model over S . Then W is obtained as a double cover of $P_S(\mathcal{O} \oplus \mathcal{L}^2)$ branched over $B = \{ X^3 + aXZ^2 + bZ^3 = 0 \}$. If W has singularities, then we can use the following.

(1.4) Canonical Resolutions

Let Y be a smooth variety and B a reduced Cartier divisor on it. Assume that $\mathcal{O}_Y(B) = L^{\otimes 2}$ for a line bundle L on Y . Then we have a double cover X of Y branched along B . To resolve the singularities on X , we consider the following process:

Perform a succession of monoidal transformations ν_i ($1 \leq i \leq m$) with smooth centres $D_i \subset B_i \subset Y_i$ ($1 \leq i \leq m$), where $Y_1 = Y$, $B_1 = B$, $Y_{j+1} \xrightarrow{\nu_j} Y_j$ and $B_{j+1} = \nu_j^* B_j$ for each j . And if we write $B_m = \bar{B} + \sum_{k=1}^m \mu_k E_k$, where $\bar{B}$ is a proper transform of B by $\nu := \nu_m \circ \dots \circ \nu_1$ and E_k 's are ν -exceptional divisors, then $\bar{B} + \sum_{k, \mu_k: \text{odd}} E_k$ is a smooth divisor. If the above process is possible, then we have a double cover branched along $\bar{B} + \sum_{i, \mu_i: \text{odd}} E_i$ and obtain a smooth variety $\tilde{X}$ which is a resolution of X . We call the above process a canonical resolution.

Let $W = W(K_S, a, b)$ be a general Weierstrass model over $S = \sum_i (3 \leq i \leq 12)$ in the sense of Definition(1.3). Then we can perform a canonical resolution on W . In our case, it is easily verified that $\text{Sing}(W) = \{ q \in \mathbb{P} ; q \in p^{-1}(D_0), X = 0, Y = 0 \}$, where $\mathbb{P} = \mathbb{P}_S(\mathcal{O} \oplus K_S^2 \oplus K_S^3)$, D_0 : negative section, and that the singularities are locally trivial deformations of a rational double points except for a finite number of points which are so-called *dissident* points. So the problem is how to overcome the difficulties which arise at these dissident points. For example, consider the case where $i = 5$. (In the case where $i = 3, 4, 6, 8, 12$ there are no dissident points.) Since G and H never vanish simultaneously on a point q of $\text{Sing}(W)$ in Definition(1.3), we may consider two cases: (1) only G vanishes at q , and (2) only H

vanishes at q . But it follows that q is dissident only in the case (2). So we may consider the situation where $q = (0,0,0,0)$, $W: y^2 = x^3 + t^3x + st^4$ in (x,y,s,t) -space ($= \mathbb{C}^4$). Then the process of a canonical resolution will be found in (Figure 1). As a consequence, we have the following Proposition.

(1.5) Proposition

Let $W = W(K_S, a, b)$ be a Weierstrass model over S , where S is one of the following: $\mathbb{P}^2, \Sigma_i$ ($0 \leq i \leq 12$). Then:

$$(0) K_W = \mathcal{O}_W$$

(1) In the case $S = \mathbb{P}^2$ or Σ_i ($0 \leq i \leq 2$), a general Weierstrass model W is smooth and $\text{Pic}(W) = \pi^* \text{Pic}(S) \oplus \mathbb{Z}[\Sigma]$. Moreover W is simply-connected.

(2) In the case $S = \Sigma_i$ ($3 \leq i \leq 12$), a general Weierstrass model W has canonical singularities such that $\text{Sing}(W) \simeq \mathbb{P}^1$ and that they are locally trivial deformations of rational double points except for finite number of points. Moreover W has a canonical resolution $\mu: \tilde{W} \rightarrow W$ such that a) $\tilde{W} \rightarrow S$ is a flat morphism if $3 \leq i \leq 8$ or $i = 12$, b) if we view $\tilde{W}$ and W as fibre spaces over $\mathbb{P}^1$ by means of the ruling $S \rightarrow \mathbb{P}^1$, then, for a general point $t \in \mathbb{P}^1$, $\mu_t: \tilde{W}_t \rightarrow W_t$ is a minimal resolution of a surface with rational double points, and c) $K_{\tilde{W}} = \mathcal{O}_{\tilde{W}}$.

Remark. In the case where $9 \leq i \leq 11$, $\tilde{W}$ is not flat over S . But we can factorize μ into $\tilde{W} \rightarrow \bar{W} \rightarrow W$, where $\bar{W}$ is a normal variety with the singularities which are locally trivial deformations of a rational double point of A_1 -type along C_i ($1 \leq i \leq r$), where C_i ($1 \leq i \leq r$) denote mutually disjoint smooth rational curves on $\bar{W}$. Moreover $\bar{W}$ is

flat over S , and $\tilde{W} \rightarrow \bar{W}$ is a trivial resolution of the above singularities. For details, see (Figures 1,2).

(3) For an arbitrary point $t \in P^1$ except for a countable number of points, $\tilde{W}_t$ is naturally an elliptic K3 surface and its Mordell Weil group is trivial.

(4) Let E_j ($1 \leq j \leq m$) be μ -exceptional divisors. Then $\text{Pic}(\tilde{W}) = (\pi_0\mu)^*\text{Pic}(S) \oplus \bigoplus_{j=1}^m \mathbb{Z}[E_j]$.

(5) $\tilde{W}$ is simply-connected.

Theorem A'

Let W and $\tilde{W}$ be a general Weierstrass model and its resolution as above. Then we have:

(1) In the case $S = P^2$ or Σ_i ($0 \leq i \leq 2$), there are mutually disjoint $(-1, -1)$ -curves $C_1, \dots, C_4$ on W such that $i_* : \bigoplus_{i=1}^4 H_2(C_i, \mathbb{C}) \rightarrow H_2(W, \mathbb{C})$ is surjective and that one can obtain, by the procedure of (1.1), a smooth compact non-Kähler 3-fold with $K = 0$, $b_2 = 0$ and $q = 0$.

(2) In the case $S = \Sigma_i$ ($3 \leq i \leq 12$), there are mutually disjoint $(-1, -1)$ -curves $C_1, \dots, C_{n(i)}$ on the variety $\tilde{W}'$ which is obtained from $\tilde{W}$ by the composite of flops of $(-1, -1)$ -curves. For C_i 's, $i_* : \bigoplus_{j=1}^{n(i)} H_2(C_j, \mathbb{C}) \rightarrow H_2(\tilde{W}', \mathbb{C})$ is surjective, and one can obtain, by the procedure of (1.1), a smooth compact non-Kähler 3-fold with $K = 0$, $b_2 = 0$ and $q = 0$.

Example (without proof)

Set $S = \Sigma_0$. Let $p: S \rightarrow P$ denote one of its ruling. Then $\text{po}n: W \rightarrow P^1$ is a K3-fibration. Let l_1 and l_2 be mutually distinct

fibre of p , and let D_1 and D_2 be mutually distinct section of p with

$(D_i)^2 = 0$. Note that $\pi^{-1}(l_i) \xrightarrow{g_i} l_i$ ($i = 1, 2$) and $\pi^{-1}(D_i) \xrightarrow{h_i} D_i$ ($i = 1, 2$) are elliptic $K3$ with canonical sections which comes from Σ , respectively. Let us consider the following four mutually disjoint $(-1, -1)$ -curves:

C_1 : a section of g_1 with $(C_1, \Sigma)_W = 0$

C_2 : a section of g_2 with $(C_2, \Sigma)_W = 1$

C_3 : a section of h_1 with $(C_3, \Sigma)_W = 0$

C_4 : a section of h_2 with $(C_4, \Sigma)_W = 1$.

Then the condition in (1.1) is satisfied.

As for Theorem A' (in particular Theorem A'(2)), it is impossible to give a proof in this report. Details will be found in [10].

The aim of this report is to explain how to derive Theorem A from Theorem A'. Let X be a Calabi-Yau 3-fold which has an elliptic fibration with a rational section. Then, as is mentioned before, X is birational equivalent to a Weierstrass model W with only canonical singularities. But W is not general in the sense of Definition(1.3). Though W has only canonical singularities, its singularities are possibly worse than the ones described in Definition(1.3). Let us consider the complete linear system $|Z|$ on $P_S(\mathcal{O} \oplus K_S^2 \oplus K_S^3)$, where $Z = \mathcal{O}_P(1) \oplus \pi^* K_S^{-6}$ and $\mathcal{O}_P(1)$ is a tautological line bundle of $P(\mathcal{O} \oplus K_S^2 \oplus K_S^3)$. Let Λ be a sublinear system of $|Z|$ which consists of the elements of the following form :

$$\varphi_1 Y^2 Z + \varphi_2 X^3 + \varphi_3 X Z^2 + \varphi_4 Z^3 = 0 ,$$

where $\varphi_1, \varphi_2 \in H^0(S, \mathcal{O}_S)$, $\varphi_3 \in H^0(S, K_S^{-4})$ and $\varphi_4 \in H^0(S, K_S^{-6})$. Then consider the universal family over $T = \mathbb{P}(\Lambda)$, $g: \mathbb{W} \rightarrow T$. Assume that $g^{-1}(t_0) = W$. If we choose a general point t on T , then $\mathbb{W}_t = g^{-1}(t)$ has the property in Theorem A'. Let C be a curve in T passing through t_0 and t . Then we have a family of Weierstrass models over C , which we denote again by $g: \mathbb{W} \rightarrow C$. In the case where $S = \mathbb{P}^2$ or Σ_i ($0 \leq i \leq 2$), a general fibre of g is smooth. But in the case where $S = \Sigma_i$ ($3 \leq i \leq 12$), a general fibre has singularities by Proposition (1.5)(2). In this case we have the following proposition.

(1.6) Proposition

Let S be a surface isomorphic to Σ_i ($3 \leq i \leq 12$) and C a curve. Consider the following flat family of Weierstrass models over S :

$$\begin{array}{ccccc} P_S(\mathcal{O} \oplus K_S^2 \oplus K_S^3) \times C & & & & \\ \cup & & \searrow & & \\ \mathbb{W} & \xrightarrow{\quad} & S \times C & & \\ g \searrow & & \swarrow p_1 \quad \searrow p_2 & & \\ & C & & S & \end{array}$$

$$\mathbb{W}: Y^2 Z = X^3 + a X Z^2 + b Z^3, \quad a \in H^0(S \times C, p_2^* K_S^{-4}), \\ b \in H^0(S \times C, p_2^* K_S^{-6})$$

Assume that $\mathbb{W}_t$ is general for every $t \in C$ except for a finite number of points $\{t_1, \dots, t_n\}$. Then there is a projective resolution $\Xi: \tilde{\mathbb{W}} \rightarrow \mathbb{W}$ such that $\Xi_t: \tilde{\mathbb{W}}_t \rightarrow \mathbb{W}_t$ becomes the resolution in Proposition(1.5) for every $t \notin \{t_1, \dots, t_n\}$.

So we have a flat projective morphism $\tilde{g} : \tilde{W} \rightarrow C$ whose general fibre is smooth. For a general point $t \in C$, $\tilde{W}_t$ satisfies Theorem A'(2), that is, there is a sequence of flops of $(-1, -1)$ -curves $D_j \subset \tilde{W}_t^{(j)}$:

$$\begin{array}{ccccccc} \tilde{W}_t^{(0)} & \dashrightarrow & \tilde{W}_t^{(1)} & \dashrightarrow & \tilde{W}_t^{(2)} & \dots\dots\dots & \dashrightarrow & \tilde{W}_t^{(m)} \\ || & & & & & & & || \\ \tilde{W}_t & & & & & & & \tilde{W}_t' \end{array}$$

and there are $(-1, -1)$ -curves on $\tilde{W}_t'$ to be contracted. Let us consider the irreducible component H of $\text{Hilb } \tilde{W} / C$ which contains $[D_0]$. Note that $\text{Hilb } \tilde{W} / C$ is etale over C at $[D_0]$ because D_0 is a $(-1, -1)$ -curve on $\tilde{W}_t$. Hence H is determined uniquely, and H is etale over C at $[D_0]$. Taking a suitable finite cover of C , we may assume that H_{red} is birational to C . Then we have the following diagram:

$$\begin{array}{ccc} \mathcal{D} & \subset & \tilde{W} \\ \swarrow & & \searrow \\ & C & \end{array} ,$$

where $\mathcal{D}_t$ is a $(-1, -1)$ -curve on $\tilde{W}_t$ for every point $t \in C^*$: a Zariski open subset of C . Restrict $\tilde{g} : \tilde{W} \rightarrow C$ to C^* , and consider $\tilde{g}^* : \tilde{W}^* \rightarrow C^*$. Then by [1] (Corollary 6.10), we can perform a flop of $\mathcal{D}^*$ relatively over C^* , and get $\tilde{g}^{(1)*} : \tilde{W}^{(1)*} \rightarrow C^*$. Here $\tilde{W}^{(1)*}$ is in general not a scheme, but an algebraic space. We can compactify $\tilde{W}^{(1)*}$ by [1] and have a proper surjective map $\tilde{g}^{(1)} : \tilde{W}^{(1)} \rightarrow C$. $\tilde{W}^{(1)}$ is assumed to be smooth by [4], and $\tilde{W}^{(1)}$ is birational to $\tilde{W}$ over C . Since $\tilde{W}_{t_0}$ contains an irreducible component birational to W and both $\tilde{W}$ and $\tilde{W}^{(1)}$ are smooth, $\tilde{W}_{t_0}^{(1)}$ also contains an irreducible

component birational to W . As a consequence, by repeating this process, we may assume from the first that there are $(-1, -1)$ -curves to be contracted on $\tilde{V}_t$. In the case where $S = \Sigma_i (3 \leq i \leq 12)$, we consider $\tilde{g} : \tilde{V} \rightarrow C$ which is obtained by repeating above process. And in the case $S = P^2$ or $\Sigma_i (0 \leq i \leq 2)$, we consider the original $g : V \rightarrow C$. Then we can use the following, if necessary, after a base change by a finite cover of C :

(1.7) Proposition

Let $h: \mathcal{Y} \rightarrow C$ be a proper flat morphism with connected fibres of an irreducible smooth 4-dimensional algebraic space $\mathcal{Y}$ to a smooth curve C . Let $t_0 \in C$ be a fixed point and W an irreducible component of $\mathcal{Y}_{t_0}$. Assume that there is a proper flat family of curves in $\mathcal{Y}$:

$$\begin{array}{ccc} \mathcal{Y}_i & \subset & \mathcal{Y} \\ \text{flat} \searrow & & \swarrow \\ \text{proper} & C & \end{array} \quad (1 \leq i \leq n)$$

such that for a general point $t \in C$, (1) $\mathcal{Y}_{i,t} (1 \leq i \leq n)$ are mutually disjoint $(-1, -1)$ -curves on $\mathcal{Y}_t$, (2) these curve satisfy the condition in (1.1), and (3) we can obtain from $\mathcal{Y}_t$ a non-Kähler 3-fold with $K = 0$, $b_2 = 0$ and $q = 0$ by the process in (1.1). Then there is a proper surjective map of a 4-dimensional complex manifold $\mathcal{X}$ to a 1-dimensional disc Δ such that

- 1) $f^{-1}(t)$ is a compact non-Kähler 3-fold with $K = 0$, $b_2 = 0$ and $q = 0$, for $t \in \Delta^*$,
- 2) $f^{-1}(0) = \sum_{i=1}^n W_i$ is a normal crossing divisor of $\mathcal{X}$, and
- 3) W_0 is bimeromorphic to W .

Proof) Let C^* be a suitable Zariski open subset in C . Then by [1] (Corollary 6.10), we can contract $\mathcal{G}_t^*$'s on $\mathcal{Y}^*$ relatively over C^* , and obtain $\bar{\mathcal{Y}}^*$. We can compactify $\bar{\mathcal{Y}}^*$ and have proper flat map of a normal algebraic space $\bar{\mathcal{Y}}$ to C . Then $\bar{\mathcal{Y}}$ is birational to $\mathcal{Y}$ over C . Consider the function field K of $\mathcal{Y}$, and let v be a discrete valuation ring which corresponds to W . Let L be a suitable Galois extension of K , then the normalizations of $\mathcal{Y}$ and $\bar{\mathcal{Y}}$ in L become schemes by the argument of [1] (proposition 1). Denote them by $\mathcal{X}$ and $\bar{\mathcal{X}}$, respectively. Then $\mathcal{Y}$ (resp. $\bar{\mathcal{Y}}$) is the quotient of $\mathcal{X}$ (resp. $\bar{\mathcal{X}}$) by the Galois group $G = \text{Gal}(L/K)$. Let $v_1, \dots, v_k$ be the extension of v in L . Then each element $g \in G$ induces a permutation of v_i 's. If g sends v_i to v_j , then we will write $j = g(i)$. By [7] (p.153), for v_1 , there is a variety $\bar{\mathcal{X}}_{v_1}$ birational to $\bar{\mathcal{X}}$ such that (1) $\bar{\mathcal{X}}_{v_1}$ is projective over $\bar{\mathcal{X}}$, (2) $\bar{\mathcal{X}}_{v_1} \supset \{z \in \bar{\mathcal{X}}; \bar{\mathcal{X}} \text{ is isomorphic to } \mathcal{X} \text{ at } z\}$, and (3) if v_1 dominates a point y of $\bar{\mathcal{X}}_{v_1}$ and a point y' on $\mathcal{X}$, then $\mathcal{O}_y$ dominates $\mathcal{O}_{y'}$. In this case we may assume that $\bar{\mathcal{X}}_{v_1}$ and $\bar{\mathcal{X}}$ are isomorphic at every point except for points over $t_0 \in C$. We denote $\bar{\mathcal{X}}_{v_1}$ by $\bar{\mathcal{X}}_1$, and define $\bar{\mathcal{X}}_g$ for each $g \in G$ by the following fibre product:

$$\begin{array}{ccc}
 \bar{\mathcal{X}}_1 & \xleftarrow{\sim} & \bar{\mathcal{X}}_g \\
 \downarrow & & \downarrow \\
 \bar{\mathcal{X}} & \xleftarrow[\bar{g}_*]{\sim} & \bar{\mathcal{X}}
 \end{array}$$

On the other hand, viewing $\bar{\mathcal{X}}_g$'s as $\bar{\mathcal{X}}$ -schemes, we have :

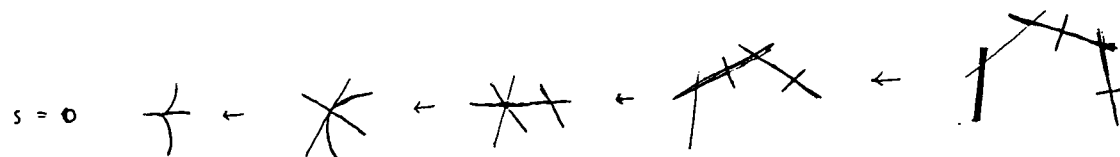
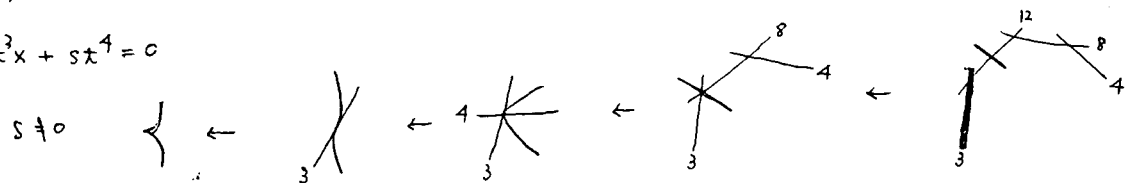
$$\begin{array}{c}
 \text{birat.} \\
 \bar{\mathcal{X}}_1 \rightarrow \bar{\mathcal{X}}_{g_1} \rightarrow \dots \rightarrow \bar{\mathcal{X}}_{g_l} \\
 \swarrow \quad \searrow \quad \swarrow \\
 \bar{\mathcal{X}} \quad G = \{1, g_1, \dots, g_l\}.
 \end{array}
 \quad (*)$$

Take a closure $\tilde{\mathcal{X}}$ of the graph of $(*)$ in $\prod_g \bar{\mathcal{X}}_g$, and embed $\tilde{\mathcal{X}}$ into $\prod_g \bar{\mathcal{X}}$ in such a way that $z \rightarrow \prod_g z$. Then the natural projection from $\prod_g \bar{\mathcal{X}}_g$ to $\prod_g \bar{\mathcal{X}}$ induces a projective morphism of $\tilde{\mathcal{X}}$ to $\bar{\mathcal{X}}$. Consider the action of G to $\prod_g \bar{\mathcal{X}}$ defined in such a way that g sends g_i -th factor $\bar{\mathcal{X}}$ of $\prod_g \bar{\mathcal{X}}$ to gg_i -th factor $\bar{\mathcal{X}}$ of $\prod_g \bar{\mathcal{X}}$ and that this map of $\tilde{\mathcal{X}}$ to itself coincides with the natural g -action of $\bar{\mathcal{X}}$. Clearly $\tilde{\mathcal{X}}$ is stable by this G -action, and this action coincides with the original G -action of $\bar{\mathcal{X}}$. So, the natural G -action is induced on $\tilde{\mathcal{X}}$. If we take a normalization of $\tilde{\mathcal{X}}$, then the action of G naturally extends. So we may assume that $\tilde{\mathcal{X}}$ is normal. Then the quotient $\bar{\mathcal{Y}}$ of $\tilde{\mathcal{X}}$ by G is an algebraic space by [6] (p. 183-184) [15], and we have a birational morphism of $\bar{\mathcal{Y}}$ to $\bar{\mathcal{X}}$. This morphism is an isomorphism over a general point $t \in C$. But by the construction, $\bar{\mathcal{Y}}_{t_0}$ contains an irreducible component birational to W . So, from the first, we may assume that $\bar{\mathcal{Y}}_{t_0}$ has an irreducible component birational to W . Now let us consider the Kuranishi space $(\mathcal{U}, u_0)$ of $\bar{\mathcal{Y}}_{t_0}$, which is a complex space and has versal property at every point u near u_0 [13][14]. On the other hand, $\bar{\mathcal{Y}}_t$ can be deformed to a non-Kähler 3-fold with $K = 0$, $b_2 = 0$ and $q = 0$ for every point t near t_0 , which implies that there is a flat deformation $f: \mathcal{X} \rightarrow \Delta$ such that $f^{-1}(0) = \bar{\mathcal{Y}}_{t_0}$ and that $f^{-1}(t)$ is a non-Kähler 3-fold with $K = 0$, $b_2 = 0$ and $q = 0$. Then the semi-stable reduction for f is a desired one. Q. E. D.

Figure 1.

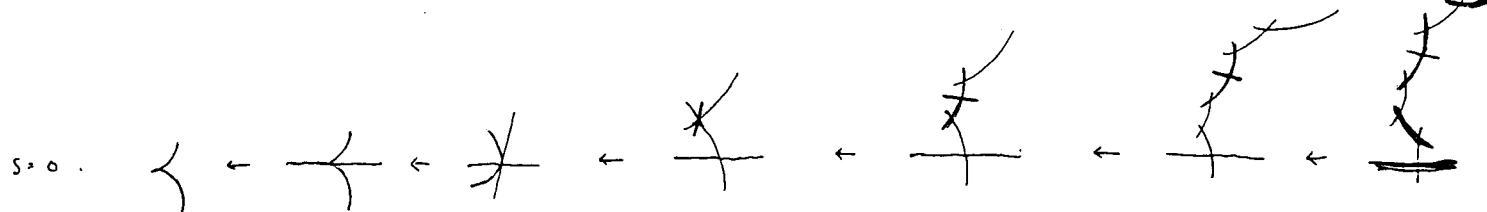
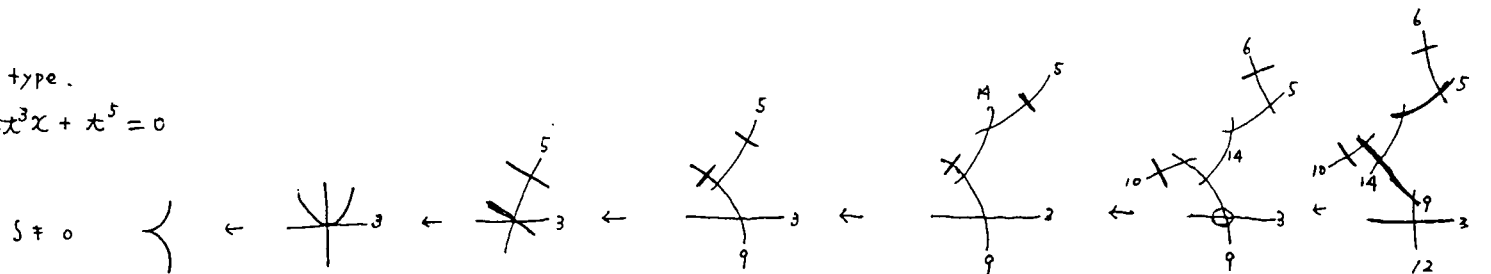
 E_6 -type

$$B: \quad x^3 + tx + st^4 = 0$$



E₇ - type.

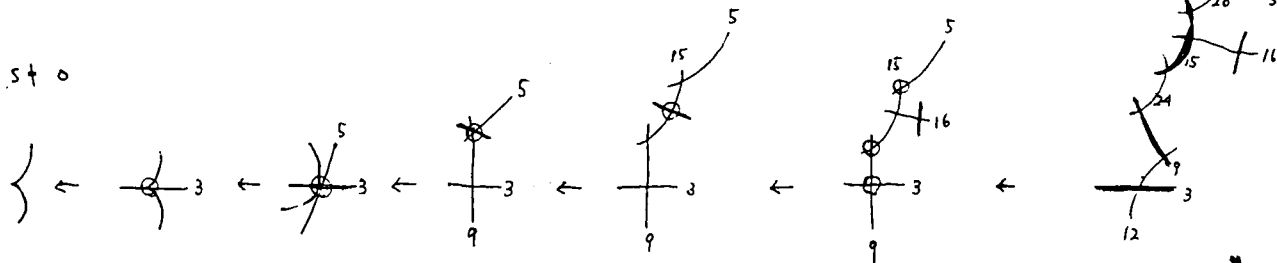
13: $x^3 + 5x^3x + x^5 = 0$



E_8 -type

B: $x^3 + x^4 + sx^5 = 0$

$s \neq 0$



$s = 0$

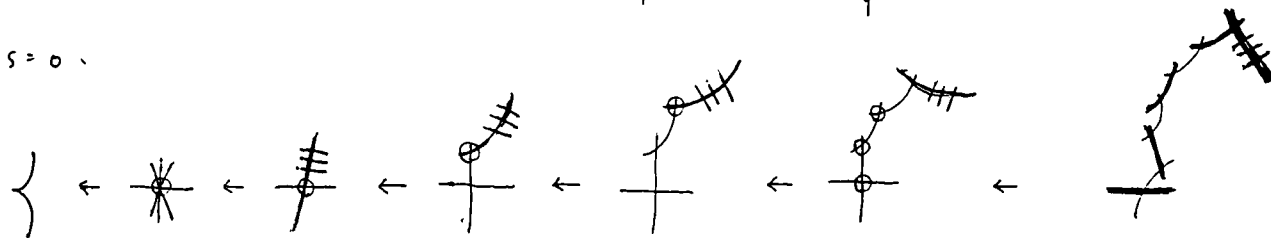
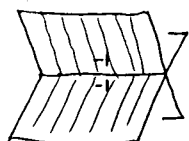
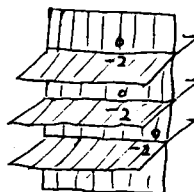


Figure 2. (μ -exceptional divisors)

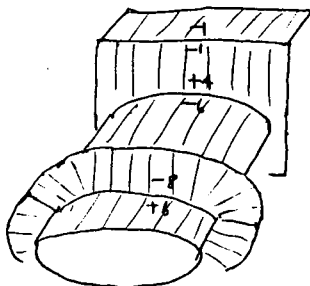
$\tilde{i} = 3$ (A_2 -type)



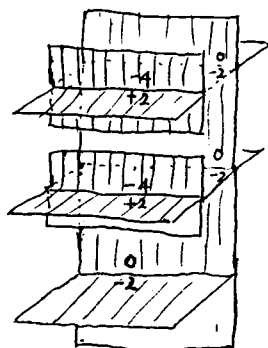
$\tilde{i} = 4$ (D_4 -type)



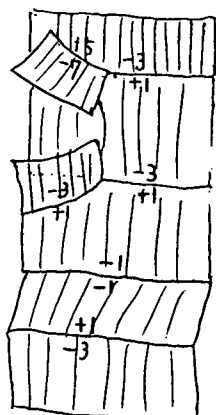
$\tilde{i} = 5$ (E_6 -type)



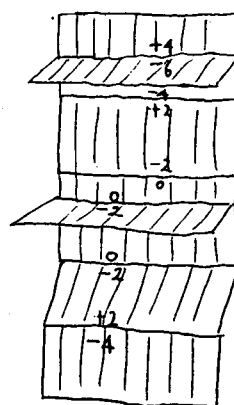
$\tilde{i} = 6$ (E_6 -type)



$\tilde{i} = 7$ (E_7 -type)



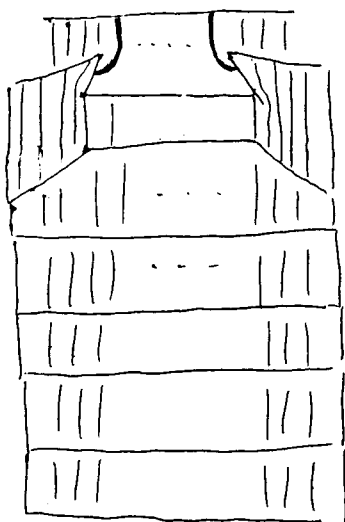
$\tilde{i} = 8$ (E_7 -type)



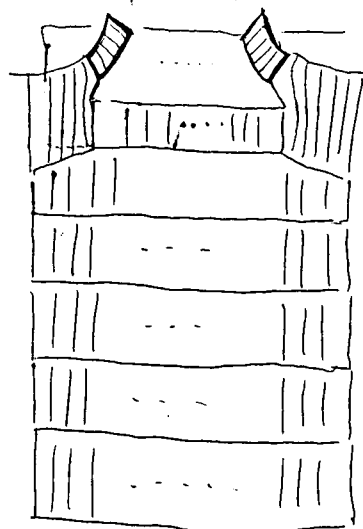
$i = 9, 10, 11$. (E₈-type)

There are 3, 2 or 1 dissident points according to whether $i = 9, 10$ or 11 .

$\equiv$
 $\tilde{W}$

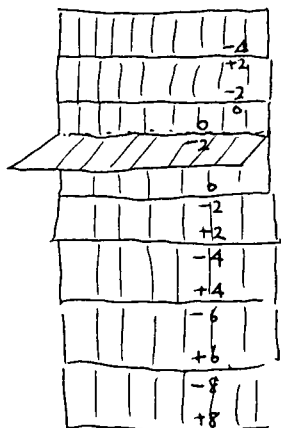


$\sim$
 $\tilde{W}$



$i = 12$. (E₈-type)

$\equiv$
 $\tilde{W} = \tilde{W}$



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