

Constructions of bounded weak approximate identities for Segal algebras on R^n .

北海道大学理学研究科 井上純治 (Jyunji Inoue)

山形大学工学部 高橋眞映 (Sin-Ei Takahasi)

In this paper we study bounded weak approximate identities for Segal algebras on R^n . In particular we show that, if a Segal algebra A on R^n belongs to some familiar class, we can construct bounded weak approximate identities for A of norm 1. This results imply at once that Segal algebras in this class are BSE-algebras. Examples of Segal algebras which have no bounded weak approximate identities are also given.

1. Introduction. In the paper of Lahr [2], the bounded weak approximate identities for commutative Banach algebras were defined and used successfully to characterize the existence of the unit element in the structure space of a commutative convolution measure algebra. Further, an example of a convolution measure algebra which has bounded weak approximate identities without no bounded (norm) approximate identities was presented by Jones and Lahr [1].

On the other hand, it was discovered by the second author and Hatori [9] that the bounded weak approximate identities were also important to decide whether a commutative Banach algebra is a **BSE**-algebra or not. In particular, a Segal algebra A on G is **BSE** if and only if A has bounded weak approximate identities.

In this paper we investigate bounded weak approximate identities for Segal algebras A on R^n , and get the following results.

Theorem 3 *For each $1 < p \leq \infty$, $S^p(R^n)$ has bounded weak approximate identities of norm 1.*

Theorem 5 *For each $1 \leq p < \infty$, $A_p(R^n)$ has bounded weak approximate identities of norm 1.*

Theorem 6 *Suppose that G is a non-discrete LCA group, $1 \leq p < \infty$, and ν is a positive Radon measure on $\hat{G}$ which has an unbounded discrete part; $\sum_{\gamma \in \hat{G}} \nu(\{\gamma\}) = \infty$. Then $A_{p,\nu}(G)$ has no bounded weak approximate identities.*

2. Preliminaries. Let A be a commutative semisimple Banach algebra with the maximal ideal space Δ_A . A net $\{e_\lambda\}_{\lambda \in \Lambda}$ in A is called a bounded weak approximate identity for A of norm C if (i) $\sup\{\|e_\lambda\|_A : \lambda \in \Lambda\} = C < \infty$ and (ii) $\lim_\lambda \|\phi(ae_\lambda) - \phi(a)\| = 0$ ($\phi \in \Delta_A$).

It is easy to see that a net $\{e_\lambda\}_{\lambda \in \Lambda}$ in A is an bounded weak approximate identity for A if and only if the relation $\lim_\lambda \phi(e_\lambda) = 1$ ($\phi \in \Delta_A$) holds.

Let G be a locally compact abelian group with the dual group $\hat{G}$, and let $L^1(G)$ and $M(G)$ denote the group algebra and the measure algebra on G , respectively.

A subspace $S(G)$ of $L^1(G)$ is called a Segal algebra if (i) $S(G)$ is dense in $L^1(G)$, (ii) $S(G)$ is a Banach space under norm $\|\cdot\|_{S(G)}$ satisfying $\|f\|_{S(G)} \geq \|f\|_1$ ($f \in S(G)$), (iii) $S(G)$ is invariant under the translation; $f_a \in S(G)$ ($f \in S(G), a \in G$), where $f_a(x) = f(x - a)$. (iv) $\|f_a\|_{S(G)} = \|f\|_{S(G)}$ ($f \in S(G), a \in G$), (v) for each $f \in S(G)$, the map $a \rightarrow f_a$ is continuous from G into $S(G)$.

Further, it is known that a Segal algebra $S(G)$ satisfies the following additional properties: (vi) $\|f * \mu\|_{S(G)} \leq \|f\|_{S(G)} \|\mu\|$ ($f \in S(G), \mu \in M(G)$), (vii) $S(G)$ is a

Bancach algebra whose maximal ideal space can be identified with $\hat{G}$, (viii) $S(G)$ has a bounded (norm) approximate identity if and only if $S(G)$ equals to the group algebra $L^1(G)$.

Examples ([8]). (i) For each $1 < p \leq \infty$, we put $S^p(G) = L^1(G) \cap L^p(G)$. Define norm by

$$\|f\|_{S^p(G)} := \max\{\|f\|_1, \|f\|_p\} \quad (f \in S^p(G)).$$

Then $S^p(G)$ is a Segal algebra.

(ii) Let G be a non-discrete LCA group, and let ν be an unbounded positive Radon measure on $\hat{G}$ and $1 \leq p < \infty$. We put

$$A_{p,\nu}(G) = \{f \in L^1(G) : \|\hat{f}\|_{p,\nu} := \left(\int_{\hat{G}} |\hat{f}(\gamma)|^p d\nu(\gamma) \right)^{1/p} < \infty\},$$

where $\hat{f}$ is the Fourier transform of f . Then $A_{p,\nu}(G)$ is a Segal algebra with norm

$$\|f\|_{A_{p,\nu}(G)} = \max\{\|f\|_1, \|\hat{f}\|_{p,\nu}\} \quad (f \in A_{p,\nu}(G)).$$

In particular, when ν is the Haar measure $m_{\hat{G}}$ on $\hat{G}$, we simply denote $A_{p,m_{\hat{G}}}(G)$ by $A_p(G)$.

$S^p(G)$ and $A_p(G)$ are typical and well known example of Segal algebras, which were studied by several authors ([3], [4], [5], [6], [7], [8], [10]).

3. Bounded weak approximate identities for Segal algebras

Lemma 1 (Jones and Lahr [1]). *Let G be an infinite discrete abelian group. Then there exists a net $\{g_\lambda\} \subseteq G \setminus \{0\}$, such that $\lim_\lambda (g_\lambda, \gamma) = 1$ ($\gamma \in \hat{G}$).*

Lemma 2 *Let $\{x_1, \dots, x_m\}$ be a finite subset of R , and let $\varepsilon > 0$ be arbitrary. Then there exists a sequence $\{n_1, n_2, \dots\}$ of natural numbers such that $n_1 < n_2 < \dots$ and $|e^{in_k x_j} - 1| < \varepsilon$ ($1 \leq j \leq m$, $k = 1, 2, \dots$).*

Proof. By Lemma 1, there exists a net of non-zero integers $\{n_\lambda\}_{\lambda \in \Lambda}$, such that $e^{in_\lambda x} \rightarrow 1$ for each $x \in R$. Since $e^{-in_\lambda x} \rightarrow 1$ for each $x \in R$, we can assume that all n_λ are positive integers. In this case, we have $n_\lambda \rightarrow \infty$. In fact if not, then there exists a subnet $\{n_{\lambda'}\}$ of $\{n_\lambda\}$ which converges to some integer $n_0 \neq 0$, hence $e^{in_0 x} = 1$ for each $x \in R$, a contradiction. Then the elementary convergence argument implies the desired result. Q.E.D.

Theorem 3 For each $1 < p \leq \infty$, $S^p(R^n)$ has weak approximate identities of norm 1.

Proof. First we prove the case $n = 1$. Let $\varepsilon > 0$ and a finite subset $F = \{\xi_1, \xi_2, \dots, \xi_{m_0}\}$ of R be arbitrary. Then we can find $M > 0$ and a natural number n_0 such that

$$F \subseteq [-M, M], \quad (1)$$

$$\left| n_0 \hat{\mathcal{X}}_E(\xi) - 1 \right| \leq \varepsilon/2 \quad (\xi \in [-M, M]), \quad (2)$$

where $\hat{\mathcal{X}}_E$ is Fourier transform of the characteristic function $\mathcal{X}_E$ of the interval $E = [-\frac{1}{2n_0}, \frac{1}{2n_0}]$. Further, by Lemma 2, we can choose positive integers $N_1, N_2, \dots, N_{n_0}$ such that

$$(E - N_k) \cap (E - N_\ell) = \emptyset \quad (1 \leq k < \ell \leq n_0), \quad (3)$$

$$\left| e^{iN_k \xi_j} - 1 \right| \leq \frac{\varepsilon}{\varepsilon + 2} \quad j = 1, \dots, m_0, \quad k = 1, \dots, n_0. \quad (4)$$

Put $e = e(F, \varepsilon) := \sum_{j=1}^{n_0} \mathcal{X}_{E-N_j}$, then it follows from (2) and (3) that

$$\left| \hat{\mathcal{X}}_E(\xi_j) \right| \leq \frac{2 + \varepsilon}{2n_0} \quad (j = 1, 2, \dots, m_0), \quad (5)$$

$$\|e\|_{S^p(R)} = 1. \quad (6)$$

Further, by (2), (4) and (5), we have

$$\left| \hat{e}(\xi_j) - 1 \right| = \left| \sum_{k=1}^{n_0} \hat{\mathcal{X}}_{E-N_k}(\xi_j) - 1 \right|$$

$$\begin{aligned}
&= \left| \sum_{k=1}^{n_0} e^{i\xi_j N_k} \hat{\mathcal{X}}_E(\xi_j) - n_0 \hat{\mathcal{X}}_E(\xi_j) + n_0 \hat{\mathcal{X}}(\xi_j) - 1 \right| \\
&\leq \sum_{k=1}^{n_0} \left| e^{i\xi_j N_k} - 1 \right| \left| \hat{\mathcal{X}}_E(\xi_j) \right| + \left| n_0 \hat{\mathcal{X}}_E(\xi_j) - 1 \right| \\
&\leq n_0 \frac{\varepsilon}{2+\varepsilon} \frac{2+\varepsilon}{2n_0} + \frac{\varepsilon}{2} = \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \quad (j = 1, 2, \dots, m_0). \quad (7)
\end{aligned}$$

Let $\Lambda := \{e(F, \varepsilon) : F \subseteq R \text{ (finite set)}, \varepsilon > 0\}$, and define partial order in Λ by

$$(F_1, \varepsilon_1) \leq (F_2, \varepsilon_2) \text{ if and only if } F_1 \subseteq F_2, \text{ and } \varepsilon_2 \leq \varepsilon_1.$$

Then it is obvious by (6) and (7) that $\{e(F, \varepsilon)\}_{(F, \varepsilon) \in \Lambda}$ is a bounded weak approximate identity for $S^p(R)$ of norm 1.

Now, let us consider the general case; $n \geq 1$. Let $\varepsilon > 0$ and a finite subset $F = \{\xi_1, \dots, \xi_{m_0}\}$ of R^n be arbitrary, where $\xi_j = (\xi_{j,1}, \dots, \xi_{j,n})$ for $j = 1, \dots, m_0$. Set $F_k = \{\xi_{1,k}, \dots, \xi_{m_0,k}\}$ for $k = 1, \dots, n$. As we have proved above, we can choose $e_k \in L^1(R)$ $k = 1, 2, \dots, n$ such that

$$\max\{\|e_k\|_1, \|e_k\|_p\} \leq 1, \quad |\widehat{e_k}(\xi_{j,k}) - 1| \leq \varepsilon/n \quad (j = 1, \dots, m_0).$$

We define $e_{(F, \varepsilon)} \in L^1(R^n)$ by

$$e_{(F, \varepsilon)}((x_1, \dots, x_n)) := \prod_{k=1}^n e_k(x_k) \quad ((x_1, \dots, x_n) \in R^n).$$

Then we have

$$\max\{\|e_{(F, \varepsilon)}\|_1, \|e_{(F, \varepsilon)}\|_p\} = \max\left\{\prod_{k=1}^n \|e_k\|_1, \prod_{k=1}^n \|e_k\|_p\right\} \leq 1,$$

and

$$\begin{aligned}
|e_{(F, \varepsilon)}(\xi_j) - 1| &\leq \left| \sum_{k=2}^n \prod_{\ell=1}^k \widehat{e_\ell}(\xi_{j,\ell}) - \prod_{\ell=1}^{k-1} \widehat{e_\ell}(\xi_{j,\ell}) \right| + |\widehat{e_1}(\xi_{j,1}) - 1| \\
&\leq \sum_{k=1}^n |\widehat{e_k}(\xi_{j,k}) - 1| \leq \varepsilon \quad (j = 1, \dots, m_0).
\end{aligned}$$

Therefore we can construct a bounded weak approximate identity $\{e_{(F, \varepsilon)}\}_{(F, \varepsilon) \in \Lambda}$ for $S^p(R^n)$ of norm 1, where $\Lambda = \{(F, \varepsilon) : F \subset R^n \text{ (finite set)}, \varepsilon > 0\}$ is a directed set with the partial order defined in the same way as in the case $n = 1$. Q.E.D.

Lemma 4 For each $1 \leq p < \infty$, we have

$$A_1(R^n) \subseteq A_p(R^n) \text{ and } \|f\|_{A_p(R^n)} \leq \|f\|_{A_1(R^n)} \quad (f \in A_1(R^n)).$$

Proof. Let $f \in A_1(R^n)$ and $1 \leq p < \infty$. We consider the two cases:

(i) $\|\hat{f}\|_1 \leq \|f\|_1$. Set $g = \frac{f}{\|f\|_1}$. Then $\|\hat{g}\|_\infty \leq \|g\|_1 = 1$ and hence $\|\hat{g}\|_p^p \leq \|\hat{g}\|_1 \leq 1$. Therefore $\|g\|_{A_p(R^n)} = \|g\|_{A_1(R^n)} (= 1)$ and so $\|f\|_{A_p(R^n)} = \|f\|_{A_1(R^n)}$.

(ii) $\|\hat{f}\|_1 > \|f\|_1$. Set $g = \frac{f}{\|f\|_1}$. Then $1 = \|\hat{g}\|_1 > \|g\|_1$ and hence $\|g\|_{A_1(R^n)} = 1$. Also since $\|\hat{g}\|_\infty \leq \|g\|_1 < 1$, it follows that $\|\hat{g}\|_p^p \leq \|\hat{g}\|_1 = 1$ and so $\|\hat{g}\|_{A_p(R^n)} \leq 1$. Thus we obtain $\|g\|_{A_p(R^n)} \leq \|g\|_{A_1(R^n)}$, which implies that $\|f\|_{A_p(R^n)} \leq \|f\|_{A_1(R^n)}$. Q.E.D.

Theorem 5 For each $1 \leq p < \infty$, $A_p(R^n)$ has bounded weak approximate identities of norm 1.

Proof. By Lemma 4, it suffices to prove only for $p = 1$.

First we prove the case $n = 1$. Consider any finite set $F = \{\xi_1, \dots, \xi_{m_0}\}$ of R and any $\varepsilon > 0$. Set $u = \mathcal{X}_{[-1/2, 1/2]} \star \mathcal{X}_{[-1/2, 1/2]}$ and let ϕ be the Fourier inverse transform of u . Then $\hat{\phi} = u$ and $\|\phi\|_1 = \|u\|_1 = 1$. Choose $\delta > 0$ such that $1 - u(\xi) < \frac{\varepsilon}{2}$ for all $\xi \in [-\delta, \delta]$, and take a natural number n_0 such that $|\frac{\xi_j}{n_0}| < \delta$ ($j = 1, \dots, m_0$). Furthermore, take a sufficiently large number $L_0 > 0$ such that $|\widehat{\mathcal{X}_{[-n_0, n_0]}}(\xi)| < \varepsilon$ for all $|\xi| \geq L_0$. Also, by Lemma 2, we can choose a finite set $\{N_1, \dots, N_{n_0}\}$ of natural numbers such that $L_0 \leq N_1$, $L_0 \leq N_{j+1} - N_j$ ($j = 1, \dots, n_0 - 1$) and $|e^{i\xi_j N_k} - 1| < \frac{\varepsilon}{4}$ ($j = 1, \dots, m_0; k = 1, \dots, n_0$). Set

$$\mu := \frac{1}{n_0} \sum_{k=1}^{n_0} \frac{\delta_{N_k} + \delta_{-N_k}}{2} \text{ and } \eta_{(F, \varepsilon)} := \mu \star \mu \star \phi_{n_0},$$

where δ_{N_k} is an unit point mass at N_k , and $\phi_{n_0}(x) = n_0 \phi(n_0 x)$ ($x \in R$). Obviously we have $\hat{\delta}_x(\xi) = e^{-i\xi x}$ ($x \in R$), $\hat{\mu}(\xi) = \frac{1}{2n_0} \sum_{k=1}^{n_0} (e^{-i\xi N_k} + e^{i\xi N_k})$ and $\hat{\phi}_{n_0}(\xi) = u(\frac{\xi}{n_0})$ ($\xi \in R$). We shall show that

$$\|\eta_{(F, \varepsilon)}\|_{A_1(R)} \leq 1 + \varepsilon \text{ and } |\hat{\eta}(\xi_j) - 1| < \varepsilon \quad (j = 1, \dots, m_0).$$

To do this, note that

$$\widehat{\eta_{(F,\varepsilon)}}(\xi) = \hat{\mu}(\xi)^2 \hat{\phi}_{n_0}(\xi) = u\left(\frac{\xi}{n_0}\right) \frac{1}{4n_0^2} \sum_{k,l=1}^{n_0} (e^{-i\xi N_k} + e^{i\xi N_k})(e^{-i\xi N_l} + e^{i\xi N_l}),$$

for each $\xi \in R$. Hence we have

$$\begin{aligned} \|\widehat{\eta_{(F,\varepsilon)}}\|_1 &\leq \frac{1}{4n_0^2} \sum_{k,l=1}^{n_0} \int_{-n_0}^{n_0} (e^{-i\xi N_k} + e^{i\xi N_k})(e^{-i\xi N_l} + e^{i\xi N_l}) d\xi \\ &= \frac{1}{4n_0^2} \left[\sum_{k \neq l} \sum_{\varepsilon_1, \varepsilon_2 = \pm 1} \widehat{\mathcal{X}}_{[-n_0, n_0]}(\varepsilon_1 N_k + \varepsilon_2 N_l) \right. \\ &\quad \left. + \sum_{k=1}^{n_0} \left(\widehat{\mathcal{X}}_{[-n_0, n_0]}(-2N_k) + \widehat{\mathcal{X}}_{[-n_0, n_0]}(2N_k) + 4n_0 \right) \right] \\ &\leq \frac{1}{4n_0^2} \left(4(n_0^2 - n_0)\varepsilon + 2n_0\varepsilon + 4n_0^2 \right) < 1 + \varepsilon. \end{aligned}$$

Also $\|\eta_{(F,\varepsilon)}\|_1 \leq \|\mu\|^2 \|\phi_{n_0}\|_1 \leq \|\phi\|_1 = 1$, so we have $\|\eta_{(F,\varepsilon)}\|_{A_1(R)} \leq 1 + \varepsilon$. Moreover,

$$\begin{aligned} |\widehat{\eta_{(F,\varepsilon)}}(\xi_j) - 1| &= \left| \hat{\mu}(\xi_j)^2 u\left(\frac{\xi_j}{n_0}\right) - 1 \right| \\ &\leq |\hat{\mu}(\xi_j)^2 - 1| + \left| \hat{\mu}(\xi_j)^2 \left(u\left(\frac{\xi_j}{n_0}\right) - 1 \right) \right| \\ &\leq 2|\hat{\mu}(\xi_j) - 1| + \left| u\left(\frac{\xi_j}{n_0}\right) - 1 \right| \\ &\leq 2 \frac{1}{2n_0} \sum_{k=1}^{n_0} (|e^{-i\xi_j N_k} - 1| + |e^{i\xi_j N_k} - 1|) + \frac{\varepsilon}{2} \\ &\leq 2 \frac{1}{2n_0} n_0 \left(\frac{\varepsilon}{4} + \frac{\varepsilon}{4} \right) + \frac{\varepsilon}{2} = \varepsilon, \end{aligned}$$

for all $j = 1, \dots, m_0$.

Let $\Lambda = \{(F, \varepsilon) : F \subseteq R \text{ (finite set), } \varepsilon > 0\}$ be the directed set introduced in the proof of Theorem 3. For each $\Lambda \ni (F, \varepsilon)$, set $e_{(F,\varepsilon)} = \frac{1}{1+\varepsilon/2} \eta_{(F,\varepsilon/2)}$. Then we have $\|e_{(F,\varepsilon)}\|_{A_1(R)} \leq 1$ and

$$|e_{(F,\varepsilon)}(\xi) - 1| = \frac{1}{1+\varepsilon/2} |e_{(F,\varepsilon/2)}(\xi) - 1 - \varepsilon/2| \leq \frac{2\varepsilon/2}{1+\varepsilon/2} \leq \varepsilon \quad (\xi \in F).$$

Therefore we see that $\{e_{(F,\varepsilon)}\}_{(F,\varepsilon) \in \Lambda}$ is a bounded weak approximate identity for $A_1(R)$ of norm 1.

Now let us consider the general case; $n \geq 1$. In this case, we can construct bounded weak approximate identities for $A_1(R^n)$ of norm 1 from bounded weak approximate identities for $A_1(R)$ of norm 1 by the same way as in theorem 3 above, and the proof is complete. Q.E.D.

Although, as we have seen, $S^p(R^n)$ and $A_p(R^n)$ have weak approximate identities, it is not true that every Segal algebra on R^n has weak approximate identities, as the next theorem shows.

Theorem 6 *Suppose that G is a non-discrete LCA group, $1 \leq p < \infty$, and ν is a positive Radon measure on $\hat{G}$ which has an unbounded discrete part; $\sum_{\gamma \in \hat{G}} \nu(\{\gamma\}) = \infty$. Then $A_{p,\nu}(G)$ has no bounded weak approximate identities.*

Proof. Suppose, on the contrary, that there exists a bounded weak approximate identity $\{e_\lambda\}_{\lambda \in \Lambda}$ for $A_{p,\nu}(G)$. Since ν has an unbounded discrete part, we can choose an infinite sequence $\{\gamma_n\}$ of elements of $\hat{G}$ such that $\sum_{n=1}^{\infty} \nu(\{\gamma_n\}) = \infty$. Then by the definition of the norm of $\|e_\lambda\|_{A_{p,\nu}(G)}$, we get

$$\sup_{\lambda} \|e_\lambda\|_{A_{p,\nu}(G)} \geq \sup_{\lambda} \left(\sum_{n=1}^{\infty} |\widehat{e_\lambda}(\gamma_n)|^p \nu(\{\gamma_n\}) \right)^{1/p}. \quad (8)$$

But the right side of the inequality (8) is infinite since $\sum_n \nu(\{\gamma_n\}) = \infty$ and $\lim_{\lambda} \widehat{e_\lambda}(\gamma) = 1$ ($\gamma \in \hat{G}$). This contradicts to the assumption that $\{e_\lambda\}_{\lambda \in \Lambda}$ has a bounded norm. Q.E.D.

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Department of Mathematics,

Faculty of Science,

Hokkaido University, Sapporo 060, Japan

E-mail adress: jinoue@math.sci.hokudai.ac.jp

and

Department of Basic Technology,

Applied Mathematics and Physics,

Yamagata University, Yonezawa 992, Japan