

確率代数多項式の実数根の個数の限界について

On the Bound of the Number of the Real Roots of a Random Algebraic Polynomial

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1 Introduction

A random algebraic polynomial of degree n is of the form

$$F_n(x, \omega) = \sum_{k=0}^n a_k(\omega) x^k,$$

where the $a_k(\omega)$ are random variables and x is a complex number. Since Bloch and Polya[1] initiated the estimate of the number of real roots of a random algebraic polynomial, there has been a stream of papers on the various estimates of the zeros of random algebraic polynomials by others, like Littlewood & Offord[3] and Evans[2], although they mainly work with independent and identically distributed coefficients. For dependent coefficients, Sambandham[4] obtained asymptotic formulae for the expectation of the number of real roots of a random algebraic polynomial in the case of random coefficients are normally distributed with mean zero, variance 1 and each correlation $\rho_{ij} = \rho \in (0, 1)$ or $\rho^{|i-j|}$, $\rho \in (0, \frac{1}{2})$. Also for the upper bound of the number of real roots of a random algebraic polynomial, Sambandham[5] considered the case of constant correlation $\rho \in (0, 1)$.

We have researched the estimate with respect to the upper and lower bounds of the number of real roots of a random algebraic polynomial whose coefficients are dependent normal random variables with varying correlation.

2 Upper Bound of the Number of Real Roots

First we suppose that the coefficients are normally distributed random variables having mean zero, variance 1 and each correlation $\rho_{ij} = \rho_{|i-j|}$, where $\{\rho_k\}$ is a nonnegative decreasing sequence satisfying $\rho_1 < \frac{1}{2}$ and $\sum_{k=1}^{\infty} \rho_k < \infty$. That is to say that we consider the random coefficients $a_k(\omega)$ $k = 0, 1, \dots, n$ have joint density function

$$|M|^{\frac{1}{2}} (2\pi)^{-\frac{n+1}{2}} \exp\left(-\frac{1}{2} \mathbf{a}' M \mathbf{a}\right),$$

where M^{-1} is the moment matrix with

$$\rho_{ij} = \begin{cases} 1 & (i = j) \\ \rho_{|i-j|} & (i \neq j) \end{cases}$$

where $\{\rho_j\}$ is a nonnegative decreasing sequence satisfying $\rho_1 < \frac{1}{2}$ and $\sum_{j=1}^{\infty} \rho_j < \infty$. $\mathbf{a}'$ is the transpose of the column vector $\mathbf{a}$.

THEOREM 1 ([6]). *There exists an integer n_0 such that for each $n > n_0$, the number of real roots of the equations $F_n(z, \omega) = 0$ is at most*

$$C(\log \log n)^2 \log n$$

except for a set of measure at most

$$\frac{C'}{\log n_0 - \log \log \log n_0},$$

where C and C' are constants.

Proof. We indicate a brief outline of the proofs. We must remark that the transformation $x \rightarrow \frac{1}{x}$ makes the equation $F_n(x, \omega) = 0$ transformed to $\sum_{r=0}^n a_{n-r}(\omega)x^r = 0$ and $(a_0(\omega), a_1(\omega), \dots, a_n(\omega))$ and $(a_n(\omega), a_{n-1}(\omega), \dots, a_0(\omega))$ have the same joint density function. Therefore the number of roots and the measure of the exceptional set in the range $[-\infty, \infty]$ are twice the corresponding estimates for the range $[-1, 1]$. But we consider the range $[-1, 0]$ only. Because it can be shown that the upper bound in $[0, 1]$ is the same as in $[-1, 0]$ by using the same procedure. Thus the number of roots in the range $[-\infty, \infty]$ and the measure of the exceptional set are each four times the corresponding estimates for the range $[-1, 0]$.

The proof consists of defining circles to cover the interval $[0, 1]$ and estimating the number of zeros in each circle by the inequality proved by Jensen's theorem. Let $N(|z - z_0| < r)$ be the number of zeros of a regular function $\phi(z)$ in the circle with center z_0 and of radius r . The following is the inequality essential in order to get the theorem,

$$N(|z - z_0| < r) \leq \frac{\log \left(\frac{\sup_{|z - z_0| < R} |\phi(z)|}{|\phi(z_0)|} \right)}{\log(R/r)}$$

where $R(> r)$.

3 Lower Bound of the Number of Real Roots

Consider

$$f_n(x, \omega) = \sum_{k=0}^n a_k(\omega) b_k x^k,$$

where the b_k are positive numbers and the coefficients be m -dependent stationary Gaussian random variables with mean zero and variance 1. In other words, we assume the random coefficients $a_k(\omega)$ $k = 0, 1, \dots, n$ have joint density function

$$|M|^{\frac{1}{2}} (2\pi)^{-\frac{n+1}{2}} \exp\left(-\frac{1}{2} \mathbf{a}' M \mathbf{a}\right),$$

where M^{-1} is the moment matrix with

$$\rho_{ij} = \begin{cases} 1 & (i = j) \\ \rho_{|i-j|} \in [0, 1) & (1 \leq |i - j| \leq m) \\ 0 & (|i - j| > m) \end{cases} \quad i, j = 0, 1, \dots, n$$

Under the above condition we get the following results.

THEOREM 2 ([7]). Let b_k , $k = 0, 1, \dots, n$ be positive numbers such that

$$\frac{k_n}{t_n} = o(\log n), \quad \text{where } k_n = \max_{0 \leq k \leq n} b_k \quad \text{and} \quad t_n = \min_{0 \leq k \leq n} b_k.$$

Then for $n > n_0$, the number of real roots of the equations $f_n(x, \omega) = 0$ is at least

$$\frac{C \log n}{\log\left(\frac{k_n}{t_n} \log n\right)}$$

except for a set of measure at most

$$\frac{C' \log\left(\frac{k_n}{t_n} \log n\right)}{\log n}$$

where C, C' are positive constants.

Proof. The method of the proof consists mainly of counting the number of crossing in each interval of length δ .

As the improvement of theorem 2, we get the following estimate.

THEOREM 3. Let b_k , $k = 0, 1, \dots, n$ be positive numbers such that $\lim_{n \rightarrow \infty} \frac{k_n}{t_n}$ is finite, where

$$k_n = \max_{0 \leq k \leq n} b_k \quad \text{and} \quad t_n = \min_{0 \leq k \leq n} b_k.$$

Then for $n > n_0$, the number of real roots of most of the equations $f_n(x, \omega) = 0$ is at least

$$\epsilon_n \log n$$

except for a set of measure at most

$$\frac{C}{\epsilon_n \log n} + \left(\frac{k_n}{t_n}\right)^\beta \exp\left(-\frac{C'\beta}{\epsilon_n}\right), \beta > 0,$$

provided ϵ_n tends to zero but $\epsilon_n \log n$ tends to infinity as n tends to infinity, where C and C' are positive constants.

Proof. We borrow the method of the proof of theorem 2.

References

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