

Critical Values of the Yamabe Functional

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§1 Introduction.

Let M be a compact connected n -dimensional smooth manifold, and C a conformal class of Riemannian metrics of M , i.e., a set of Riemannian metrics which are conformally related to one another. To be more precise, C is a set which can be written as

$$C = \{e^u g; u \in C^\infty(M)\},$$

where g is an arbitrarily fixed element of C . The Yamabe functional $I: C \rightarrow \mathbb{R}$ is then defined as

$$I(g) = \frac{\int_M R_g dv_g}{\left(\int_M dv_g\right)^{\frac{n-2}{n}}} \quad \text{for } g \in C,$$

where R_g is the scalar curvature function of g , and dv_g is the canonical volume element. The exponent $(n-2)/n$ in the denominator is chosen so that I is invariant by scaling, i.e., $I(g) = I(cg)$ for any positive $c \in \mathbb{R}$ and $g \in C$. It follows from the Gauss-Bonnet formula that I is constant equal to $4\pi \chi(M)$ when $n = 2$. In general, I is bounded below, and we denote by $\mu(M, C)$ the greatest lower bound of $I: C \rightarrow \mathbb{R}$;

$$\mu(M, C) = \inf_{g \in C} I(g).$$

The first variational formula is computed as

$$\left. \frac{d}{dt} I(e^{tu} g) \right|_{t=0} = \frac{n-2}{n} \frac{\int_M (R_g - r_g) u \, dv_g}{\left(\int_M dv_g \right)^{\frac{n-2}{n}}},$$

$$r_g = \int_M R_g \, dv_g / \int_M dv_g$$

for $u \in C^\infty(M)$ and $g \in C$. Therefore, when $n \geq 3$, $g \in C$ is a critical point of $I: C \rightarrow \mathbb{R}$ if and only if $R_g = \text{const.}$, and $r \in \mathbb{R}$ is a critical value of $I: C \rightarrow \mathbb{R}$ if and only if there exists a metric $g \in C$ such that $\text{Vol}(M, g) = 1$ and $R_g = \text{const.} = r$. The Yamabe theorem (cf. [2, 5]) asserts that $\mu(M, C)$ is in fact a critical value of $I: C \rightarrow \mathbb{R}$, and we always have at least one critical value of $I: C \rightarrow \mathbb{R}$. The purpose of this paper is to report some results concerning the following

Conjecture. The number of critical values of the Yamabe functional is finite.

§ 2 Known results.

As was mentioned, $\mu(M, C)$ is the unique critical value of I if $n = 2$, and we are interested in the case when $n \geq 3$. Then C can be written as

$$C = \left\{ u^{\frac{4}{n-2}} g; u \in C^\infty(M) \text{ and } u > 0 \right\},$$

for any fixed $g \in C$. It is straightforward to see

$$I(u^{\frac{4}{n-2}}g) = \frac{\frac{4(n-1)}{n-2} \int_M |du|^2 dv_g + \int_M R_g dv_g}{\left(\int_M u^{\frac{2n}{n-2}} dv_g \right)^{\frac{n-2}{n}}},$$

and

$$\mu(M, C) = \inf_{u>0} I(u^{\frac{4}{n-2}}g).$$

From this expression, we can see $\mu(M, C) \leq \max R_g$ for any $g \in C$ with $\text{Vol}(M, g) = 1$. Moreover, from the Sobolev inequality, we have $\min R_g \leq 0$ for any $g \in C$ if $\mu(M, C) \leq 0$. Then it is easy to see the following.

Lemma. If $\mu(M, C) \leq 0$, then $\min R_g \leq \mu(M, C) \leq \max R_g$ for any $g \in C$ with $\text{Vol}(M, g) = 1$.

As a corollary we have

Fact 1. If $\mu(M, C) \leq 0$, $\mu(M, C)$ only is a critical value of $I: C \rightarrow \mathbb{R}$.

Here, the assumption $\mu(M, C) \leq 0$ is essential, and the problem becomes more difficult when $\mu(M, C) > 0$. The following fact due to M. Obata [4] is one of few results about the critical values of the Yamabe functional which are applicable to cases when $\mu(M, C) > 0$.

Fact 2. If C contains an Einstein metric, $\mu(M, C)$ is the unique critical value of $I: C \rightarrow \mathbb{R}$.

Now suppose $M = S^1 \times S^{n-1}$, $n \geq 3$, and $g_\lambda = ds^2 + h_0$, where h_0 is the standard metric of the Euclidean $(n-1)$ -sphere S^{n-1} and ds^2 is a metric of S^1 with $\text{length}(S^1, ds^2) = \lambda$. Let C_λ be the conformal class containing the metric g_λ of $S^1 \times S^{n-1}$. We can

apply a theorem of Gidas, Ni and Nirenberg [1] in order to get all critical points of $I:C_\lambda \rightarrow \mathbb{R}$. As a result, we have

Fact 3. There are only finitely many critical values of
 $I:C_\lambda \rightarrow \mathbb{R}$.

An interesting observation about this example is that the number of critical values increases to infinity as λ goes to infinity.

§ 3 Almost critical values.

So far we have seen a few results which are partial evidences to our conjecture. Here we change slightly our point of view.

Definition. $r \in \mathbb{R}$ is called an almost critical value of the Yamabe functional $I:C \rightarrow \mathbb{R}$ if for any $\varepsilon > 0$ there is a metric $g \in C$ with $\text{Vol}(M, g) = 1$ such that $|R_g(x) - r| < \varepsilon$ for all $x \in M$.

It is clear that every critical value is almost critical.

Theorem. If $\mu(M, C) > 0$ and $\dim M \geq 3$, there are infinitely many almost critical values of the Yamabe functional $I:C \rightarrow \mathbb{R}$. Otherwise, $\mu(M, C)$ only is an almost critical value of I .

Proof. The latter part follows immediately from the lemma in § 2. The first part is a restatement of Theorem 4 of [3]. $\square$

References.

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