

Modified Elastic Wave Equations on Riemannian and Kähler Manifolds

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We introduce some geometrically invariant systems of differential equations on any Riemannian manifolds and also on any Kähler manifolds, which are natural extensions of the elastic wave equations on $\mathbb{R}^3$. Further we prove the local decomposition theorems of distribution solutions for those systems. In particular, the solutions of our systems on Kähler manifolds are decomposed into 4 solutions with different propagation speeds.

Definition 1. Let $\bigwedge^{(p)} T^*M$ be a vector bundle of p -differential forms on M . Let $\mathcal{E}_M^{(p)}$ be a sheaf of p -forms on M with C^∞ coefficients, and $\mathcal{D}b_M^{(p)}$ a sheaf of p -currents on M ; that is, p -forms with distribution coefficients. In this article, we do not mean distributions the dual space of $C_0^\infty(M)$. Our distributions behave as “functions” for coordinate transformations.

Definition 2. We put $\widetilde{M} := \mathbb{R}_t \times M$. We denote by $\widetilde{\mathcal{E}}_M^{(p)}$, $\widetilde{\mathcal{D}b}_M^{(p)}$ the sheaves of sections of $\mathcal{E}_{\widetilde{M}}^{(p)}$, $\mathcal{D}b_{\widetilde{M}}^{(p)}$ which do not include the covariant vector dt . That is, setting the projection $\pi : \widetilde{M} \rightarrow M$, we define

$$\widetilde{\mathcal{E}}_M^{(p)} := \mathcal{E}_{\widetilde{M}}^{(0)} \otimes_{\pi^{-1}\mathcal{E}_M^{(0)}} \pi^{-1}\mathcal{E}_M^{(p)}, \quad \widetilde{\mathcal{D}b}_M^{(p)} := \mathcal{D}b_{\widetilde{M}}^{(0)} \otimes_{\pi^{-1}\mathcal{E}_M^{(0)}} \pi^{-1}\mathcal{E}_M^{(p)}.$$

Definition 3. The inner products $\langle \cdot, \cdot \rangle : \bigwedge^{(1)} T_x^* M \times \bigwedge^{(1)} T_x M \rightarrow \mathbb{R}$, $\langle \cdot, \cdot \rangle^* : \bigwedge^{(p)} T_x^* M \times \bigwedge^{(p)} T_x^* M \rightarrow \mathbb{R}$, are defined as follows. We choose a positive orthonormal system $(\omega^1, \dots, \omega^n)$ of C^∞ sections of T^*M concerning the Riemannian metric; that is, there is a positive number α such that $\omega^1 \wedge \dots \wedge \omega^n = \alpha \Omega_x > 0$. Then for

$$\sigma = \sum_{1 \leq i \leq n} \sigma_i dx^i, \quad \tau = \sum_{1 \leq i \leq n} \tau^i \partial_i,$$

we define

$$\langle \sigma, \tau \rangle := \sum_{1 \leq i \leq n} \sigma_i \tau^i,$$

and for

$$\begin{aligned} \phi &= \sum_{1 \leq i_1 < \dots < i_p \leq n} \phi_{i_1 \dots i_p} \omega^{i_1} \wedge \dots \wedge \omega^{i_p}, \\ \psi &= \sum_{1 \leq i_1 < \dots < i_p \leq n} \psi_{i_1 \dots i_p} \omega^{i_1} \wedge \dots \wedge \omega^{i_p}, \end{aligned}$$

we define

$$\begin{aligned} \langle \phi, \psi \rangle^* &:= \sum_{1 \leq i_1 < \dots < i_p \leq n} \phi_{i_1 \dots i_p} \psi^{i_1 \dots i_p} \\ &:= \sum_{\substack{1 \leq i_1 < \dots < i_p \leq n \\ 1 \leq j_1 < \dots < j_p \leq n}} \phi_{i_1 \dots i_p} g^{i_1 j_1} \dots g^{i_p j_p} \psi_{j_1 \dots j_p}. \end{aligned}$$

Definition 4. We denote by $d : \mathcal{D}b_M^{(p)} \rightarrow \mathcal{D}b_M^{(p+1)}$ the exterior differential operator which acts on $\mathcal{D}b_M^{(p)}$ as a sheaf morphism. Then the following formulas are well-known:

$$\begin{cases} 1. d(\phi \pm \psi) = d\phi \pm d\psi & (\phi, \psi \in \mathcal{D}b_M^{(p)}), \\ 2. d(\phi \wedge \psi) = d\phi \wedge \psi + (-1)^p \phi \wedge d\psi & (\phi \in \mathcal{D}b_M^{(p)}, \psi \in \mathcal{D}b_M^{(q)}), \\ 3. d(d\phi) = 0 & (\phi \in \mathcal{D}b_M^{(p)}), \\ 4. \text{For } f \in \mathcal{D}b_M^{(0)}, df := \sum \frac{\partial f}{\partial x_j} dx^j \in \mathcal{D}b_M^{(1)}. \end{cases}$$

Here $0 \leq p \leq n$. If $p = n$, $d\phi = 0$ holds.

Definition 5. The isomorphism $*$: $\bigwedge T^*M \rightarrow \bigwedge T^*M$ of vector bundle is defined as follows:

$$\left\{ \begin{array}{l} 1. * : \bigwedge^{(p)} T_x^*M \mapsto \bigwedge^{(n-p)} T_x^*M \text{ is a linear map,} \\ 2. * (\omega^{i_1} \wedge \dots \wedge \omega^{i_p}) = (-1)^{(i_1-1)+\dots+(i_p-p)} \omega^{j_1} \wedge \dots \wedge \omega^{j_{n-p}}, \\ \text{for any permutation } (i_1, \dots, i_p, j_1, \dots, j_{n-p}) \text{ of } (1, \dots, n). \end{array} \right.$$

Here $(i_1 \dots i_p)$ and $(j_1 \dots j_{n-p})$ are indices satisfying

$$\left\{ \begin{array}{l} 1. (i_1 \dots i_p j_1 \dots j_{n-p}) \text{ is a permutation of } (1 \dots n), \\ 2. 1 \leq i_1 < \dots < i_p \leq n, 1 \leq j_1 < \dots < j_{n-p} \leq n. \end{array} \right.$$

Remark 6. The definition above does not depend on the choice of the positive orthonormal system $\{\omega^1, \dots, \omega^n\}$.

Proposition 7. We set $\phi, \psi \in \bigwedge^{(p)} T_x^*M$. Then we obtain

$$\left\{ \begin{array}{l} 1. \phi \wedge * \psi = (*\phi) \wedge \psi = \langle \phi, \psi \rangle^* \omega^1 \wedge \dots \wedge \omega^n, \\ 2. * 1 = \omega^1 \wedge \dots \wedge \omega^n = \sqrt{g} dx^1 \wedge \dots \wedge dx^n, \\ 3. * \phi = (-1)^{(i_1-1)+\dots+(i_p-p)} \sqrt{g} g^{i_1 j_1} \dots g^{i_p j_p} \phi_{i_1 \dots i_p} dx^{j_1} \wedge \dots \wedge dx^{j_{n-p}} \\ \quad \in \bigwedge^{(n-p)} T_x^*M. \end{array} \right.$$

Here $g = \det(g_{\lambda\kappa})$.

Let $U \subset M$ be an open subset. Let $\alpha^{(p)} \in \mathcal{D}b_M^{(p)}(U)$, $\beta^{(p)} \in \mathcal{E}_M^{(p)}(U)$ be sections. We suppose that $\beta^{(p)}$ has a compact support in U . Then the following integral is well-defined.

$$(\alpha^{(p)}, \beta^{(p)}) := \int_M \langle \alpha^{(p)}, \beta^{(p)} \rangle^* \omega^1 \wedge \dots \wedge \omega^n.$$

Definition 8. Let $\alpha^{(p)} \in \mathcal{D}b_M^{(p)}$, $\beta^{(p-1)} \in \mathcal{E}_M^{(p-1)}$ be sections. We suppose $\beta^{(p-1)}$ has a compact support. Then the sheaf morphism $\delta : \mathcal{D}b_M^{(p)} \rightarrow \mathcal{D}b_M^{(p-1)}$ is defined as

$$(\delta\alpha^{(p)}, \beta^{(p-1)}) = (\alpha^{(p)}, d\beta^{(p-1)}).$$

Hence we have

$$\delta = (-1)^{n(p-1)+1} * d *.$$

Definition 9. Let $\mathfrak{X}_s^r$ be the sheaf of $\bigotimes^r T_x M \otimes \bigotimes^s T_x^* M$ -valued C^∞ functions, and $\mathcal{D}b_s^r$ the sheaf of $\bigotimes^r T_x M \otimes \bigotimes^s T_x^* M$ -valued distributions. Then, the sheaf morphisms $\nabla : \mathfrak{X}_s^r \rightarrow \mathfrak{X}_{s+1}^r$, $\mathcal{D}b_s^r \rightarrow \mathcal{D}b_{s+1}^r$ are defined as follows:

$$\left\{ \begin{array}{ll} 1. \text{ For } a(x) \in \mathfrak{X}_0^0, & \text{we have } \nabla a(x) = \frac{\partial a}{\partial x^j} dx^j. \\ 2. \text{ For } \frac{\partial}{\partial x^j} \in \mathfrak{X}_0^1, & \text{we have } \nabla \left(\frac{\partial}{\partial x^j} \right) = \Gamma_j^i{}^k \frac{\partial}{\partial x^i} \otimes dx^k. \\ 3. \text{ For } dx^j \in \mathfrak{X}_1^0, & \text{we have } \nabla (dx^j) = -\Gamma_i^j{}^k dx^i \otimes dx^k. \\ 4. \text{ For } e \in \mathfrak{X}_s^r, f \in \mathfrak{X}_{s'}^{r'}, & \text{we have } \nabla(e \otimes f) = (\nabla e) \otimes f + e \otimes \nabla f. \end{array} \right.$$

Here,

$$\left\{ \Gamma_i^j{}^k = g^{jl} \Gamma_{ilk} = g^{jl} \cdot \frac{1}{2} \left(\frac{\partial g_{il}}{\partial x^k} + \frac{\partial g_{lk}}{\partial x^i} - \frac{\partial g_{ki}}{\partial x^l} \right) \right\}$$

are the Riemann-Christoffel symbols.

Proposition 10. We set

$$e = e_{i_1 \dots i_s}^r dx^{i_1} \otimes \dots \otimes dx^{i_s} \otimes \frac{\partial}{\partial x^{j_1}} \otimes \dots \otimes \frac{\partial}{\partial x^{j_r}} \in \mathfrak{X}_s^r.$$

Then we have

$$\begin{aligned} \nabla e = & \left(\partial_k e_{i_1 \dots i_s}^r + e_{i_1 \dots i_s}^q \Gamma_q^r{}^k + e_{i_1 \dots i_{p-1} q i_{p+1} \dots i_s}^r \Gamma_{i_p}^q{}^k \right) \\ & \times dx^k \otimes dx^{i_1} \otimes \dots \otimes dx^{i_s} \otimes \frac{\partial}{\partial x^{j_1}} \otimes \dots \otimes \frac{\partial}{\partial x^{j_r}}. \end{aligned}$$

Hence we call the following *the covariant differentiation* :

$$\begin{aligned} \nabla_k e = & \left(\partial_k e_{i_1 \dots i_s}^r + e_{i_1 \dots i_s}^q \Gamma_q^r k + e_{i_1 \dots i_{p-1} q i_{p+1} \dots i_s}^r \Gamma_{i_p}^q k \right) \\ & \times dx^{i_1} \otimes \dots \otimes dx^{i_s} \otimes \frac{\partial}{\partial x^{j_1}} \otimes \dots \otimes \frac{\partial}{\partial x^{j_r}}. \end{aligned}$$

For

$$u = \sum_{1 \leq i_1 < \dots < i_p \leq n} u_{i_1 \dots i_p} dx^{i_1} \wedge \dots \wedge dx^{i_p} \in \widetilde{\mathcal{D}b}_M^{(p)},$$

we define an operator P_R for $\widetilde{\mathcal{D}b}_M^{(p)}$ on M ($1 \leq p \leq n-1$), where the coefficients $\{u_{i_1 \dots i_p}\}$ are supposed to be alternating with respect to $(i_1 \dots i_p)$.

Definition 11. We define sheaf-morphisms $P_R : \widetilde{\mathcal{D}b}_M^{(p)} \longrightarrow \widetilde{\mathcal{D}b}_M^{(p)}$ by

$$P_R u := \rho \frac{\partial^2}{\partial t^2} u + (\lambda + 2\mu) d\delta u + \mu \delta du,$$

where the density constant ρ and the Lamé constants λ, μ are positive.

For $p = 1$, this equation is the covariant form of $P_R u^i$.

When $p = 0$ or n , $P_R u = 0$ reduces to a wave equation. Therefore we suppose $1 \leq p \leq n-1$.

For $u \in \widetilde{\mathcal{D}b}_M^{(p)}$, we define equations $\mathfrak{M}^R, \mathfrak{M}_1^R, \mathfrak{M}_2^R, \mathfrak{M}_0^R$ as follows:

$$\begin{aligned} \mathfrak{M}^R : \quad & P_R u = 0, \\ \mathfrak{M}_1^R : \quad & \begin{cases} P_R u = 0, \\ du = 0, \end{cases} \iff \begin{cases} (\partial_t^2 + \alpha \Delta) u = 0, \\ du = 0, \end{cases} \\ \mathfrak{M}_2^R : \quad & \begin{cases} P_R u = 0, \\ \delta u = 0, \end{cases} \iff \begin{cases} (\partial_t^2 + \beta \Delta) u = 0, \\ \delta u = 0, \end{cases} \\ \mathfrak{M}_0^R : \quad & \begin{cases} P_R u = 0, \\ du = 0, \\ \delta u = 0, \end{cases} \iff \begin{cases} \partial_t^2 u = 0, \\ du = 0, \\ \delta u = 0. \end{cases} \end{aligned}$$

Here, $\alpha = (\lambda + 2\mu)/\rho$, $\beta = \mu/\rho$ and $\Delta = d\delta + \delta d : \widetilde{\mathcal{D}b}_M^{(p)} \rightarrow \widetilde{\mathcal{D}b}_M^{(p)}$ is the Laplacian on M .

Further we define subsheaves $Sol(\mathfrak{M}^R; p)$, $Sol(\mathfrak{M}_j^R; p)$, ($j = 0, 1, 2$) of $\widetilde{\mathcal{D}b}_M^{(p)}$ as follows: For $\mathfrak{N}^R = \mathfrak{M}^R, \mathfrak{M}_j^R$,

$$Sol(\mathfrak{N}^R; p) := \left\{ u \in \widetilde{\mathcal{D}b}_M^{(p)} \mid u \text{ satisfies } \mathfrak{N}^R \right\}.$$

Then, we have the following theorem.

Theorem 12. *For any germ $u \in Sol(\mathfrak{M}^R; p) \big|_{(\overset{\circ}{t}, \overset{\circ}{x})}$, there exist some germs $u_j \in Sol(\mathfrak{M}_j^R; p) \big|_{(\overset{\circ}{t}, \overset{\circ}{x})}$ ($j = 1, 2$) such that $u = u_1 + u_2$.*

Further, the equation $u = u_1 + u_2 = 0$ implies $u_1, u_2 \in Sol(\mathfrak{M}_0^R; p) \big|_{(\overset{\circ}{t}, \overset{\circ}{x})}$. Equivalently, we have the following exact sequence:

$$0 \longrightarrow Sol(\mathfrak{M}_0^R; p) \longrightarrow Sol(\mathfrak{M}_1^R; p) \oplus Sol(\mathfrak{M}_2^R; p) \longrightarrow Sol(\mathfrak{M}^R; p) \longrightarrow 0,$$

where $F(U) = U \oplus (-U)$, $G(U_1 \oplus U_2) = U_1 + U_2$.

Remark 13. For the case $p = 1$, the contravariant form of this decomposition means the decomposition $u^i = u_1^i + u_2^i \in \widetilde{\mathcal{D}b}_0^1$ satisfying the next conditions:

$$\nabla_i u_1^i = 0, \quad \nabla^i u_2^j - \nabla^j u_2^i = 0.$$

Let X be an n -dimensional complex manifold with a Hermitian metric, and $\bigwedge^{(q,r)} T^*X$ a vector bundle of (q, r) -type differential forms on X . Let $\mathcal{E}_X^{(q,r)}$ be a sheaf of (q, r) -forms on X with C^∞ coefficients, and $\mathcal{D}b_X^{(q,r)}$ a sheaf of (q, r) -currents on X . Setting $\tilde{X} = \mathbb{R}_t \times X$, we also define $\tilde{\mathcal{E}}_X^{(q,r)}$, $\widetilde{\mathcal{D}b}_X^{(q,r)}$ similarly to $\tilde{\mathcal{E}}_M^{(p)}$, $\widetilde{\mathcal{D}b}_M^{(p)}$.

Definition 14. We denote by $\partial : \mathcal{D}b_X^{(q,r)} \rightarrow \mathcal{D}b_X^{(q+1,r)}$ the exterior differential operator which acts on $\mathcal{D}b_X^{(q,r)}$ as a sheaf morphism and $\bar{\partial} : \mathcal{D}b_X^{(q,r)} \rightarrow \mathcal{D}b_X^{(q,r+1)}$ the conjugate exterior differential operator. For a section

$$\phi = \phi_{i_1 \dots i_q \bar{j}_1 \dots \bar{j}_r} dz^{i_1} \wedge \dots \wedge dz^{i_q} \wedge d\bar{z}^{j_1} \wedge \dots \wedge d\bar{z}^{j_r} \quad \text{of } \mathcal{D}b_X^{(q,r)},$$

the following formulas are well-known:

$$\begin{cases} d\phi = (\partial + \bar{\partial})\phi, \\ \partial\phi = \frac{\partial\phi}{\partial z^k} dz^k \wedge dz^{i_1} \wedge \dots \wedge dz^{i_q} \wedge d\bar{z}^{j_1} \wedge \dots \wedge d\bar{z}^{j_r} \in \mathcal{D}b_X^{(q+1,r)}, \\ \bar{\partial}\phi = \frac{\partial\phi}{\partial \bar{z}^k} d\bar{z}^k \wedge dz^{i_1} \wedge \dots \wedge dz^{i_q} \wedge d\bar{z}^{j_1} \wedge \dots \wedge d\bar{z}^{j_r} \in \mathcal{D}b_X^{(q,r+1)}. \end{cases}$$

Definition 15. The linear operator $*$ on X induces isomorphisms $\bigwedge^{(q,r)} T^*X \longrightarrow \bigwedge^{(n-r,n-q)} T^*X$ of vector bundle. Hence we have sheaf-morphisms $* : \mathcal{D}b_X^{(q,r)} \longrightarrow \mathcal{D}b_X^{(n-r,n-q)}$ on X as follows: For

$$\psi = \psi_{I\bar{J}} \omega^I \wedge \bar{\omega}^{\bar{J}} \in \mathcal{D}b_X^{(q,r)},$$

we have

$$*\psi = \delta \begin{pmatrix} 1 \dots n \bar{1} \dots \bar{n} \\ I \quad \bar{J} \quad \bar{J}^C \quad I^C \end{pmatrix} \psi_{I\bar{J}} \omega^{J^C} \wedge \bar{\omega}^{I^C} \in \mathcal{D}b_X^{(n-r,n-q)},$$

where $\{\omega^1, \dots, \omega^n\}$ is a local orthonormal system of C^∞ sections of T^*X concerning the Hermitian metric and $I^C := \{1, \dots, n\} \setminus I$. Here $\delta(\cdot) = \pm 1$ is the signature of the permutation $(I\bar{J}\bar{J}^C I^C)$ of $(1 \dots n \bar{1} \dots \bar{n})$.

Let $U \subset X$ be an open subset. Let $\alpha^{(q,r)} = \alpha_{I\bar{J}} \omega^I \wedge \bar{\omega}^{\bar{J}} \in \mathcal{D}b_X^{(q,r)}(U)$, $\beta^{(q,r)} = \beta_{I\bar{J}} \omega^I \wedge \bar{\omega}^{\bar{J}} \in \mathcal{E}_X^{(q,r)}(U)$ be sections. We suppose that $\beta^{(q,r)}$ has a compact support in U . Then the following integral is well-defined.

$$(\alpha^{(q,r)}, \beta^{(q,r)}) := \int_X \langle \alpha^{(q,r)}, \beta^{(q,r)} \rangle^* \omega^1 \wedge \dots \wedge \omega^n \wedge \bar{\omega}^1 \wedge \dots \wedge \bar{\omega}^n,$$

where, $\langle \alpha^{(q,r)}, \beta^{(q,r)} \rangle^* = \sum_{I, \bar{J}} \alpha_{I\bar{J}} \overline{\beta_{I\bar{J}}}$.

Definition 16. Let $\alpha^{(q,r)} \in \mathcal{D}b_X^{(q,r)}$, $\beta^{(q-1,r)} \in \mathcal{E}_X^{(q-1,r)}$, and $\gamma^{(q,r-1)} \in \mathcal{E}_X^{(q,r-1)}$ be sections. We suppose $\beta^{(q-1,r)}$ and $\gamma^{(q,r-1)}$ have compact supports. Then sheaf morphisms $\bar{\vartheta} : \mathcal{D}b_X^{(q,r)} \rightarrow \mathcal{D}b_X^{(q-1,r)}$ and $\vartheta : \mathcal{D}b_X^{(q,r)} \rightarrow \mathcal{D}b_X^{(q,r-1)}$ are defined as

$$\begin{aligned} (\bar{\vartheta}\alpha^{(q,r)}, \beta^{(q-1,r)}) &= (\alpha^{(q,r)}, \partial\beta^{(q-1,r)}), \\ (\vartheta\alpha^{(q,r)}, \gamma^{(q,r-1)}) &= (\alpha^{(q,r)}, \bar{\partial}\gamma^{(q,r-1)}). \end{aligned}$$

Further they satisfy the following equations:

$$\begin{cases} \delta &= \bar{\vartheta} + \vartheta, \\ \bar{\vartheta} &= - * \bar{\partial} *, \\ \vartheta &= - * \partial *. \end{cases}$$

Now we assume that X is a Kähler manifold; that is, for the Hermitian metric h , we have the equation $d(\sum h_{j\bar{k}}(z)dz^j \wedge d\bar{z}^k) = 0$, and we know that $h_{j\bar{k}}$ can be described as $h_{j\bar{k}} = \partial_j \bar{\partial}_k \phi$ with a smooth real function ϕ locally.

Then the following equations for operators on $\widetilde{\mathcal{D}b}_X^{(q,r)}$ are well-known:

$$\begin{cases} \square = \bar{\square} = \frac{1}{2}\Delta, \\ \partial\vartheta + \vartheta\partial = 0, \quad \bar{\partial}\bar{\vartheta} + \bar{\vartheta}\bar{\partial} = 0, \\ \partial\bar{\vartheta} + \bar{\vartheta}\partial = 0, \quad \vartheta\bar{\partial} + \bar{\partial}\vartheta = 0. \end{cases}$$

Definition 17. We define sheaf-morphisms $P_K : \widetilde{\mathcal{D}b}_X^{(q,r)} \rightarrow \widetilde{\mathcal{D}b}_X^{(q,r)}$ on $\tilde{X}$ by

$$P_K = \frac{\partial^2}{\partial t^2} + \alpha_1 \partial \bar{\vartheta} + \alpha_2 \bar{\vartheta} \partial + \alpha_3 \bar{\partial} \vartheta + \alpha_4 \vartheta \bar{\partial}.$$

Here, $\alpha_1, \alpha_2, \alpha_3$ and α_4 are positive coefficients.

When $q, r = 0$ or n , $P_K u = 0$ reduces to a wave equation. Therefore, we suppose $1 \leq q, r \leq n-1$.

For $u \in \widetilde{\mathcal{D}b}_X^{(q,r)}$, we define equations $\mathfrak{M}^K, \mathfrak{M}_i^K$ ($i = 1, 2, 3, 4$), $\mathfrak{M}_{jk}^K, \mathfrak{M}_{jk0}^K$ ($(jk) = (13), (14), (23), (24)$) as follows:

$$\mathfrak{M}^{\kappa} : P_{\kappa} u = 0,$$

$$\mathfrak{M}_1^{\kappa} : \begin{cases} P_{\kappa} u = 0, \\ \partial u = 0, \\ \bar{\partial} u = 0, \end{cases} \iff \begin{cases} \left(\partial_t^2 + \frac{\alpha_1 + \alpha_3}{2} \Delta \right) u = 0, \\ \partial u = 0, \\ \bar{\partial} u = 0, \end{cases}$$

$$\mathfrak{M}_2^{\kappa} : \begin{cases} P_{\kappa} u = 0, \\ \bar{\partial} u = 0, \\ \partial u = 0, \end{cases} \iff \begin{cases} \left(\partial_t^2 + \frac{\alpha_2 + \alpha_3}{2} \Delta \right) u = 0, \\ \bar{\partial} u = 0, \\ \partial u = 0, \end{cases}$$

$$\mathfrak{M}_3^{\kappa} : \begin{cases} P_{\kappa} u = 0, \\ \partial u = 0, \\ \vartheta u = 0, \end{cases} \iff \begin{cases} \left(\partial_t^2 + \frac{\alpha_1 + \alpha_4}{2} \Delta \right) u = 0, \\ \partial u = 0, \\ \vartheta u = 0, \end{cases}$$

$$\mathfrak{M}_4^{\kappa} : \begin{cases} P_{\kappa} u = 0, \\ \bar{\vartheta} u = 0, \\ \vartheta u = 0, \end{cases} \iff \begin{cases} \left(\partial_t^2 + \frac{\alpha_2 + \alpha_4}{2} \Delta \right) u = 0, \\ \bar{\vartheta} u = 0, \\ \vartheta u = 0, \end{cases}$$

$$\mathfrak{M}_{13}^{\kappa} : \begin{cases} P_{\kappa} u = 0, \\ \partial u = 0, \\ \bar{\partial} u = 0, \\ \vartheta u = 0, \end{cases} \iff \begin{cases} \partial_t^2 u = 0, \\ \partial u = 0, \\ \bar{\partial} u = 0, \\ \vartheta u = 0, \end{cases}$$

$$\mathfrak{M}_{24}^{\kappa} : \begin{cases} P_{\kappa} u = 0, \\ \bar{\vartheta} u = 0, \\ \bar{\partial} u = 0, \\ \vartheta u = 0, \end{cases} \iff \begin{cases} \partial_t^2 u = 0, \\ \bar{\vartheta} u = 0, \\ \bar{\partial} u = 0, \\ \vartheta u = 0, \end{cases}$$

$$\mathfrak{M}_{12}^{\kappa} : \begin{cases} P_{\kappa} u = 0, \\ \bar{\partial} u = 0, \\ \bar{\vartheta} u = 0, \\ \partial u = 0, \end{cases} \iff \begin{cases} \partial_t^2 u = 0, \\ \bar{\partial} u = 0, \\ \bar{\vartheta} u = 0, \\ \partial u = 0, \end{cases}$$

$$\mathfrak{M}_{34}^{\mathbf{K}} : \begin{cases} P_{\mathbf{K}} u = 0, \\ \vartheta u = 0, \\ \bar{\vartheta} u = 0, \\ \partial u = 0, \end{cases} \iff \begin{cases} \partial_t^2 u = 0, \\ \vartheta u = 0, \\ \bar{\vartheta} u = 0, \\ \partial u = 0, \end{cases}$$

$$\mathfrak{M}_0^{\mathbf{K}} : \begin{cases} P_{\mathbf{K}} u = 0, \\ \partial u = 0, \\ \bar{\partial} u = 0, \\ \vartheta u = 0, \\ \bar{\vartheta} u = 0, \end{cases} \iff \begin{cases} \partial_t^2 u = 0, \\ \partial u = 0, \\ \bar{\partial} u = 0, \\ \vartheta u = 0, \\ \bar{\vartheta} u = 0. \end{cases}$$

Further we define subsheaves $Sol(\mathfrak{M}^{\mathbf{K}}; q, r)$, $Sol(\mathfrak{M}_i^{\mathbf{K}}; q, r)$ ($i = 1, 2, 3, 4$), $Sol(\mathfrak{M}_{jk}^{\mathbf{K}}; q, r)$, $Sol(\mathfrak{M}_{jk0}^{\mathbf{K}}; q, r)$ ($(jk) = (13), (23), (14), (24)$) of $\widetilde{\mathcal{D}b}_X^{(q,r)}$ as the sheaves of $\widetilde{\mathcal{D}b}_X^{(q,r)}$ -solutions, respectively.

Then, we have the following theorem.

Theorem 18. *For any germ $u \in Sol(\mathfrak{M}^{\mathbf{K}}; q, r) \Big|_{\left(\overset{\circ}{t}, \overset{\circ}{z}\right)}$, there exist some germs $u_{ij} \in Sol(\mathfrak{M}_{ij}^{\mathbf{K}}; q, r) \Big|_{\left(\overset{\circ}{t}, \overset{\circ}{z}\right)}$ ($(ij) = (13), (23), (14), (24)$) such that $u = u_{13} + u_{23} + u_{14} + u_{24}$.*

Further, we find that $u = u_{13} + u_{23} + u_{14} + u_{24} = 0$ implies

$$u_{jk} \in Sol(\mathfrak{M}_{jk0}^{\mathbf{K}}; q, r) \quad ((jk) = (13), (23), (14), (24)).$$

Equivalently, we have the following exact sequence:

$$0 \longrightarrow \bigoplus_{(ij)}' Sol(\mathfrak{M}_{ij0}^{\mathbf{K}}; q, r) \xrightarrow{G} \bigoplus_{(ij)} Sol(\mathfrak{M}_{ij}^{\mathbf{K}}; q, r) \xrightarrow{H} Sol(\mathfrak{M}^{\mathbf{K}}; q, r) \longrightarrow 0.$$

Here,

$$\bigoplus_{(ij)}' Sol(\mathfrak{M}_{ij0}^{\mathbf{K}}; q, r) := \left\{ (u_{ij}) \in \bigoplus_{(ij)} Sol(\mathfrak{M}_{ij0}^{\mathbf{K}}; q, r) \mid \sum_{(ij)} u_{ij} = 0 \right\},$$

$$G(U_{13} \oplus U_{23} \oplus U_{14} \oplus U_{24}) = U_{13} \oplus U_{23} \oplus U_{14} \oplus U_{24}, H(U_{13} \oplus U_{23} \oplus U_{14} \oplus U_{24}) = U_{13} + U_{23} + U_{14} + U_{24}.$$

Example 19. We assume $X = \mathbb{C}^2$. Then, X is a Kähler manifold with the complex Euclidean metric. We find a solution $u \in \widetilde{\mathcal{D}b}_X^{(1,1)}$ of the form with $\zeta \equiv \zeta_1 dz^1 + \zeta_2 dz^2$ where $(\zeta_1, \zeta_2) \in \mathbb{C}^2 \setminus \{0\}$;

$$u(t, z) = U(t) e^{i(z \cdot \zeta + \bar{z} \cdot \bar{\zeta})}.$$

Then,

$$\begin{aligned} P_K u &= U'' + (\alpha_1 - \alpha_2) \zeta \wedge \left(* (\bar{\zeta} \wedge * U) \right) + \alpha_2 |\zeta|^2 U \\ &\quad + (\alpha_3 - \alpha_4) \bar{\zeta} \wedge \left(* (\zeta \wedge * U) \right) + \alpha_4 |\zeta|^2 U = 0. \end{aligned}$$

We put

$$U(t) = c_1(t) \zeta \wedge \bar{\zeta} + c_2(t) \zeta \wedge \bar{\zeta}^\perp + c_3(t) \zeta^\perp \wedge \bar{\zeta} + c_4(t) \zeta^\perp \wedge \bar{\zeta}^\perp,$$

where $\zeta^\perp = \bar{\zeta}_2 dz^1 - \bar{\zeta}_1 dz^2$, $|\zeta| = |\zeta^\perp|$ hold. Then, we get

$$\begin{aligned} &\left(c_1'' + (\alpha_1 + \alpha_3) |\zeta|^2 c_1 \right) \zeta \wedge \bar{\zeta} + \left(c_2'' + (\alpha_1 + \alpha_4) |\zeta|^2 c_2 \right) \zeta \wedge \bar{\zeta}^\perp \\ &\left(c_3'' + (\alpha_2 + \alpha_3) |\zeta|^2 c_3 \right) \zeta^\perp \wedge \bar{\zeta} + \left(c_4'' + (\alpha_2 + \alpha_4) |\zeta|^2 c_4 \right) \zeta^\perp \wedge \bar{\zeta}^\perp = 0. \end{aligned}$$

Hence, we obtain

$$\begin{aligned} c_1(t) &= A_{13}^+ \exp(i\sqrt{\alpha_1 + \alpha_3} |\zeta| t) + A_{13}^- \exp(-i\sqrt{\alpha_1 + \alpha_3} |\zeta| t), \\ c_2(t) &= A_{14}^+ \exp(i\sqrt{\alpha_1 + \alpha_4} |\zeta| t) + A_{14}^- \exp(-i\sqrt{\alpha_1 + \alpha_4} |\zeta| t), \\ c_3(t) &= A_{23}^+ \exp(i\sqrt{\alpha_2 + \alpha_3} |\zeta| t) + A_{23}^- \exp(-i\sqrt{\alpha_2 + \alpha_3} |\zeta| t), \\ c_4(t) &= A_{24}^+ \exp(i\sqrt{\alpha_2 + \alpha_4} |\zeta| t) + A_{24}^- \exp(-i\sqrt{\alpha_2 + \alpha_4} |\zeta| t). \end{aligned}$$

Since

$$\begin{aligned}
U(0) &= (A_{13}^+ + A_{13}^-) \zeta \wedge \bar{\zeta} + (A_{14}^+ + A_{14}^-) \zeta \wedge \bar{\zeta}^\perp \\
&\quad + (A_{23}^+ + A_{23}^-) \zeta^\perp \wedge \bar{\zeta} + (A_{24}^+ + A_{24}^-) \zeta^\perp \wedge \bar{\zeta}^\perp, \\
\frac{\partial}{\partial t} U(0) &= i\sqrt{\alpha_1 + \alpha_3} |\zeta| (A_{13}^+ - A_{13}^-) \zeta \wedge \bar{\zeta} \\
&\quad + i\sqrt{\alpha_1 + \alpha_4} |\zeta| (A_{14}^+ - A_{14}^-) \zeta \wedge \bar{\zeta}^\perp \\
&\quad + i\sqrt{\alpha_2 + \alpha_3} |\zeta| (A_{23}^+ - A_{23}^-) \zeta^\perp \wedge \bar{\zeta} \\
&\quad + i\sqrt{\alpha_2 + \alpha_4} |\zeta| (A_{24}^+ - A_{24}^-) \zeta^\perp \wedge \bar{\zeta}^\perp,
\end{aligned}$$

we get

$$\begin{aligned}
A_{13}^+ &= \frac{\langle U(0), \zeta \wedge \bar{\zeta} \rangle^*}{2|\zeta|^4} - i \frac{\langle \frac{\partial}{\partial t} U(0), \zeta \wedge \bar{\zeta} \rangle^*}{2\sqrt{\alpha_1 + \alpha_3} |\zeta|^5}, \\
A_{13}^- &= \frac{\langle U(0), \zeta \wedge \bar{\zeta} \rangle^*}{2|\zeta|^4} + i \frac{\langle \frac{\partial}{\partial t} U(0), \zeta \wedge \bar{\zeta} \rangle^*}{2\sqrt{\alpha_1 + \alpha_3} |\zeta|^5}, \\
A_{14}^+ &= \frac{\langle U(0), \zeta \wedge \bar{\zeta}^\perp \rangle^*}{2|\zeta|^4} - i \frac{\langle \frac{\partial}{\partial t} U(0), \zeta \wedge \bar{\zeta}^\perp \rangle^*}{2\sqrt{\alpha_1 + \alpha_4} |\zeta|^5}, \\
A_{14}^- &= \frac{\langle U(0), \zeta \wedge \bar{\zeta}^\perp \rangle^*}{2|\zeta|^4} + i \frac{\langle \frac{\partial}{\partial t} U(0), \zeta \wedge \bar{\zeta}^\perp \rangle^*}{2\sqrt{\alpha_1 + \alpha_4} |\zeta|^5}, \\
A_{23}^+ &= \frac{\langle U(0), \zeta^\perp \wedge \bar{\zeta} \rangle^*}{2|\zeta|^4} - i \frac{\langle \frac{\partial}{\partial t} U(0), \zeta^\perp \wedge \bar{\zeta} \rangle^*}{2\sqrt{\alpha_2 + \alpha_3} |\zeta|^5}, \\
A_{23}^- &= \frac{\langle U(0), \zeta^\perp \wedge \bar{\zeta} \rangle^*}{2|\zeta|^4} + i \frac{\langle \frac{\partial}{\partial t} U(0), \zeta^\perp \wedge \bar{\zeta} \rangle^*}{2\sqrt{\alpha_2 + \alpha_3} |\zeta|^5}, \\
A_{24}^+ &= \frac{\langle U(0), \zeta^\perp \wedge \bar{\zeta}^\perp \rangle^*}{2|\zeta|^4} - i \frac{\langle \frac{\partial}{\partial t} U(0), \zeta^\perp \wedge \bar{\zeta}^\perp \rangle^*}{2\sqrt{\alpha_2 + \alpha_4} |\zeta|^5}, \\
A_{24}^- &= \frac{\langle U(0), \zeta^\perp \wedge \bar{\zeta}^\perp \rangle^*}{2|\zeta|^4} + i \frac{\langle \frac{\partial}{\partial t} U(0), \zeta^\perp \wedge \bar{\zeta}^\perp \rangle^*}{2\sqrt{\alpha_2 + \alpha_4} |\zeta|^5}.
\end{aligned}$$