

ON THE CONFLUENT HYPERGEOMETRIC FUNCTIONS IN 2 VARIABLES

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§0. Introduction.

Let $M(r, n)$ be the set of complex matrices, $\lambda = (\lambda_0, \dots, \lambda_{l-1})$ a partition of n . We consider the action of $GL(r, \mathbb{C}) \times H_\lambda$ on $Z_{r,n} := \{z \in M(r, n) : \text{rank } z = r\}$ defined by

$$(*) \quad \begin{aligned} GL(r, \mathbb{C}) \times Z_{r,n} \times H_\lambda &\longrightarrow Z_{r,n} \\ (g, z, h) &\longmapsto gzh, \end{aligned}$$

where $H_\lambda = J(\lambda_0) \times \dots \times J(\lambda_{l-1}) \subset GL(n)$ be the associated maximal abelian subgroup with respect to λ , $J(m) = \left\{ \sum_{i=0}^{m-1} h_i \Lambda^i : h_0 \in \mathbb{C}^\times, h_1, \dots, h_{m-1} \in \mathbb{C} \right\}$

$$\Lambda := \begin{pmatrix} 0 & 1 & & 0 \\ & 0 & \ddots & \\ & \ddots & \ddots & 1 \\ 0 & & & 0 \end{pmatrix}$$

Let $\iota : H_\lambda \longrightarrow \prod_i (\mathbb{C}^\times \times \mathbb{C}^{\lambda_i - 1})$. For $z \in Z_{r,n}$, the generalized confluent hypergeometric function (CHG function, for short) is defined as

$$(0.1) \quad \Phi(z; \alpha) = \int_{\Delta} \chi(\iota^{-1}(tz); \alpha) \cdot \omega$$

where α be an n -tuple of complex numbers satisfying $\sum_{i=0}^{l-1} \alpha_0^{(i)} = -r$, λ the character of the universal covering group of H_λ , and Δ is a *twisted cycle* in the

t -space depending on z and α . The function Φ admits the following symmetries:

$$(0.2) \quad \Phi(gz; \alpha) = (\det g)^{-1} \Phi(z; \alpha) \quad g \in GL(r, \mathbb{C})$$

$$(0.3) \quad \Phi(zh_\lambda; \alpha) = \chi(h_\lambda) \Phi(z; \alpha) \quad h_\lambda \in H_\lambda.$$

$$(0.4) \quad \Phi(zw_\lambda; \alpha) = \Phi(z; \alpha^t w_\lambda) \quad w_\lambda \in W_\lambda,$$

where W_λ is an analogue of the Weyl group, see [K-K] and Section 1.

The CHG functions Φ on $Z_{2,4}$ and $Z_{2,5}$ for various partitions λ of 4 and 5 were investigated in the papers [K-H-T], [O-K] and [K-K]. It is known that the functions Φ are generalizations of Gauss', Kummer's, Bessel's, Hermite's, Airy's functions and the classical hypergeometric functions of two variables, i.e., $F_1, \phi_1, \phi_2, \phi_3, G_2, \Gamma_1, \Gamma_2$ in Horn's list ([Erd 1]). In this talk, we study the hypergeometric functions of type λ in two variables on the strata of the set $M(3, 6)$ of 3×6 complex matrices

§1. Construction of the group W_λ .

Set $\lambda^{(0)} = (1, 1, 1, 1, 1, 1)$, $\lambda^{(1)} = (2, 1, 1, 1, 1)$, $\lambda^{(2)} = (2, 2, 1, 1)$, $\lambda^{(3)} = (2, 2, 2)$, $\lambda^{(4)} = (3, 1, 1, 1)$, $\lambda^{(5)} = (3, 2, 1)$, $\lambda^{(6)} = (3, 3)$, $\lambda^{(7)} = (4, 1, 1)$ and $\lambda^{(8)} = (4, 2)$. We set

$$\begin{aligned} P_{\lambda^{(0)}} &= \mathfrak{S}_6, \quad P_{\lambda^{(1)}} = \left\{ \begin{pmatrix} I_2 & 0 \\ 0 & \mathfrak{S}_4 \end{pmatrix} \right\} \\ P_{\lambda^{(2)}} &= \left\{ \begin{pmatrix} I_2 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & \mathfrak{S}_2 \end{pmatrix}, \begin{pmatrix} 0 & I_2 & 0 \\ I_2 & 0 & 0 \\ 0 & 0 & \mathfrak{S}_2 \end{pmatrix} \right\} \\ P_{\lambda^{(3)}} &= \left\{ \begin{pmatrix} I_2 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_2 \end{pmatrix}, \begin{pmatrix} I_2 & 0 & 0 \\ 0 & 0 & I_2 \\ 0 & I_2 & 0 \end{pmatrix}, \begin{pmatrix} 0 & I_2 & 0 \\ I_2 & 0 & 0 \\ 0 & 0 & I_2 \end{pmatrix}, \right. \\ &\quad \left. \begin{pmatrix} 0 & I_2 & 0 \\ 0 & 0 & I_2 \\ I_2 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & I_2 \\ I_2 & 0 & 0 \\ 0 & I_2 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & I_2 \\ 0 & I_2 & 0 \\ I_2 & 0 & 0 \end{pmatrix} \right\} \\ P_{\lambda^{(4)}} &= \left\{ \begin{pmatrix} I_3 & 0 \\ 0 & \mathfrak{S}_3 \end{pmatrix} \right\}, \quad P_{\lambda^{(5)}} = \left\{ \begin{pmatrix} I_3 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\} \\ P_{\lambda^{(6)}} &= \left\{ \begin{pmatrix} I_3 & 0 \\ 0 & I_3 \end{pmatrix}, \begin{pmatrix} 0 & I_3 \\ I_3 & 0 \end{pmatrix} \right\} \\ P_{\lambda^{(7)}} &= \left\{ \begin{pmatrix} I_4 & 0 \\ 0 & \mathfrak{S}_2 \end{pmatrix} \right\}, \quad P_{\lambda^{(8)}} = \left\{ \begin{pmatrix} I_4 & 0 \\ 0 & I_2 \end{pmatrix} \right\} \end{aligned}$$

where I_i is the $i \times i$ identity matrix, $\mathfrak{S}_i$ is the group of $i \times i$ permutation matrices and

$$\left\{ \begin{pmatrix} * & 0 \\ 0 & \mathfrak{S}_i \end{pmatrix} \right\} := \left\{ \begin{pmatrix} * & 0 \\ 0 & A \end{pmatrix} : A \in \mathfrak{S}_i \right\}.$$

Then we have the following proposition (see [K-K]).

Proposition 1.1. *For the partitions $\lambda^{(\nu)}$, the Weyl groups $W_{\lambda^{(\nu)}}$ ($\nu = 0, \dots, 8$) are given by*

$$W_{\lambda^{(\nu)}} = R_{\lambda^{(\nu)}} \rtimes P_{\lambda^{(\nu)}},$$

where

$$\begin{aligned} R_{\lambda^{(0)}} &= I_6 & R_{\lambda^{(1)}} &= \text{diag}(W(2), I_4) \\ R_{\lambda^{(2)}} &= \text{diag}(W(2), W(2), I_2) & R_{\lambda^{(3)}} &= \text{diag}(W(2), W(2), W(2)) \\ R_{\lambda^{(4)}} &= \text{diag}(W(3), I_3) & R_{\lambda^{(5)}} &= \text{diag}(W(3), W(2), 1) \\ R_{\lambda^{(6)}} &= \text{diag}(W(3), W(3)) & R_{\lambda^{(7)}} &= \text{diag}(W(4), I_2) \\ R_{\lambda^{(8)}} &= \text{diag}(W(4), W(2)). \end{aligned}$$

§2. Orbital decomposition of the set of strata.

Set $D(i, j, k) = \det(z_i, z_j, z_k)$ for $z = (z_0, z_1, \dots, z_5) \in M(3, 6)$.

Definition 2.1. *Let λ be a Young diagram of weight 6, $(i, j, k), (i, m, n)$ two subdiagrams of λ , where i, j, k, m, n are mutually distinct. We denote by the symbol $\{(i, j, k), (i, m, n)\}$ the set*

$$\left\{ z \in M(3, 6) \mid \begin{array}{l} D(i, j, k) = D(i, m, n) = 0, \\ D(p, q, r) \neq 0 \text{ for any} \\ \text{other subdiagram } (p, q, r) \end{array} \right\}$$

and call it a general stratum of type $(3, 6)$ associated to λ (for short, a stratum).

Let S_λ denote the set of strata $\{(i, j, k), (i, m, n)\}$ associated to the Young diagram λ . We simply write S for $S_{\lambda^{(0)}}$.

Proposition 2.1. (1) *The Weyl group W acts transitively on S .*

(2) $\#S = 90$.

Proposition 2.2. Under the action of $P_{\lambda(\nu)}$, the orbital decomposition of $S_{\lambda(\nu)}$ is described as $S_{\lambda(\nu)} = \coprod_i O_{P_{\lambda(\nu)}}(s_\nu^i)$, where

$$\begin{aligned} s_1^1 &= \{(0, 1, 2), (0, 4, 5)\}, s_1^2 = \{(4, 0, 1), (4, 2, 3)\}, s_1^3 = \{(4, 0, 5), (4, 2, 3)\}, \\ s_1^4 &= \{(0, 2, 3), (0, 4, 5)\}; s_2^1 = \{(0, 1, 2), (0, 4, 5)\}, s_2^2 = \{(2, 0, 1), (2, 3, 4)\}, \\ s_2^3 &= \{(4, 0, 1), (4, 2, 3)\}, s_2^4 = \{(0, 2, 3), (0, 4, 5)\}, s_2^5 = \{(0, 1, 4), (0, 2, 5)\}, \\ s_2^6 &= \{(4, 0, 5), (4, 2, 3)\}; s_3^1 = \{(0, 1, 2), (0, 4, 5)\}, s_3^2 = \{(4, 0, 1), (4, 2, 3)\}; \\ s_4^1 &= \{(0, 1, 2), (0, 3, 4)\}, s_4^2 = \{(0, 1, 3), (0, 4, 5)\}, s_4^3 = \{(3, 0, 1), (3, 4, 5)\}; \\ s_5^1 &= \{(0, 1, 2), (0, 3, 4)\}, s_5^2 = \{(0, 1, 2), (0, 3, 5)\}, s_5^3 = \{(3, 0, 1), (3, 4, 5)\}, \\ s_5^4 &= \{(5, 0, 1), (5, 3, 4)\}; s_6^1 = \{(0, 1, 2), (0, 3, 4)\}; s_7^1 = \{(0, 1, 2), (0, 4, 5)\}; \\ s_8^1 &= \{(0, 1, 2), (0, 4, 5)\}. \end{aligned}$$

§3. Normal forms of matrices.

Set $Z^\nu = \bigcup_{s \in S_{\lambda(\nu)}} s$. For simplicity we write Z for Z^0 . By the action of $GL(3) \times H_{\lambda(\nu)}$ on Z^ν defined by (*), We first describe how to take the elements of $GL(3) \setminus Z/H$ as the normal forms of $z \in Z$. We fix one stratum $s_0 = \{(4, 0, 1), (4, 2, 3)\} \in S$.

Proposition 3.1. For each $i = 0, \dots, 14$, the following assertion holds:
For any $z \in s_0$ there exists a unique $(x, y) \in \mathbb{C}^2$ such that

$$(1) \quad f_i(x, y) \neq 0, \quad (2) \quad z \in GL(3) \vec{z}_i(x, y) H,$$

where $\vec{z}_i = \vec{z}_i(x, y)$ and $f_i = f_i(x, y)$ are given by

$$\begin{aligned} \vec{z}_0(x, y) &= \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & y \\ 0 & 0 & 1 & 1 & 0 & x \\ 0 & 1 & 0 & 1 & 1 & 1 \end{pmatrix} & \vec{z}_1(x, y) &= \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & y \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & x & 1 & 1 \end{pmatrix} \\ \vec{z}_2(x, y) &= \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & y & 0 & x & 1 & 1 \end{pmatrix} & \vec{z}_3(x, y) &= \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & x \\ 0 & y & 0 & 1 & 1 & 1 \end{pmatrix} \\ \vec{z}_4(x, y) &= \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & y & 0 & 1 & 1 & x \end{pmatrix} & \vec{z}_5(x, y) &= \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & y \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & x \end{pmatrix} \\ \vec{z}_6(x, y) &= \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & x & 1 & y \end{pmatrix} & \vec{z}_7(x, y) &= \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & x \\ 0 & 1 & 0 & 1 & 1 & y \end{pmatrix} \end{aligned}$$

$$\begin{aligned}
\vec{z}_8(x, y) &= \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & x \\ 0 & 0 & 1 & y & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 \end{pmatrix} & \vec{z}_9(x, y) &= \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & y & 0 & 1 \\ 0 & x & 0 & 1 & 1 & 1 \end{pmatrix} \\
\vec{z}_{10}(x, y) &= \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & y & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & x \end{pmatrix} & \vec{z}_{11}(x, y) &= \begin{pmatrix} 1 & y & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & x \\ 0 & 1 & 0 & 1 & 1 & 1 \end{pmatrix} \\
\vec{z}_{12}(x, y) &= \begin{pmatrix} 1 & y & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & x & 1 & 1 \end{pmatrix} & \vec{z}_{13}(x, y) &= \begin{pmatrix} 1 & y & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & x & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 \end{pmatrix} \\
\vec{z}_{14}(x, y) &= \begin{pmatrix} 1 & x & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & y \end{pmatrix}
\end{aligned}$$

and

$$\begin{aligned}
f_i(x, y) &= xy(1-x)(1-y)(1-x-y) & \text{for } i = 0, \dots, 3 \\
f_i(x, y) &= xy(1-x)(x-y)(1-x+y) & \text{for } i = 4, 5 \\
f_i(x, y) &= xy(y-x)(1-y)(1+x-y) & \text{for } i = 6, 7 \\
f_i(x, y) &= xy(1-x)(xy-1)(1-xy+y) & \text{for } i = 8, 9 \\
f_i(x, y) &= xy(1-x)(xy-1)(1-xy+y) & \text{for } i = 10 \\
f_i(x, y) &= xy(1-x)(1-y)(1+xy-y) & \text{for } i = 11, 12 \\
f_i(x, y) &= xy(1-x)(1-y)(xy-x-y) & \text{for } i = 13 \\
f_i(x, y) &= xy(xy-1)(1-y)(1-xy+x) & \text{for } i = 14.
\end{aligned}$$

We set $N = \{\vec{z}_i : 0 \leq i \leq 14\}$. For the stratum s_0 , we denote by N_{s_0} the set $\{\vec{z} = \sigma \vec{z}_i : \sigma \in \mathfrak{S}_3, \vec{z}_i \in N\}$ and call $\vec{z} \in N_{s_0}$ a *normal form* of $z \in s_0$.

Proposition 3.2. *The normal forms of the matrices in any other stratum $s \in S$ can be obtained by the action of Weyl group $W \simeq \mathfrak{S}_6$ on N_{s_0} .*

Using the same method as above, we can obtain the normal forms of $z \in Z_\nu$ for $\nu = 1, 2, \dots, 8$.

§4 CHG functions and the classical CHG functions of 2 variables.

Definition 4.1. For $0 \leq \nu \leq 8$, the function Φ given by (0.1), i.e.,

$$(4.1) \quad \Phi(z; \alpha) = \int_{\Delta} \chi(\iota^{-1}(tz); \alpha) \cdot \omega \quad \text{for } z \in Z^\nu$$

is called the confluent hypergeometric function of type $\lambda^{(\nu)}$ (for short CHG function).

By the symmetry

$$\Phi(zw_\lambda; \alpha) = \Phi(z; \alpha^t w_\lambda) \quad \text{for } w_\lambda \in W_\lambda,$$

we can list up the functions Φ on Z^ν with normalized parameters (see Table III). On the other hand, there is a list of the classical CHG functions of two variables, which is known as Horn's list (see [Erd 1]). Integral representations of these functions have been investigated by M. Kita [Ki], M. Yoshida [Y], B. Dwork and F. Loeser [D-L].

We reinterpret some of the functions $F_2, \Psi_1, \Psi_2, F_3, \Xi_1, \Xi_2, H_2, \mathbf{H}_k (k = 2, 3, 4, 5, 11)$ in list in terms of the CHG functions. The changes of variables

$$\begin{aligned} (u, v) &\longrightarrow (-1/u, -1/v) && \text{for } F_2, \Psi_1 \text{ and } \Psi_2, \\ (u, v) &\longrightarrow (-u, -v) && \text{for } F_3 \text{ and } \Xi_1, \\ (u, v) &\longrightarrow (-u, 1/v) && \text{for } \Xi_2, \\ (u, v) &\longrightarrow (-1/u, v) && \text{for } H_2 \text{ and } \mathbf{H}_k (k = 2, 3, 4, 5, 11) \end{aligned}$$

transform the integral representations of these functions into the following:

$$\begin{aligned} F_2 : & \quad v^{\beta'-\gamma'}(v+y)^{-\beta'}u^{\beta-\gamma}(u+x)^{-\beta}(1+u+v)^{\gamma+\gamma'-\alpha-2}dudv \\ \Psi_1 : & \quad v^{-\gamma'}\exp(-\frac{y}{v})u^{\beta-\gamma}(u+x)^{-\beta}(1+u+v)^{\gamma+\gamma'-\alpha-2}dudv \\ \Psi_2 : & \quad v^{-\gamma'}\exp(-\frac{y}{v})u^{-\gamma}\exp(-\frac{x}{u})(1+u+v)^{\gamma+\gamma'-\alpha-2}dudv \\ F_3 : & \quad (1+yv)^{-\beta'}v^{\alpha'-1}(1+u+v)^{\gamma-\alpha-\alpha'-1}u^{\alpha-1}(1+xu)^{-\beta}dudv \\ \Xi_1 : & \quad \exp(-yv)v^{\alpha'-1}(1+u+v)^{\gamma-\alpha-\alpha'-1}u^{\alpha-1}(1+xu)^{-\beta}dudv \\ \Xi_2 : & \quad \exp(-yv)v^{\gamma-\beta-2}\exp(-\frac{u+1}{v})u^{\beta-1}(1+xu)^{-\alpha}dudv \\ H_2 : & \quad v^{\gamma-1}(1+u+v)^{\delta-\alpha-\gamma-1}(1-yu)^{-\beta'}u^{\beta-\delta}(u+x)^{-\beta}dudv \\ \mathbf{H}_2 : & \quad v^{\delta-\alpha-2}\exp(-\frac{u+1}{v})(1-yv)^{-\beta'}u^{\beta-\delta}(u+x)^{-\beta}dudv \\ \mathbf{H}_3 : & \quad v^{\delta-\alpha-2}\exp(-\frac{u+1}{v})\exp(yv)u^{\beta-\delta}(u+x)^{-\beta}dudv \\ \mathbf{H}_5 : & \quad v^{\delta-\alpha-2}\exp(-\frac{u+1}{v})\exp(yv)u^{-\delta}\exp(-\frac{x}{u})dudv \end{aligned}$$

$$H_2 : u^{\beta-\delta}(u+x)^{-\beta}(1-yv)^{-\beta'}v^{\gamma-1}(1+u+v)^{\delta-\alpha-\gamma-1}dudv$$

$$\mathbf{H}_{11} : u^{-\delta}\exp\left(-\frac{x}{u}\right)(1-yv)^{-\beta'}v^{\gamma-1}(1+u+v)^{\delta-\alpha-\gamma-1}dudv$$

$$\mathbf{H}_4 : u^{-\delta}\exp\left(-\frac{x}{u}\right)\exp(yv)v^{\gamma}(1+u+v)^{\delta-\alpha-\gamma-1}dudv$$

For these functions, the corresponding partitions λ and normal forms $\vec{x}_i = \vec{x}_i(x, y)$ are tabulated in the following:

TABLE II

Function (λ)	Normal form $\vec{x}_i = \vec{x}_i(x, y)$	$g_i(x, y)$
$F_2 (\lambda^{(0)})$	$\vec{x}_1 = \begin{pmatrix} 0 & y & 0 & x & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 1 \end{pmatrix}$	$xy(1-x-y)$ $(1-x)(1-y)$
$\Psi_1 (\lambda^{(1)})$	$\vec{x}_2 = \begin{pmatrix} 0 & y & 0 & x & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$	$xy(x-1)$
$\Psi_2 (\lambda^{(2)})$	$\vec{x}_3 = \begin{pmatrix} 0 & y & 0 & x & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$	xy
$F_3 (\lambda^{(0)})$	$\vec{x}_4 = \begin{pmatrix} 1 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & x \\ 0 & y & 1 & 1 & 0 & 0 \end{pmatrix}$	$xy(xy-x-y)$ $(1-x)(1-y)$
$\Xi_1 (\lambda^{(1)})$	$\vec{x}_5 = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & x \\ 0 & y & 1 & 1 & 0 & 0 \end{pmatrix}$	$xy(x-1)$
$\Xi_2 (\lambda^{(2)})$	$\vec{x}_6 = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & x \\ 0 & y & 1 & 0 & 0 & 0 \end{pmatrix}$	$xy(x-1)$
$H_2 (\lambda^{(0)})$	$\vec{x}_7 = \begin{pmatrix} 0 & 1 & 1 & 1 & 0 & x \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & y & 0 & 0 \end{pmatrix}$	$xy(xy-y+1)$ $(1-x)(1-y)$
$\mathbf{H}_2 (\lambda^{(1)})$	$\vec{x}_8 = \begin{pmatrix} 0 & 1 & 1 & 1 & 0 & x \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & y & 0 & 0 \end{pmatrix}$	$xy(x-1)$

Function (λ)	Normal form $\vec{x}_i = \vec{x}_i(x, y)$	$g_i(x, y)$
$\mathbf{H}_3 (\lambda^{(2)})$	$\vec{x}_9 = \begin{pmatrix} 0 & 1 & 1 & 0 & 0 & x \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & y & 0 & 0 \end{pmatrix}$	$xy(x-1)$
$\mathbf{H}_5 (\lambda^{(3)})$	$\vec{x}_{10} = \begin{pmatrix} 0 & 1 & 1 & 0 & 0 & x \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & y & 0 & 0 \end{pmatrix}$	xy
$H_2 (\lambda^{(0)})$	$\vec{x}_{11} = \begin{pmatrix} 0 & x & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & y & 1 & 1 \end{pmatrix}$	$xy(xy-y+1)$ $(1-x)(1-y)$
$\mathbf{H}_{11}(\lambda^{(1)})$	$\vec{x}_{12} = \begin{pmatrix} 0 & x & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & y & 1 & 1 \end{pmatrix}$	$xy(y-1)$
$\mathbf{H}_4 (\lambda^{(2)})$	$\vec{x}_{13} = \begin{pmatrix} 0 & x & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & y & 1 & 1 \end{pmatrix}$	xy

For the normal forms $\vec{x}_i = \vec{x}_i(x, y)$, the variables $(x, y) \in \mathbb{C}^2$ are subject to the condition $g_i(x, y) \neq 0$.

Proposition 4.1. Let $\lambda^{(\nu)}$ and the normal forms $\vec{x}_i = \vec{x}_i(x, y)$ be given in Table III. The CHG functions on $GL(3) \setminus Z^\nu / H_{\lambda^{(\nu)}}$ with the normalized parameters β_ν ($0 \leq \nu \leq 3$) are related with the classical hypergeometric functions of two variables, for instance, as

$$\begin{aligned}
\Phi_{\lambda^{(0)}}(\vec{x}_1; \beta_0) &= \int_{\Delta_1} v^{\alpha_0} (v+y)^{\alpha_1} u^{\alpha_2} (u+x)^{\alpha_3} (1+u+v)^{\alpha_5} du dv \\
&= C_1 F_2(\alpha_4+1, -\alpha_3, -\alpha_1, -\alpha_2-\alpha_3, -\alpha_0-\alpha_1; x, y) \\
\Phi_{\lambda^{(1)}}(\vec{x}_2; \beta_1) &= \int_{\Delta_2} v^{\alpha_0} \exp\left(-\frac{y}{v}\right) u^{\alpha_2} (u+x)^{\alpha_3} (1+u+v)^{\alpha_5} du dv \\
&= C_2 \Psi_1(\alpha_4+1, -\alpha_3, -\alpha_3-\alpha_2, -\alpha_0; x, y) \\
\Phi_{\lambda^{(2)}}(\vec{x}_3; \beta_2) &= \int_{\Delta_3} v^{\alpha_0} \exp\left(-\frac{y}{v}\right) u^{\alpha_2} \exp\left(-\frac{x}{u}\right) (1+u+v)^{\alpha_5} du dv \\
&= C_3 \Psi_2(\alpha_4+1, -\alpha_2, -\alpha_0; x, y).
\end{aligned}$$

The properties of those functions can be described in the following Table.

TABLE III

Type	Function	Stratum	Orbit
(1, 1, 1, 1, 1, 1)	$G_1 = F_2$	$\{(4, 0, 1), (4, 2, 3)\}$	S
(2, 1, 1, 1, 1)	$G_2 = \Psi_1$	"	$O_{P_{\lambda(1)}}(s_1^2)$
(2, 2, 1, 1)	$G_3 = \Psi_2$	"	$O_{P_{\lambda(2)}}(s_2^3)$
(1, 1, 1, 1, 1, 1)	$G_4 = F_3$	$\{(0, 1, 2), (0, 4, 5)\}$	S
(2, 1, 1, 1, 1)	$G_5 = \Xi_1$	"	$O_{P_{\lambda(1)}}(s_1^1)$
(2, 2, 1, 1)	$G_6 = \Xi_2$	"	$O_{P_{\lambda(2)}}(s_2^1)$
(1, 1, 1, 1, 1, 1)	$G_7 = H_2$	$\{(2, 0, 3), (2, 4, 5)\}$	S
(2, 1, 1, 1, 1)	$G_8 = \mathbf{H}_2$	"	$O_{P_{\lambda(1)}}(s_1^3)$
(2, 2, 1, 1)	$G_9 = \mathbf{H}_3$	"	$O_{P_{\lambda(2)}}(s_2^1)$
(2, 2, 2)	$G_{10} = \mathbf{H}_5$	"	$O_{P_{\lambda(3)}}(s_3^2)$
(1, 1, 1, 1, 1, 1)	$G_{11} = H_2$	$\{(2, 0, 1), (2, 3, 4)\}$	S
(2, 1, 1, 1, 1)	$G_{12} = \mathbf{H}_{11}$	"	$O_{P_{\lambda(1)}}(s_1^2)$
(2, 2, 1, 1)	$G_{13} = \mathbf{H}_4$	"	$O_{P_{\lambda(2)}}(s_2^2)$

G_i is a multi-valued holomorphic function in the domain:

$$(4.2) \quad X_i = \{(x, y) \in \mathbb{C}^2 : g_i(x, y) \neq 0\},$$

where $g_i(x, y)$ ($1 \leq i \leq 13$) are given in Table II.

Note that the functions $\{F_2, F_3, H_2\}$, $\{\Xi_2, \mathbf{H}_3\}$, $\{\Psi_1, \mathbf{H}_{11}\}$ belong to the same orbits, respectively.

§5 Transformation formulae.

We systematically deduce some transformation formulae for the systems of partial differential equations from the symmetries for the function Φ .

$$\begin{aligned}
& F_3(\alpha, \alpha', \beta, \beta', \gamma; x, y) \\
&= x^{-\beta} y^{-\beta'} F_2(\beta + \beta' - \gamma + 1, \beta, \beta', \beta - \alpha + 1, \beta' - \alpha' + 1; \frac{1}{x}, \frac{1}{y}). \\
& H_2(\alpha, \beta, \beta', \gamma, \delta; x, y) \\
&= y^{-\beta'} F_2(\alpha + \beta', \beta, \beta', \delta, \beta' - \gamma + 1; x, -\frac{1}{y}), \\
& F_3(\alpha, \alpha', \beta, \beta', \gamma; x, y) \\
&= x^{-\alpha} H_2(\alpha - \gamma + 1, \alpha, \alpha', \beta', \alpha - \beta + 1; \frac{1}{x}, -y), \\
& H_{11}(\alpha, \beta', \gamma, \delta; x, y) \\
&= y^{-\beta'} \Psi_1(\alpha + \beta', \beta', \beta' - \gamma + 1, \delta; -\frac{1}{y}, x), \\
& H_3(\alpha, \beta, \delta; x, y) \\
&= x^{-\beta} \Xi_2(\beta, \beta - \delta + 1, -\alpha + \beta + 1; \frac{1}{x}, -y).
\end{aligned}$$

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