

On the Abhyankar's question for affine plane curves with one place at infinity

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1 Introduction

Let C be an irreducible algebraic curve in complex affine plane $\mathbb{C}^2$. We say that C has *one place at infinity*, if the closure of C intersects with the ∞ -line in $\mathbb{P}^2$ at only one point P and C is locally irreducible at that point P .

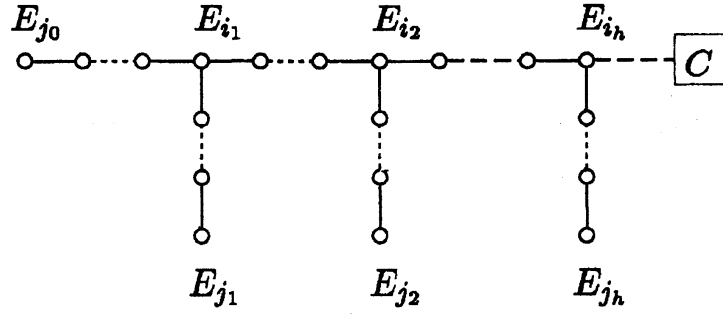
The problem of finding the canonical models of curves with one place at infinity under the polynomial transformations of the coordinates of $\mathbb{C}^2$ has been studied by many mathematicians since Suzuki [10] and Abhyankar–Moh [2] proved independently that the canonical model of C is a line when C is non-singular and simply connected.

Sathaye [8] introduce the Abhyankar's question for curves with one place at infinity and Sathaye–Stenerson [9] suggested a candidate of counter example for this question. However, they could not give the answer to the question since the root computation for a huge polynomial system was required.

We found a counter example for the Abhyankar's question using computer algebra system. In this report, we give the details.

2 Preliminaries

Let C be a curve with one place at infinity defined by a polynomial equation $f(x, y) = 0$ in the complex affine plane $\mathbb{C}^2$. Assume that $\deg_x f = m$, $\deg_y f = n$ and $d = \gcd(m, n)$. The dual graph corresponding to the minimal resolution of the singularity of C at infinity is the following [11]:

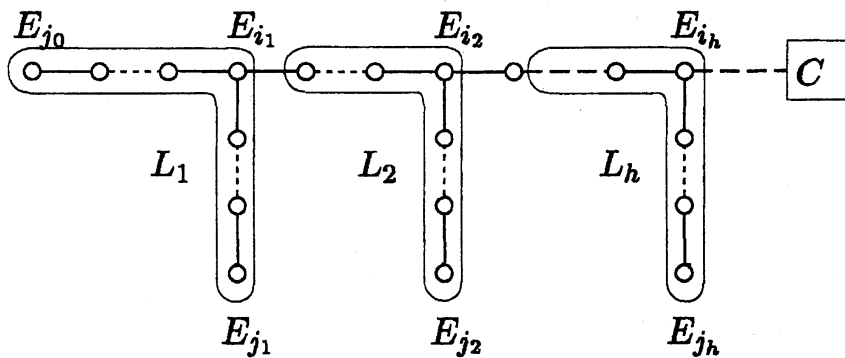


DEFINITION 1 (δ -sequence) *Let f be the defining polynomial of a curve C with one place at infinity. Let δ_k ($0 \leq k \leq h$) be the order of the pole of f on E_{j_k} in the above dual graph. We shall call the sequence $\{\delta_0, \delta_1, \dots, \delta_h\}$ the δ -sequence of C (or of f).*

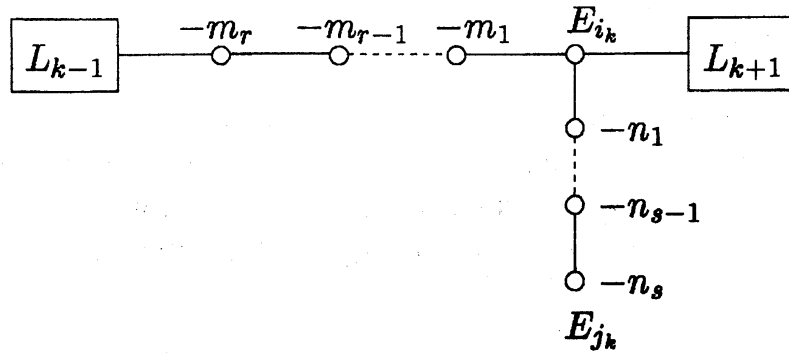
We have the following fact since $\deg_x f = m$ and $\deg_y f = n$.

Fact 1 $\delta_0 = n, \delta_1 = m$

We set L_k for each k ($1 \leq k \leq h$) like the following figure:



DEFINITION 2 ((p, q) -sequence) *Now, we assume that the weights of L_k is of the following form:*



We define the natural numbers p_k, a_k, q_k, b_k satisfying

$$(p_k, a_k) = 1, (q_k, b_k) = 1, 0 < a_k < p_k, 0 < b_k < q_k,$$

$$\frac{p_k}{a_k} = m_1 - \frac{1}{m_2 - \frac{1}{m_3 - \dots - \frac{1}{m_r}}} \quad \text{and} \quad \frac{q_k}{b_k} = n_1 - \frac{1}{n_2 - \frac{1}{n_3 - \dots - \frac{1}{n_s}}}.$$

We shall call the sequence $\{(p_1, q_1), (p_2, q_2), \dots, (p_h, q_h)\}$ the (p, q) -sequence of C (or of f).

There are the following Abhyankar–Moh’s semigroup theorem and its converse theorem by Sathaye–Stenerson as results for δ -sequence. We set $\mathbb{N} = \{n \in \mathbb{Z} \mid n \geq 0\}$ and $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$.

Theorem 1 (Abhyankar–Moh [1, 3, 4]) Let C be an affine plane curve with one place at infinity. Let $\{\delta_0, \delta_1, \dots, \delta_h\}$ be the δ -sequence of C and $\{(p_1, q_1), \dots, (p_h, q_h)\}$ be the (p, q) -sequence of C . We set $d_k = \gcd\{\delta_0, \delta_1, \dots, \delta_{k-1}\}$ ($1 \leq k \leq h+1$). We have then,

- (i) $q_k = d_k/d_{k+1}, d_{h+1} = 1$ ($1 \leq k \leq h$),
- (ii) $d_{k+1}p_k = \begin{cases} \delta_1 & (k=1) \\ q_{k-1}\delta_{k-1} - \delta_k & (2 \leq k \leq h) \end{cases},$
- (iii) $q_k\delta_k \in \mathbb{N}\delta_0 + \mathbb{N}\delta_1 + \dots + \mathbb{N}\delta_{k-1}$ ($1 \leq k \leq h$).

Theorem 2 (Sathaye–Stenerson [9]) Let $\{\delta_0, \delta_1, \dots, \delta_h\}$ ($h \geq 1$) be the sequence of $h+1$ natural numbers. We set $d_k = \gcd\{\delta_0, \delta_1, \dots, \delta_{k-1}\}$ ($1 \leq k \leq h+1$) and $q_k = d_k/d_{k+1}$ ($1 \leq k \leq h$). Furthermore, suppose that the following conditions are satisfied :

- (1) $\delta_0 < \delta_1$,
- (2) $q_k \geq 2$ ($1 \leq k \leq h$),
- (3) $d_{h+1} = 1$,
- (4) $\delta_k < q_{k-1}\delta_{k-1}$ ($2 \leq k \leq h$),
- (5) $q_k\delta_k \in \mathbb{N}\delta_0 + \mathbb{N}\delta_1 + \cdots + \mathbb{N}\delta_{k-1}$ ($1 \leq k \leq h$).

Then, there exists a curve with one place at infinity of the δ -sequence $\{\delta_0, \delta_1, \dots, \delta_h\}$.

Suzuki [11] gave an algebrico-geometric proof of the above two theorem by the consideration of the resolution graph at infinity. Further, Suzuki gave an algorithm for mutual conversion of a dual graph and a δ -sequence.

3 Construction of defining polynomials of curves

We shall assume that $f(x, y)$ is monic in y . We define approximate roots by Abhyankar's definition.

DEFINITION 3 (approximate roots) *Let $f(x, y)$ be the defining polynomial, monic in y , of a curve with one place at infinity. Let $\{\delta_0, \delta_1, \dots, \delta_h\}$ be the δ -sequence of f . We set $n = \deg_y f$, $d_k = \gcd\{\delta_0, \delta_1, \dots, \delta_{k-1}\}$ and $n_k = n/d_k$ ($1 \leq k \leq h+1$). Then, for each k ($1 \leq k \leq h+1$), a pair of polynomials $(g_k(x, y), \psi_k(x, y))$ satisfying the following conditions is uniquely determined:*

- (i) g_k is monic in y and $\deg_y g_k = n_k$,
- (ii) $\deg_y \psi_k < n - n_k$,
- (iii) $f = g_k^{d_k} + \psi_k$.

We call this g_k the k -th approximate root of f .

We can easily get the following fact from the definition of approximate roots.

Fact 2 *We have*

$$g_1 = y + \sum_{j=0}^{\lfloor p/q \rfloor} c_j x^j, \quad g_{h+1} = f$$

where $c_k \in \mathbb{C}$, $p = \deg_x f/d$, $q = \deg_y f/d$, $d = \gcd\{\deg_x f, \deg_y f\}$ and $\lfloor p/q \rfloor$ is the maximal integer ℓ such that $\ell \leq p/q$.

DEFINITION 4 (Abhyankar-Moh's condition) *We shall call the conditions (1) – (5) concerning $\{\delta_0, \delta_1, \dots, \delta_h\}$ in Theorem 2 Abhyankar-Moh's condition.*

The following theorem gives normal forms of defining polynomials of curves with one place at infinity and the method of construction of their defining polynomials.

Theorem 3 ([5]) *Let $\{\delta_0, \delta_1, \dots, \delta_h\}$ ($h \geq 1$) be a sequence of natural numbers satisfying Abhyankar-Moh's condition (see DEFINITION 4). Set $d_k = \gcd\{\delta_0, \delta_1, \dots, \delta_{k-1}\}$ ($1 \leq k \leq h+1$) and $q_k = d_k/d_{k+1}$ ($1 \leq k \leq h$).*

(1) *We define g_k ($0 \leq k \leq h+1$) as follows:*

$$\left\{ \begin{array}{l} g_0 = x, \\ g_1 = y + \sum_{j=0}^{\lfloor p/q \rfloor} c_j x^j, \quad c_j \in \mathbb{C}, p = \delta_1/d_2, q = \delta_0/d_2, \\ g_{i+1} = g_i^{q_i} + a_{\bar{\alpha}_0 \bar{\alpha}_1 \dots \bar{\alpha}_{i-1}} g_0^{\bar{\alpha}_0} g_1^{\bar{\alpha}_1} \dots g_{i-1}^{\bar{\alpha}_{i-1}} \\ \quad + \sum_{(\alpha_0, \alpha_1, \dots, \alpha_i) \in \Lambda_i} c_{\alpha_0 \alpha_1 \dots \alpha_i} g_0^{\alpha_0} g_1^{\alpha_1} \dots g_i^{\alpha_i}, \\ \quad a_{\bar{\alpha}_0 \bar{\alpha}_1 \dots \bar{\alpha}_{i-1}} \in \mathbb{C}^*, c_{\alpha_0 \alpha_1 \dots \alpha_i} \in \mathbb{C} \quad (1 \leq i \leq h), \end{array} \right.$$

where $(\bar{\alpha}_0, \bar{\alpha}_1, \dots, \bar{\alpha}_{i-1})$ is the sequence of i non-negative integers satisfying

$$\sum_{j=0}^{i-1} \bar{\alpha}_j \delta_j = q_i \delta_i, \quad \bar{\alpha}_j < q_j \quad (0 < j < i)$$

and

$$\Lambda_i = \left\{ (\alpha_0, \alpha_1, \dots, \alpha_i) \in \mathbb{N}^{i+1} \mid \alpha_j < q_j \quad (0 < j < i), \alpha_i < q_i - 1, \sum_{j=0}^i \alpha_j \delta_j < q_i \delta_i \right\}.$$

Then, $g_0, g_1, \dots, g_h$ are approximate roots of $f(= g_{h+1})$, and f is the defining polynomial, monic in y , of a curve with one place at infinity of the δ -sequence $\{\delta_0, \delta_1, \dots, \delta_h\}$.

(2) *The defining polynomial f , monic in y , of a curve with one place at infinity of the δ -sequence $\{\delta_0, \delta_1, \dots, \delta_h\}$ is obtained by the procedure of (1), and the values of parameters $\{a_{\bar{\alpha}_0 \bar{\alpha}_1 \dots \bar{\alpha}_{i-1}}\}_{1 \leq i \leq h}$ and $\{c_{\alpha_0 \alpha_1 \dots \alpha_i}\}_{0 \leq i \leq h}$ are uniquely determined for f .*

4 Abhyankar's Question

DEFINITION 5 (planar semigroup) *Let $\{\delta_0, \delta_1, \dots, \delta_h\}$ ($h \geq 1$) be a sequence of natural numbers satisfying Abhyankar-Moh's condition. A semigroup generated by $\{\delta_0, \delta_1, \dots, \delta_h\}$ is said to be a planar semigroup.*

DEFINITION 6 (polynomial curve) *Let C be an algebraic curve defined by $f(x, y) = 0$, where $f(x, y)$ is an irreducible polynomial in $\mathbb{C}[x, y]$. We call C a polynomial curve, if C has a parametrisation $x = x(t)$, $y = y(t)$, where $x(t)$ and $y(t)$ are polynomials in $\mathbb{C}[t]$.*

Abhyankar's Question: Let Ω be a planar semigroup. Is there a polynomial curve with δ -sequence generating Ω ?

Moh [6] showed that there is no polynomial curve with δ -sequence $\{6, 8, 3\}$. But there is a polynomial curve $(x, y) = (t^3, t^8)$ with δ -sequence $\{3, 8\}$ which generates the same semigroup as above. Sathaye-Stenerson [9] proved that the semigroup generated by $\{6, 22, 17\}$ has no other δ -sequence generating the same semigroup, and proposed the following conjecture for this question.

Sathaye-Stenerson's Conjecture: There is no polynomial curve having the δ -sequence $\{6, 22, 17\}$.

By Theorem 3, the defining polynomial of the curve with one place at infinity of the δ -sequence $\{6, 22, 17\}$ as follows:

$$f = (g_2^2 + a_{2,1}x^2g_1) + c_{5,0,0}x^5 + c_{4,0,0}x^4 + c_{3,0,0}x^3 + c_{2,0,0}x^2 + c_{1,1,0}xg_1 + c_{1,0,0}x + c_{0,1,0}g_1 + c_{0,0,0}$$

where

$$\begin{aligned} g_1 &= y + c_3x^3 + c_2x^2 + c_1x + c_0, \\ g_2 &= (g_1^3 + a_{11}x^{11}) + c_{10,0}x^{10} + c_{9,0}x^9 + c_{8,0}x^8 + (c_{7,1}g_1 + c_{7,0})x^7 \\ &\quad + (c_{6,1}g_1 + c_{6,0})x^6 + (c_{5,1}g_1 + c_{5,0})x^5 + (c_{4,1}g_1 + c_{4,0})x^4 \\ &\quad + (c_{3,1}g_1 + c_{3,0})x^3 + (c_{2,1}g_1 + c_{2,0})x^2 + (c_{1,1}g_1 + c_{1,0})x + c_{0,1}g_1 + c_{0,0}. \end{aligned}$$

Since C has one place at infinity and genus zero if and only if C has polynomial parametrization (Abhyankar), $\{6, 22, 17\}$ is a counter example if it can be shown that the above type curve does not include a polynomial curve.

5 Approach by using a computer algebra system

We assume that C is a polynomial curve and has the δ -sequence $\{6, 22, 17\}$. Therefore C has the following polynomial parametrization:

$$\begin{cases} x &= t^6 + a_1t^5 + a_2t^4 + a_3t^3 + a_4t^2 + a_5t + a_6 \\ y &= t^{22} + b_1t^{21} + b_2t^{20} + b_3t^{19} + \dots + b_{21}t + b_{22} \end{cases}$$

It follows that $\deg_t g_2(x(t), y(t)) = 17$ from the form of f and g_2 in the previous section. We can get the polynomial system I with 11 variables and 17 polynomials after eliminating variables from the coefficients of all terms of t -degree more than 18 in $g_2(x(t), y(t))$.

$\{6, 22, 17\}$ is a counter example of Abhyankar's question if I does not have a root. For such a huge polynomial system it is suitable to compute the Gröbner basis of the ideal. However, it was impossible to compute the Gröbner basis of I even if using a computer with 8GB memory.

We classified δ -sequences with genus ≤ 50 into groups which generate the same semigroup. Furthermore, we listed δ -sequences with the following three properties: (i) There is no other δ -sequence which generates the same semigroup. (ii) The number of generators is 3. (iii) k -number ≥ -1 . Then, we obtained $\{6, 15, 4\}$, $\{4, 14, 9\}$, $\{6, 15, 7\}$, $\{6, 21, 4\}$, $\dots$. The Gröbner basis computations for the polynomial systems corresponding to these δ -sequences showed that $\{6, 21, 4\}$ was a counter example of Abhyankar's question.

The defining polynomial of the curve with one place at infinity of the δ -sequence $\{6, 21, 4\}$ as follows:

$$f = g_2^3 + a_{2,0}x^2 + c_{1,0,1}xg_2 + c_{1,0,0}x + c_{0,0,1}g_2 + c_{0,0,0}$$

where

$$\begin{aligned} g_2 &= g_1^2 + a_7x^7 + c_{6,0}x^6 + c_{5,0}x^5 + c_{4,0}x^4 + c_{3,0}x^3 \\ &\quad + c_{2,0}x^2 + c_{1,0}x + c_{0,0} \\ g_1 &= y + c_3x^3 + c_2x^2 + c_1x + c_0 \end{aligned}$$

Let the following be the polynomial parametrization of the polynomial curve with δ -sequence $\{6, 21, 4\}$:

$$\begin{cases} x &= t^6 + a_1t^5 + a_2t^4 + a_3t^3 + a_4t^2 + a_5t + a_6 \\ y &= t^{21} + b_1t^{20} + b_2t^{19} + b_3t^{18} + \dots + b_{20}t + b_{21} \end{cases}$$

By the same operation as the case of $\{6, 22, 17\}$ we can get the polynomial system J with 7 variables $\{a_2, a_3, a_4, a_5, a_6, b_{12}, b_{18}\}$ and 13 polynomials from $\deg_t g_2(x(t), y(t)) = 4$.

We used the total degree reverse lexicographic ordering (DRL) with $a_2 \succ a_3 \succ a_4 \succ a_5 \succ a_6 \succ b_{12} \succ b_{18}$ to the Gröbner basis computation. CPU time for the computation is 3 hours 40 minutes and the required memory is 850MB. The computer is a PC AthlonMP 2200+ with 4GB memory. The computer algebra system is Risa/Asir [7] on FreeBSD 4.7.

The obtained Gröbner basis G of J was not $\{1\}$. However, the normal form of the coefficient p of the term with t -degree = 4 in $g_2(x(t), y(t))$ with respect to G is 0. This shows that $p \in J$. Thus, we get $\deg_t g_2(x(t), y(t)) < 4$. Since this is contradictory for $\deg_t g_2(x(t), y(t)) = 4$, there is no polynomial curve with δ -sequence $\{6, 21, 4\}$.

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