

GROUPS OF MEASURE-PRESERVING HOMEOMORPHISMS AND VOLUME-PRESERVING DIFFEOMORPHISMS OF NONCOMPACT MANIFOLDS AND MASS FLOW TOWARD ENDS

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1. SPACES OF MEASURES AND GROUPS OF MEASURE-PRESERVING HOMEOMORPHISMS

Suppose M is a connected n -manifold possibly with boundary. The symbol $\mathcal{B}(M)$ denotes the σ -algebra of Borel subsets of M .

Definition 1.1. A Radon measure on M is a Borel measure μ on M such that $\mu(K) < \infty$ for any compact subset K of M . A Radon measure μ is said to be good if

- (i) $\mu(p) = 0$ for any point p of M and
- (ii) $\mu(U) > 0$ for any nonempty open subset U of M .

Definition 1.2.

- (1) $\mathcal{M}_g^\partial(M)$ denotes the set of good Radon measures on M with $\mu(\partial M) = 0$.
- (2) The weak topology w on $\mathcal{M}_g^\partial(M)$ is the weakest topology such that the function

$$\Phi_f : \mathcal{M}_g^\partial(M) \rightarrow \mathbb{R} : \Phi_f(\mu) = \int_M f d\mu$$

is continuous for any continuous function $f : M \rightarrow \mathbb{R}$ with compact support.

Let $\mathcal{H}(M)$ denote the group of homeomorphisms of M with the compact-open topology. Any subgroup $\mathcal{G}$ of $\mathcal{H}(M)$ is equipped with the subspace topology. $\mathcal{G}_0$ and $\mathcal{G}_1$ denote the connected component and the path-component of the identity in $\mathcal{G}$.

Definition 1.3. Suppose μ is a good Radon measures on M . The subgroups $\mathcal{H}(M; \mu) \subset \mathcal{H}(M; \mu\text{-reg}) \subset \mathcal{H}(M)$ are defined as follows:

- (1) $h \in \mathcal{H}(M)$ is μ -preserving if $\mu(h(B)) = \mu(B)$ for any $B \in \mathcal{B}(M)$.
 $\mathcal{H}(M; \mu)$ denotes the subgroup of $\mathcal{H}(M)$ consisting of μ -preserving homeomorphisms of M .

(2) $h \in \mathcal{H}(M)$ is μ -biregular if “ $\mu(h(B)) = 0$ iff $\mu(B) = 0$ for any $B \in \mathcal{B}(M)$ ”.

$\mathcal{H}(M; \mu\text{-reg})$ denotes the subgroup of $\mathcal{H}(M)$ consisting of μ -biregular homeomorphisms of M .

The topological group $\mathcal{H}(M)$ acts continuously on the space $\mathcal{M}_g^\partial(M)_w$ by $h \cdot \mu = h_*\mu$, where $h_*\mu \in \mathcal{M}_g^\partial(M)$ is defined by $(h_*\mu)(B) = \mu(h^{-1}(B))$ ($B \in \mathcal{B}(M)$). The subgroup $\mathcal{H}(M; \mu)$ coincides with the stabilizer of μ under this action.

We also use the following terminologies.

Definition 1.4. Suppose X is a space and A is a subspace of X .

- (1) A is a SDR (strong deformation retract) of X if there exists a homotopy $\varphi_t : X \rightarrow X$ such that $\varphi_0 = id_X$, $\varphi_1(X) = A$ and $\varphi_t|_A = id_A$ ($0 \leq t \leq 1$).
- (2) A is HD (homotopy dense) in X if there exists a homotopy $\varphi_t : X \rightarrow X$ such that $\varphi_0 = id_X$ and $\varphi_t(X) \subset A$ ($0 < t \leq 1$).

In both cases the inclusion map $A \subset X$ is a homotopy equivalence with a homotopy inverse $\varphi_1 : X \rightarrow A$.

2. COMPACT CASE — FATHI’S RESULTS

Suppose M is a compact connected n -manifold. The von Neumann-Oxtoby-Ulam theorem [10] asserts that the above action is essentially transitive.

Theorem 2.1. (von Neumann-Oxtoby-Ulam) *Suppose M is compact and $\mu, \nu \in \mathcal{M}_g^\partial(M)$ with $\nu(M) = \mu(M)$. Then there exists $h \in \mathcal{H}_\partial(M)_0$ such that $h_*\mu = \nu$.*

A parametrized version of this theorem was obtained by A. Fathi [6]. Let $\mu \in \mathcal{M}_g^\partial(M)$. We need to restrict ourselves to the following subspace of $\mathcal{M}_g^\partial(M)$.

Definition 2.1. $\mathcal{M}_g^\partial(M; \mu\text{-reg})$ denotes the subset of $\mathcal{M}_g^\partial(M)$ consisting of $\nu \in \mathcal{M}_g^\partial(M)$ which has the same total mass and the same null sets as μ .

The action of $\mathcal{H}(M)$ on $\mathcal{M}_g^\partial(M)$ restricts to the action of the subgroup $\mathcal{H}(M; \mu\text{-reg})$ on the subspace $\mathcal{M}_g^\partial(M; \mu\text{-reg})_w$. We obtain the orbit map

$$\pi : \mathcal{H}(M; \mu\text{-reg}) \longrightarrow \mathcal{M}_g^\partial(M; \mu\text{-reg})_w \quad : \quad \pi(h) = h_*\mu.$$

Theorem 2.2. (A. Fathi [6], 1980) *Suppose M is a compact connected n -manifold.*

(1) *The orbit map π admits a section $\sigma : \mathcal{M}_g^\partial(M; \mu\text{-reg})_w \rightarrow \mathcal{H}_\partial(M; \mu\text{-reg})_1 \subset \mathcal{H}(M; \mu\text{-reg})$.*

(2) $\mathcal{H}(M; \mu\text{-reg}) \cong \mathcal{H}(M; \mu) \times \mathcal{M}_g^\partial(M; \mu\text{-reg})_w$

(3)
$$\begin{array}{ccccc} & \text{SDR} & & \text{Weak HD} & \\ \mathcal{H}(M; \mu) & \subset & \mathcal{H}(M, \mu\text{-reg}) & \subset & \mathcal{H}(M) \end{array}$$

(4) $n = 2$

$$\begin{array}{ccccc} & & \text{SDR} & & \\ & \swarrow & & \searrow & \\ \mathcal{H}(M; \mu) & \xrightarrow{\text{SDR}} & \mathcal{H}(M, \mu\text{-reg}) & \xrightarrow{\text{HD}} & \mathcal{H}(M) \\ \text{ANR} & & \text{ANR} & & \text{ANR} \end{array}$$

Corollary 2.1. (Yagasaki [13]) *$\mathcal{H}(M; \mu)$ is an ℓ_2 -manifold.*

Corollary 2.1 easily follows from the next topological characterization of ℓ_2 -manifold.

Theorem 2.3. (T. Dobrowolski - H. Toruńczyk [5])

A topological group G is a ℓ_2 -manifold iff G is a separable, non locally compact, completely metrizable ANR.

3. NON-COMPACT CASE — R. BERLANGA'S RESULTS

Suppose M is a noncompact connected n -manifold possibly with boundary. First we introduce some notations on the ends of M .

Definition 3.1.

(1) An end e of M is a function which assigns to each compact subset K of M a connected component $e(K)$ of $M - K$ such that $e(K_1) \supset e(K_2)$ if $K_1 \subset K_2$.

(2) $E(M)$ denotes the space of ends of M .

$\overline{M} = M \cup E(M)$ denotes the end compactification of M .

(3) The topology of $\overline{M}$ is described by the following conditions:

(i) M is an open subspace of $\overline{M}$.

(ii) Fundamental open neighborhoods of $e \in E(M)$ is given by

$$N(e, K) = e(K) \cup \{e' \in E(M) \mid e'(K) = e(K)\} \quad (K \subset M : \text{compact})$$

$\overline{M}$ is a compact metrizable space and $E(M)$ is a 0-dim compact subset of $\overline{M}$.

Let $\mu \in \mathcal{M}_g^\partial(M)$.

Definition 3.2.

- (1) $e \in E(M)$ is μ -finite if $\mu(e(K)) < \infty$ for some compact subset K of M (i.e., e has a neighborhood with finite μ -mass).
- (2) $E_f(M; \mu)$ denotes the subspace of μ -finite ends of M .

The von Neumann-Oxtoby-Ulam theorem is extended to the non-compact case in the following form.

Theorem 3.1. (R. Berlanga [1], 1983)

Suppose $\mu, \nu \in \mathcal{M}_g^\partial(M)$ has same total mass and same finite ends. Then there exists $h \in \mathcal{H}_\partial(M)_1$ with $h_*\mu = \nu$.

A parametrized version of this theorem is obtained recently by R. Berlanga [3]. Simple examples show that the weak topology w on $\mathcal{M}_g^\partial(M; \mu\text{-reg})$ is not enough to extend the section theorem (Theorem 2.2 (1)) to the noncompact case. R. Berlanga introduces a little stronger topology called the finite-end weak topology, which turns out to be the correct topology for this purpose.

Definition 3.3. (Finite-end weak topology) Let $\mu \in \mathcal{M}_g^\partial(M)$.

- (1) $\mathcal{M}_g^\partial(M; \mu\text{-end-reg})$ denotes the subset of $\nu \in \mathcal{M}_g^\partial(M)$ which has the same total mass, same null sets and same finite ends as μ .
- (2) Consider the inclusions $M \stackrel{\iota}{\subset} M \cup E_f(M; \mu) \subset \overline{M}$.

The map ι induces the natural map

$$\iota_* : \mathcal{M}_g^\partial(M; \mu\text{-end-reg}) \longrightarrow \mathcal{M}_g^\partial(M \cup E_f(M; \mu))_w : \nu \longmapsto \overline{\nu} = \iota_*\nu$$

- (3) The finite-end weak topology ew on $\mathcal{M}_g^\partial(M; \mu\text{-end-reg})$ is the weakest topology such that ι_* is continuous.

The space $\mathcal{M}_g^\partial(M; \mu\text{-end-reg})_{ew}$ admits the contraction $\varphi_t(\nu) = (1-t)\nu + t\mu$ ($0 \leq t \leq 1$).

Definition 3.4. $\mathcal{H}(M; \mu\text{-end-reg})$ denotes the subgroup of $\mathcal{H}(M)$ consisting of $h \in \mathcal{H}(M)$ which preserves μ -null sets and μ -finite ends of M .

The group $\mathcal{H}(M; \mu\text{-end-reg})$ acts continuously on $\mathcal{M}_g^\partial(M; \mu\text{-end-reg})_{ew}$ by $h \cdot \nu = h_*\nu$ and we obtain the orbit map

$$\pi : \mathcal{H}(M; \mu\text{-end-reg}) \longrightarrow \mathcal{M}_g^\partial(M; \mu\text{-end-reg})_{ew} : \pi(h) = h_*\mu.$$

Theorem 3.2. (R. Berlanga [3], 2003)

(1) *The orbit map π has a section*

$$\sigma : \mathcal{M}_g^\partial(M; \mu\text{-end-reg})_{ew} \longrightarrow \mathcal{H}_\partial(M; \mu\text{-end-reg})_1 \subset \mathcal{H}(M; \mu\text{-end-reg}).$$

(2) $\mathcal{H}(M; \mu\text{-end-reg}) \cong \mathcal{H}(M; \mu) \times \mathcal{M}_g^\partial(M; \mu\text{-end-reg})_{ew}$

(3)
$$\begin{array}{c} SDR \\ \mathcal{H}(M; \mu) \subset \mathcal{H}(M, \mu\text{-end-reg}) \subset \mathcal{H}(M) \end{array}$$

The relation between the two groups $\mathcal{H}(M, \mu\text{-end-reg}) \subset \mathcal{H}(M)$ is not known for $n \geq 3$. In $n = 2$ we can apply our results on homeomorphism groups of noncompact 2-manifolds [11, 12] to obtain the following conclusions.

Theorem 3.3. (Yagasaki [13])

$$\begin{array}{c} n = 2 \\ \begin{array}{ccccc} & & SDR & & \\ & \swarrow & & \searrow & \\ \mathcal{H}(M; \mu)_0 & \subset & \mathcal{H}(M, \mu\text{-end-reg})_0 & \subset & \mathcal{H}(M)_0 \\ \ell_2\text{-MFD} & & ANR & & ANR \end{array} \end{array}$$

The main statement $\mathcal{H}(M, \mu\text{-end-reg})_0 \subset \mathcal{H}(M)_0$ can be derived by the following arguments. When M is a PL n -manifold, $\mathcal{H}^{\text{PL}}(M)$ denotes the subgroup of $\mathcal{H}(M)$ consisting of PL-homeomorphisms of M .

(1) Suppose M is a noncompact connected 2-manifold. Then

(i) M admits a PL-structure.

(ii) $\mathcal{H}^{\text{PL}}(M)_0$ is HD in $\mathcal{H}(M)_0$ for any PL-structure on M [12], cf. [7].

(2) Suppose M is a PL n -manifold and $\mu \in \mathcal{M}_g^\partial(M)$. Then the PL-structure on M can be isotoped to a new PL-structure so that $\mathcal{H}^{\text{PL}}(M) \subset \mathcal{H}(M; \mu\text{-reg})$ [15].

4. MASS FLOW TOWARD ENDS ON NON-COMPACT n -MANIFOLDS

Suppose M is a noncompact connected n -manifold and $\mu \in \mathcal{M}_g^\partial(M)$.

4.1. Topological Vector Space $V_\mu(M)$.

First we define a topological vector space $V_\mu(M)$, which parametrizes mass flows toward ends by μ -preserving homeomorphisms.

Definition 4.1.

- (1) $\mathcal{B}_c(M) = \{B \in \mathcal{B}(M) \mid \text{Fr } B : \text{Compact}\}$
- (2) $W(M)$ denotes the space of all functions $a : \mathcal{B}_c(M) \rightarrow \mathbb{R}$.
 - (i) $W(M)$ is a real vector space under the addition and the scalar product of real valued functions.
 - (ii) $W(M)$ is equipped with the product topology,
i.e., the topology induced by the projections

$$\pi_C : W(M) \rightarrow \mathbb{R} : \pi_C(a) = a(C) \quad (C \in \mathcal{B}_c(M)).$$

- (3) $V(M) = \{a : \mathcal{B}_c(M) \rightarrow \mathbb{R} \mid (*)_1, (*)_2, (*)_3\}$

$$(*)_1 \quad C, D \in \mathcal{B}_c(M), \quad Cl(C - D), Cl(D - C) : \text{compact} \implies a(C) = a(D)$$

$$(*)_2 \quad C, D \in \mathcal{B}_c(M), \quad C \cap D = \emptyset \implies a(C \cup D) = a(C) + a(D)$$

$$(*)_3 \quad a(M) = 0$$

$$V_\mu(M) = \{a \in V(M) \mid (*)_4\}$$

$$(*)_4 \quad C \in \mathcal{B}_c(M), \quad \mu(C) < \infty \implies a(C) = 0$$

$V(M)$ and $V_\mu(M)$ are linear subspaces of $W(M)$, which are equipped with the subspace topology.

4.2. Mass flow homomorphism toward ends $J : \mathcal{H}_E(M, \mu) \rightarrow V_\mu(M)$.

Next we define a continuous group homomorphism $J : \mathcal{H}_E(M, \mu) \rightarrow V_\mu(M)$, which measures a mass moved toward ends by each $h \in \mathcal{H}_E(M, \mu)$. Let $E = E(M)$. Each $h \in \mathcal{H}(M)$ has a unique extension $\bar{h} \in \mathcal{H}(\bar{M})$.

Definition 4.2.

- (1) $\mathcal{H}_E(M, \mu) = \{h \in \mathcal{H}(M, \mu) \mid \bar{h}|_E = id_E\}$ (a subgroup of $\mathcal{H}(M, \mu)$)
- (2) $J : \mathcal{H}_E(M, \mu) \ni h \mapsto J_h \in V_\mu(M)$

$$J_h(C) = \mu(C - h(C)) - \mu(h(C) - C) \quad (C \in \mathcal{B}_c(M))$$

The group $\mathcal{H}_E(M, \mu)$ acts continuously on $V_\mu(M)$ by $h \cdot a = J_h + a$ and the homomorphism $J : \mathcal{H}_E(M, \mu) \rightarrow V_\mu(M)$ coincides with the orbit map at $0 \in V_\mu(M)$.

Theorem 4.1. (Yagasaki [14])

- (1) *The map J admits a section $s : V_\mu(M) \rightarrow \mathcal{H}_\partial(M, \mu)_1 \subset \mathcal{H}_E(M, \mu)$ (i.e., $Js = id$) with $s(0) = id_M$.*
- (2) (i) $\mathcal{H}_E(M; \mu) \cong \text{Ker } J \times V_\mu(M)$ (ii) $\text{Ker } J \subset \mathcal{H}_E(M; \mu) : a \text{ SDR}$

$\text{Ker } J$ contains the subgroup $\mathcal{H}^c(M; \mu)$ of μ -preserving homeomorphisms with compact support. Our next aim is the study of relation between these groups.

5. SPACES OF VOLUME FORMS AND

GROUPS OF VOLUME-PRESERVING DIFFEOMORPHISMS

Suppose M is a connected oriented C^∞ n -manifold without boundary.

Definition 5.1.

- (1) $\mathcal{D}^+(M)$ deotes the group of orientation-preserving diffeomorphisms of M with the compact-open C^∞ -topology.
- (2) For a positive volume form ω on M ,
 $\mathcal{D}(M; \omega)$ denotes the subgroup of ω -preserving diffeomorphisms of M .
- (3) $\mathcal{V}^+(M)_w$ denotes the space of positive volume forms on M equipped with the weak C^∞ topology.

For $m \in (0, \infty]$, $\mathcal{V}^+(M, m)_w = \{\mu \in \mathcal{V}^+(M) \mid \mu(M) = m\}$ (the weak C^∞ topology).

Each $\mu \in \mathcal{V}^+(M)$ determines a unique good Radon measure on M , which is denoted by the same symbol μ . This defines an inclusion $\mathcal{V}^+(M) \subset \mathcal{M}_g^\partial(M)$.

The topological group $\mathcal{D}^+(M)$ acts continuously on $\mathcal{V}^+(M)_w$ and $\mathcal{V}^+(M, m)_w$ by $h \cdot \mu = h_*\mu (= (h^{-1})^*\mu)$. The subgroup $\mathcal{D}(M; \omega)$ coincides with the stabilizer of ω under this action.

5.1. Compact case.

Suppose M is a compact connected oriented C^∞ n -manifold without boundary. Moser's theorem [9] implies the transitivity of this action and its parametrized version.

Theorem 5.1. *Suppose M is a compact connected oriented C^∞ n -manifold.*

- (1) (Transitivity) *For any $\mu, \nu \in \mathcal{V}^+(M, m)$ there exists $h \in \mathcal{D}(M)_1$ such that $h_*\mu = \nu$.*
- (2) (Parametrized version) *Let $\omega \in \mathcal{V}^+(M; m)$. Then the orbit map $\pi : \mathcal{D}^+(M) \rightarrow \mathcal{V}^+(M; m)_w$, $\pi(h) = h_*\omega$, admits a section $\sigma : \mathcal{V}^+(M; m)_w \rightarrow \mathcal{D}(M)_1 \subset \mathcal{D}^+(M)$.*

5.2. Non-compact case.

Suppose M is a non-compact connected C^∞ n -manifold without boundary. Recall that $E = E(M)$ is the space of ends of M and $\overline{M} = M \cup E(M)$ is the end compactification of M . Each $h \in \mathcal{D}(M)$ has a unique extension $\bar{h} \in \mathcal{H}(\overline{M})$. For $\mu \in \mathcal{V}^+(M)$, $E_f(M, \mu)$ denotes the subspace of $E(M)$ consisting of μ -finite ends of M .

Definition 5.2. Suppose $F \subset E(M)$ is an open subset.

- (1) $\mathcal{D}^+(M; F) = \{h \in \mathcal{D}^+(M) \mid \bar{h}(F) = F\}$ (a subgroup of $\mathcal{D}^+(M)$)
- (2) $\mathcal{V}^+(M; F) = \{\mu \in \mathcal{V}^+(M) \mid E_f(M, \mu) = F\}$
 $\mathcal{V}^+(M; m, F) = \mathcal{V}^+(M; m) \cap \mathcal{V}^+(M; F)$
 $\mathcal{M}_g^\partial(M; F) = \{\mu \in \mathcal{M}_g^\partial(M) \mid E_f(M, \mu) = F\}$
- (3) (Finite-end weak topology)

The inclusion $M \xhookrightarrow{\iota} M \cup F (\subset \overline{M})$ induces the injection

$$\begin{array}{ccccc} \iota_\# & : & \mathcal{V}^+(M; m, F) & \subset & \mathcal{M}_g^\partial(M; F) & \xrightarrow{\iota_*} & \mathcal{M}_g(M \cup F)_w \\ & & \nu & \longmapsto & \nu & \longmapsto & \bar{\nu} = \iota_*\nu \end{array}$$

The finite-end weak topology ew on $\mathcal{V}^+(M; m, F)$ is the weakest topology such that the maps $\iota_\#$ and $id : \mathcal{V}^+(M; m, F) \rightarrow \mathcal{V}^+(M; m, F)_w$ are continuous.

The group $\mathcal{D}^+(M; F)$ acts continuously on $\mathcal{V}^+(M; m, F)_{ew}$ by $h \cdot \mu = h_*\mu$ and the stabilizer of $\omega \in \mathcal{V}^+(M; m, F)_w$ coincides with the subgroup $\mathcal{D}(M; \omega)$. Transitivity of this action was verified by R. E. Greene - K. Shiohama [8].

Theorem 5.2. (R. E. Greene - K. Shiohama [8])

For any $\mu, \nu \in \mathcal{V}^+(M; m, F)$ there exists $h \in \mathcal{D}(M)_1$ such that $h_\mu = \nu$.*

A C^∞ -modification of R. Berlanga's argument [3] leads to the parametrized version of this theorem.

Theorem 5.3. (Yagasaki [15])

Suppose P is a paracompact Hausdorff space and $\mu, \nu : P \rightarrow \mathcal{V}^+(M; F)_{ew}$ are maps such that $\mu_p(M) = \nu_p(M)$ ($p \in P$). Then there exists a map $h : P \rightarrow \mathcal{D}(M)_1$ such that

- (i) $h_{p*}\mu_p = \nu_p$ ($p \in P$) and (ii) if $p \in P$ and $\mu_p = \nu_p$, then $h_p = id_M$.

Corollary 5.1. Let $\omega \in \mathcal{V}^+(M; m, F)$.

- (1) The orbit map $\pi : \mathcal{D}^+(M; F) \rightarrow \mathcal{V}^+(M; m, F)_{ew}$, $\pi(h) = h_*\omega$, admits a section $\sigma : \mathcal{V}^+(M; m, F)_{ew} \rightarrow \mathcal{D}(M)_1 \subset \mathcal{D}^+(M; F)$.

- (2) (i) $\mathcal{D}^+(M; F) \cong \mathcal{V}^+(M; m, F)_{ew} \times \mathcal{D}(M; \omega)$ (ii) $\mathcal{D}(M; \omega) \subset \mathcal{D}^+(M; F) : a \text{ SDR}$

5.3. Mass flow toward ends on non-compact C^∞ n -manifolds.

Suppose M is a non-compact connected C^∞ n -manifold without boundary and $\omega \in \mathcal{V}^+(M)$. The topological vector space $V(M)$, $V_\omega(M)$ and a continuous group homomorphism $J^\omega : \mathcal{D}_E(M, \omega) \rightarrow V_\omega(M)$ are defined as in § 4.1 and § 4.2. For $h \in \mathcal{D}_E(M; \omega)$

$$J_h^\omega : \mathcal{B}_c(M) \rightarrow \mathbb{R} : J_h^\omega(C) = \omega(C - h(C)) - \omega(h(C) - C) \quad (C \in \mathcal{B}_c(M)).$$

The group $\mathcal{D}_E(M, \omega)$ acts continuously on $V_\omega(M)$ by

$$h \cdot a = J_h^\omega + a \quad (h \in \mathcal{D}_E(M, \omega), a \in V_\omega(M)).$$

The map $J^\omega : \mathcal{D}_E(M, \omega) \rightarrow V_\omega(M)$ coincides with the orbit map at $0 \in V_\omega(M)$.

Definition 5.3. For two maps $\mu, \nu : P \rightarrow \mathcal{V}^+(M)$ we write as $\mu \underset{c}{\sim} \nu$ if

for any $p \in P$ there exists a neighborhood U of p in P and a compact subset $K \subset M$ such that $\mu_q = \nu_q$ on $M - K$ for any $q \in U$.

Theorem 5.4. Suppose P is a paracompact Hausdorff space and $\mu, \nu : P \rightarrow \mathcal{V}^+(M)_w$, $a : P \rightarrow V(M)$ are maps such that $\mu \underset{c}{\sim} \nu$, $(\mu - \nu)(M) = 0$ and $a_p \in V_{\mu_p}(M)$ ($p \in P$).

Then there exists a map $h : P \rightarrow \mathcal{D}(M)_1$ such that

- (i) $h_{p*}\mu_p = \nu_p$ ($p \in P$) and (ii) if $p \in P$ and $\mu_p = \nu_p$, then $J_{h_p}^{\mu_p} = a_p$.

Corollary 5.2. Let $\omega \in \mathcal{V}^+(M)$.

- (1) The map $J^\omega : \mathcal{D}_E(M, \omega) \rightarrow V_\omega(M)$ admits a section $s : V_\omega(M) \rightarrow \mathcal{D}(M, \omega)_1 \subset \mathcal{D}_E(M, \omega)$ ($J^\omega s = id_{V_\omega(M)}$) with $s(0) = id_M$.

- (2) (i) $\mathcal{D}_E(M; \omega) \cong \text{Ker } J^\omega \times V_\omega(M)$ (ii) $\text{Ker } J^\omega \subset \mathcal{D}_E(M; \omega) : a \text{ SDR}$

Our next aim is to study the relation between two groups $\mathcal{D}^c(M; \omega) \subset \text{Ker } J^\omega$.

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