

Smoothing effect in Gevrey classes for Schrödinger equations

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Introduction

We shall investigate Gevrey smoothing effects of the solutions to the Cauchy problem for Schrödinger type equations. Roughly speaking, we shall prove that if the initial data decay as $e^{-c\langle x \rangle^\kappa}$ ($0 < \kappa \leq 1, c > 0$), then the solutions belong to Gevrey class $\gamma^{1/\kappa}$ with respect to the space variables. Let $T > 0$. We consider the following Cauchy problem,

$$(1) \quad \frac{\partial}{\partial t} u(t, x) - i\Delta u(t, x) - b(t, x, D)u(t, x) = 0, t \in [-T, T], x \in R^n,$$

$$(2) \quad u(0, x) = u_0(x), x \in R^n,$$

where

$$(3) \quad b(t, x, D)u = \sum_{j=1}^n b_j(t, x)D_j u + b_0(t, x)u,$$

and $D_j = -i\frac{\partial}{\partial x_j}$. We assume that the coefficients $b_j(t, x)$ satisfy

$$(4) \quad |D_x^\alpha b_j(t, x)| \leq C_b(\rho_b \langle x \rangle)^{-|\alpha|} |\alpha|!^s,$$

for $(t, x) \in [-T, T] \times R^n, \alpha \in N^n$, where $\langle x \rangle = (1 + |x|^2)^{1/2}$. Moreover we assume that there is $\kappa \in (0, 1]$ such that

$$(5) \quad \lim_{|x| \rightarrow \infty} \text{Re} b_j(t, x) \langle x \rangle^{1-\kappa} = 0, \text{ uniformly in } t \in [-T, T].$$

For $\rho \geq 0$ let define a exponential operator $e^{\rho \langle D \rangle^\kappa}$ as follows,

$$e^{\rho \langle D \rangle^\kappa} u(x) = \int_{R^n} e^{ix\xi + \rho \langle \xi \rangle^\kappa} \hat{u}(\xi) d\xi$$

where $\hat{u}(\xi)$ stands for a Fourier transform of u and $d\xi = (2\pi)^{-n} d\xi$. For $\varepsilon \in R$ denote $\phi_\varepsilon = x\xi - i\varepsilon x\xi \langle x \rangle^{\sigma-1} \langle \xi \rangle^{\delta-1}$, where $\sigma + \delta = \kappa$ and we define

$$I_{\phi_\varepsilon}(x, D)u(x) = \int_{R^n} e^{i\phi_\varepsilon(x, \xi)} \hat{u}(\xi) d\xi.$$

Then our main theorem follows.

Theorem. Assume (4)-(5) are valid and there is $\varepsilon > 0$ such that $I_{\phi_\varepsilon} u_0 \in L^2(R^n)$. Then if $d\kappa \leq 1$, there exists a solution of (1)-(2) satisfying that there are $C > 0$, $\rho > 0$ and $\delta > 0$ such that

$$(6) \quad |\partial_x^\alpha u(t, x)| \leq C(\rho|t|)^{-|\alpha|} |\alpha|!^s e^{\delta \langle x \rangle^\kappa},$$

for $(t, x) \in [-T, T] \setminus \{0\} \times R^n$, $\alpha \in N^n$.

Remark. (i) Kato T. and Yajima in [12] considered the smoothing effect phenomena. A. Jensen in [6] and Hayashi, Nakamitsu & Tsutsumi in [5] showed that if $\langle x \rangle^\kappa u_0(x) \in L^2(R^n)$, the solution u of (1)-(2) belongs to H_{loc}^κ for $t \neq 0$, Hayashi & Saitoh in [4] proved that if $e^{\delta \langle x \rangle^2} u_0$ ($\delta > 0$) is in $L^2(R^n)$, the solution u is analytic in x for $t \neq 0$ and De Bouard, Hayashi & Kato in [1], Kato & Taniguti in [11] show that if u_0 satisfies $\|(x \cdot \nabla)^j u_0\| \leq C^{j+1} j!^s$ for $j = 0, 1, 2, \dots$, then the solution belongs to Gevrey $\gamma^{s/2}$ with respect to x for $t \neq 0$. Theorem 1 is proved by Kajitani in [8] and [10], when $\sigma = \kappa = 1$.

1 Weighted Sobolev spaces

We introduce some Sobolev spaces with weights. Let ρ, δ be real numbers and $\kappa \in (0, 1]$. Define

$$\hat{H}_\delta^\kappa = \{u \in L_{loc}^2(R^n); e^{\delta \langle x \rangle^\kappa} u(x) \in L^2(R^n)\}.$$

For $\rho \geq 0$ let define

$$H_\rho^\kappa = \{u \in L^2(R^n); Fu(\xi) \in \hat{H}_\rho(R_\xi^n)\},$$

where Fu stands for the Fourier transform of u . For $\rho < 0$ we define H_ρ^κ as the dual space of $H_{-\rho}^\kappa$. Then the Fourier transform F becomes bijective from H_ρ^κ to $\hat{H}_{\rho^\kappa}$. We define the operator $e^{\rho \langle D \rangle^\kappa}$ mapping continuously from $H_{\rho_1}^\kappa$ to $H_{\rho_1 - \rho}^\kappa$ as follows;

$$e^{\rho \langle D \rangle^\kappa} u(x) = F^{-1}(e^{\rho \langle \xi \rangle^\kappa} Fu(\xi))(x),$$

for $u \in H_{\rho_1}^\kappa$ and $e^{\delta \langle x \rangle^\kappa}$ maps continuously from $\hat{H}_{\delta_1}^\kappa$ to $\hat{H}_{\delta_1 - \delta}^\kappa$. We define for $\delta \geq 0$ and $\rho \in R$

$$(1.1) \quad H_{\rho, \delta}^\kappa = \{u \in H_\rho^\kappa; e^{\rho \langle D \rangle^\kappa} u \in \hat{H}_\delta^\kappa\}.$$

For $\delta < 0$ we define $H_{\rho, \delta}^\kappa$ as the dual space of $H_{-\rho, -\delta}^\kappa$. We note that $H_{\rho, 0}^\kappa = H_\rho^\kappa$, $H_{0, \delta}^\kappa = \hat{H}_\delta^\kappa$ and $H_{0, 0}^\kappa = L^2(R^n)$. Furthermore we define for $\rho \geq 0$ and $\delta \in R$

$$(1.2) \quad \tilde{H}_{\rho, \delta}^\kappa = \{u \in \hat{H}_\delta^\kappa; e^{\delta \langle x \rangle^\kappa} u \in H_\rho^\kappa\}$$

and for $\rho < 0$ define $\tilde{H}_{\rho, \delta}^\kappa$ as the dual space of $\tilde{H}_{-\rho, -\delta}^\kappa$. Denote by H' the dual space of a topological space H . Then $H_{\rho, \delta}^{\kappa'} = H_{-\rho, -\delta}^\kappa$ and $\tilde{H}_{\rho, \delta}^{\kappa'} = \tilde{H}_{-\rho, -\delta}^\kappa$ hold for any ρ and $\delta \in R$. We shall prove $H_{\rho, \delta}^\kappa = \tilde{H}_{\rho, \delta}^\kappa$ later on (see Proposition 3.8).

Lemma 1.1. Let $\rho, \delta \in R$. Then

$$(i) \quad H_{\rho, \delta}^\kappa = e^{-\rho \langle D \rangle^\kappa} e^{-\delta \langle x \rangle^\kappa} L^2 = e^{-\rho \langle D \rangle^\kappa} \hat{H}_\delta^\kappa.$$

$$(ii) \quad \tilde{H}_{\rho, \delta}^\kappa = e^{-\delta \langle x \rangle^\kappa} e^{-\rho \langle D \rangle^\kappa} L^2 = e^{-\delta \langle x \rangle^\kappa} H_\rho^\kappa.$$

Lemma 1.2 Let $1 > \rho > 0$, $\delta \in R$ and $u \in \tilde{H}_{\rho, \delta}^\kappa$. Then

$$(1.6) \quad |D_x^\alpha u(x)| \leq C_n (1 - \epsilon)^{-n/2} \|u\|_{\tilde{H}_{\rho, \delta}^\kappa} (\epsilon \rho)^{-|\alpha|} |\alpha|! e^{\delta \langle x \rangle^\kappa}$$

for $x \in R^n$, $\alpha \in N^n$ and $0 < \epsilon < 1$.

We can prove these lemmas analogously to the case of $\kappa = 1$ which is proved in [10].

2 Almost analytic extension of symbols

Following Hörmander's notation we define the symbol classes of pseudo-differential operators. Let $m(x, \xi)$, $\varphi(x, \xi)$, $\psi(x, \xi)$ a weight and $g = \varphi^{-2}dx^2 + \psi^{-2}d\xi^2$ a Riemann metric. We denote by $S(m, g)$ the set of symbols $a(x, \xi)$ satisfying

$$|a_{(\beta)}^{(\alpha)}(x, \xi)| \leq C_{\alpha\beta} m(x, \xi) \psi^{-|\alpha|} \theta^{-|\beta|},$$

for $(x, \xi) \in R^{2n}$, $\alpha, \beta \in N^n$, where $a_{(\beta)}^{(\alpha)} = \partial_\xi^\alpha D_x^\beta a$. Let $d \geq 1$. Moreover we call that a function $a(x, \xi) \in S(m, g)$ belongs to $\gamma^d S(m, g)$, if $a(x, \xi)$ satisfies that there are $C_a > 0$, $\rho_a > 0$ such that

$$(2.1) \quad |a_{(\beta)}^{(\alpha)}(x, \xi)| \leq C_a \rho_a^{-|\alpha+\beta|} |\alpha + \beta|!^d \psi^{-|\beta|} \varphi^{-|\alpha|}$$

for $(x, \xi) \in R^{2n}$, $\alpha, \beta \in N^n$. We denote $g_0 = dx^2 + d\xi^2$ and $g_1 = \langle x \rangle^{-2} dx^2 + \langle \xi \rangle^{-2} d\xi^2$. We remark that the symbol class $\gamma^1 S(m, g_i)$ ($i = 0, 1$) is introduced in [10] when $d = 1$. Here we consider the case of $d > 1$.

Let $d > 1$ and $\chi(t) \in C_0^\infty((0, \infty))$ satisfying that $\chi(t) = 0$, $t \leq 1/2$, $\chi(t) = 1$, $t \leq 1$, and

$$(2.2) \quad |D_t^k \chi(t)| \leq C_0 \rho_0^{-k} k!^d,$$

for $t \in R$, $k \in N$. Then for a weight $w(x, \xi) \in \gamma^d S(m, g_1)$ and a parameter $b > 0$ we can see easily that $\chi(bw(x, \xi)) \in \gamma^d S(1, g_1)$ satisfying

$$(2.3) \quad |D_x^\beta D_\xi^\alpha \chi(bw(x, \xi))| \leq C_1 \rho_1^{-|\alpha+\beta|} |\alpha + \beta|!^d \langle x \rangle^{-|\beta|} \langle \xi \rangle^{-|\alpha|},$$

for $(x, \xi) \in R^{2n}$, $\alpha, \beta \in N^n$, $b \geq 1$.

Lemma 2.1. Let $d \geq 1$ and $\{p_k(x, \xi)\}_{k=1}^\infty$ be a series of symbols satisfying

$$(2.4) \quad |p_{k(\beta)}^{(\alpha)}(x, \xi)| \leq m(x, \xi) \langle x \rangle \langle \xi \rangle^k \rho_p^{-|\alpha+\beta|-k} |\alpha + \beta|!^d k!^d \langle x \rangle^{-|\beta|} \langle \xi \rangle^{-|\alpha|},$$

for $(x, \xi) \in R^{2n}$, $\alpha, \beta \in N^n$ and $k \geq 0$. Then there is $p(x, \xi) \in \gamma^{(d)} S(m, g_1)$ such that

$$(2.5) \quad p(x, \xi) - \sum_{k=0}^{N-1} p_k(x, \xi) \in \gamma^{(d)} S(m(\langle x \rangle \langle \xi \rangle \rho_p)^{-N} N!^d, g_1),$$

for any integer $N \geq 0$.

Proof This lemma is essentially a result of [2]. The case of $d = 1$ is explained in [10]. Here we prove the lemma in the case of $d > 1$. Let $b_k = \rho_p^{-1} k!^{\frac{d}{k}} M$ and $M \geq 2$. Define

$$(2.6) \quad p(x, \xi) = \sum_{k=0}^{\infty} p_k(x, \xi) \chi(b_k(\langle x \rangle \langle \xi \rangle)^{-1}),$$

Then we have

$$\begin{aligned} |p_{(\beta)}^{(\alpha)}(x, \xi)| &= \left| \sum_k \sum_{\alpha', \beta'} \binom{\alpha}{\alpha'} \binom{\beta}{\beta - \beta'} p_{k(\beta')}^{(\alpha')} (\chi(b_k(\langle x \rangle \langle \xi \rangle)^{-1}))_{(\beta - \beta')}^{(\alpha - \alpha')} \right| \\ &\leq \sum_k \sum_{\alpha', \beta'} \binom{\alpha}{\alpha'} \binom{\beta}{\beta - \beta'} m(x, \xi) \rho_k^{-|\alpha' + \beta'|} |\alpha' + \beta'|!^d \langle x \rangle^{-|\beta|} \langle \xi \rangle^{-|\alpha|} \\ &\quad \times M^{-k} C_0 \rho_0^{-|\alpha - \alpha' + \beta - \beta'|} |\alpha - \alpha' + \beta - \beta'|!^d \end{aligned}$$

$$\leq 2 \frac{\rho_0}{\rho_0 - \rho_p} m(x, \xi) \rho^{-|\alpha+\beta|} |\alpha + \beta|!^d \langle x \rangle^{-|\beta|} \langle \xi \rangle^{-|\alpha|},$$

for $(x, \xi) \in R^{2n}$, $\alpha, \beta \in N^n$. Here we used the following inequality

$$(2.7) \quad \sum_{\alpha' \leq \alpha} \binom{\alpha}{\alpha'} \rho_p^{-|\alpha'|} |\alpha'|!^d \rho_0^{-|\alpha-\alpha'|} |\alpha - \alpha'|!^d \leq \frac{\rho_0}{\rho_0 - \rho_p} |\alpha|!^d,$$

for $\rho_0 > \rho_p$. Moreover we can write

$$\begin{aligned} p(x, \xi) &= \sum_{k=0}^{N-1} p_k(x, \xi) \\ &= \sum_{k=N}^{\infty} p_k(x, \xi) \chi(b_k(\langle x \rangle \langle \xi \rangle)^{-1}) + \sum_{k=0}^{N-1} p_k(x, \xi) (1 - \chi(b_k(\langle x \rangle \langle \xi \rangle)^{-1})) \\ &=: I + II. \end{aligned}$$

Noting that $\rho_p^{-k} k!^d (M \langle x \rangle \langle \xi \rangle)^{-N} \leq 1$ on $\text{supp} \chi(b_k(\langle x \rangle \langle \xi \rangle)^{-1})$ for $k \geq N$ and $\rho_p^{-k} k!^d (M \langle x \rangle \langle \xi \rangle)^{-N} \geq 1/2$ on $\text{supp}(1 - \chi(b_k(\langle x \rangle \langle \xi \rangle)^{-1}))$ for $k \leq N-1$ respectively, we can see that I and II belong to $\gamma^d S(m(\langle x \rangle \langle \xi \rangle \rho_p)^{-N} N!^d, g)$. Q.E.D.

Let $a(x, \xi) \in \gamma^d(m, g_1)$, that is, $a(x, \xi)$ satisfies (2.1). Denote $b_\alpha(x) = B \rho_a^{-1} 4^n \langle x \rangle^{-1} |\alpha|!^{\frac{d-1}{|\alpha|}}$ for $x \in R^n$. We define an almost analytic extension of $a(x, \xi)$ as follows,

$$(2.8) \quad a(x + iy, \xi + i\eta) = \sum_{\alpha, \beta} a_{(\beta)}^{(\alpha)}(x, \xi) (-y)^\beta (i\eta)^\alpha \chi(b_\beta(x) |y|) \chi(b_\alpha(\xi) |\eta|) (\alpha! \beta!)^{-1},$$

for $x, y, \xi, \eta \in R^n$, where $a_{(\beta)}^{(\alpha)}(x, \xi) = \partial_\xi^\alpha (-i \partial_x)^\beta a(x, \xi)$. Then we can prove easily

Proposition 2.2 *Let $a(x, \xi) \in \gamma^d S(m, g_1)$. Then the function $a(x + iy, \xi + i\eta)$ defined by (2.8) satisfies the following properties.*

$$(i) \quad |D_x^\beta \partial_\xi^\alpha D_y^\gamma \partial_\eta^\delta a(x + iy, \xi + i\eta)| \leq C m(x, \xi) (C \rho_a)^{-|\alpha+\beta+\gamma+\delta|} \langle x \rangle^{-|\beta|} \langle \xi \rangle^{-|\alpha|} \langle y \rangle^{-|\gamma|} \langle \eta \rangle^{-|\delta|} |\alpha + \beta + \gamma + \delta|!^d.$$

$$\begin{aligned} (ii) \quad & |(\partial_{x_j} + i \partial_{y_j}) D_x^\beta \partial_\xi^\alpha D_y^\gamma \partial_\eta^\delta a(x + iy, \xi + i\eta)| \\ & \leq C m(x, \xi) (C \rho_a)^{-|\alpha+\beta+\gamma+\delta|} e^{-c_0(\frac{\langle x \rangle}{|y|})^{\frac{1}{d-1}}} \langle x \rangle^{-|\beta|} \langle \xi \rangle^{-|\alpha|} \langle y \rangle^{-|\gamma|} < \eta >^{-|\delta|} |\alpha + \beta + \gamma + \delta|!^d. \end{aligned}$$

$$\begin{aligned} (iii) \quad & |(\partial_{\xi_j} + i \partial_{\eta_j}) D_x^\beta \partial_\xi^\alpha D_y^\gamma \partial_\eta^\delta a(x + iy, \xi + i\eta)| \\ & \leq C m(x, \xi) (C \rho_a)^{-|\alpha+\beta+\gamma+\delta|} e^{-c_0(\frac{\langle \xi \rangle}{|\eta|})^{\frac{1}{d-1}}} \langle x \rangle^{-|\beta|} \langle \xi \rangle^{-|\alpha|} < y >^{-|\gamma|} \langle \eta \rangle^{-|\delta|} |\alpha + \beta + \gamma + \delta|!^d. \end{aligned}$$

For simplicity denote $\gamma^{1/\kappa} S(e^{\delta \langle x \rangle^\kappa + \rho \langle \xi \rangle^\kappa}, g_0)$ by $A_{\rho, \delta}^\kappa$, where $g_0 = dx^2 + d\xi^2$. For $a_i \in A_{\rho_i, \delta_i}^\kappa$ ($i = 1, 2$) we define a product of a_1 and a_2 as follows,

$$\begin{aligned} (2.9) \quad (a_1 \circ a_2)(x, \xi) &= os - \int \int_{R^{2n}} e^{-iy\eta} a_1(x, \xi + \eta) a_2(x + y, \xi) dy d\eta, \\ &= \lim_{\epsilon \rightarrow 0} \int \int_{R^{2n}} e^{-iy\eta - \epsilon(|y|^2 + |\eta|^2)} a_1(x, \xi + \eta) a_2(x + y, \xi) dy d\eta, \end{aligned}$$

where $d\eta = (2\pi)^{-n} d\eta$. Then we can show the proposition below.

Proposition 2.3. (i) Let $\kappa \leq 1$ and $a_i \in A_{\rho_i, \delta_i}^\kappa, i = 1, 2$. Then there is $\epsilon_0 > 0$ such that if $|\rho_1|, |\delta_2| \leq \epsilon_0$, the product $a_1 \circ a_2$ belongs to $A_{\rho_1 + \rho_2, \delta_1 + \delta_2}^\kappa$.

(ii) Let $a_i \in A_{\rho_i, \delta_i}^\kappa, i = 1, 2, 3$. Then if $|\rho_i| (i = 1, 2), |\delta_i| (i = 2, 3) \leq \epsilon_0/2$, we have $(a_1 \circ a_2) \circ a_3 = a_1 \circ (a_2 \circ a_3)$.

Proposition 2.4 Let $d \geq 1$ and $a_i \in \gamma^d S(\langle x \rangle^{m_i} \langle \xi \rangle^{\ell_i}, g_1), i = 1, 2$. Then $a_1 \circ a_2$ belongs to $S(\langle x \rangle^{m_1+m_2} \langle \xi \rangle^{\ell_1+\ell_2}, g_1)$ and moreover we can decompose

$$(2.10) \quad a_1 \circ a_2(x, \xi) = p(x, \xi) + r(x, \xi),$$

where $p(x, \xi) \in \gamma^d S(\langle x \rangle^{m_1+m_2} \langle \xi \rangle^{\ell_1+\ell_2}, g_1)$ satisfies that there are $C > 0$ and $\epsilon_0 > 0$ such that

$$(2.11) \quad p(x, \xi) - \sum_{|\gamma| < N} \gamma!^{-1} a_1^{(\gamma)}(x, \xi) a_{2(\gamma)}(x, \xi) \in \gamma^d S(C^{1+N} N! \langle x \rangle^{m_1+m_2-N} \langle \xi \rangle^{\ell_1+\ell_2-N}, g),$$

for any non negative integer N , and $r(x, \xi)$ belongs to $A_{-\epsilon_0, -\epsilon_0}^{1/d}$.

3 Pseudo-differential operators

Let $\kappa \leq 1$. Now we want to define a pseudo differential operator $a(x, D)$ for a symbol $a(x, \xi) \in A_{\rho, \delta}^\kappa$, which operates from $H_{\rho', \delta'}^\kappa$ to $H_{\rho' - \rho, \delta' - \delta}^\kappa$. When ρ and δ are non positive, since $A_{\rho, \delta}^\kappa$ is contained in the usual symbol class $S_{0,0}^0$ (denote by $S_{\rho, \delta}^m$ the Hörmander's class), we can define

$$(3.1) \quad a(x, D)u(x) = \int e^{ix\xi} a(x, \xi) \hat{u}(\xi) \bar{d}\xi,$$

for $u \in L^2(R^n)$ and for $a \in A_{\rho, \delta}^\kappa$. Moreover for $a_i \in A_{\rho_i, \delta_i}^\kappa, i = 1, 2$ (ρ_i and δ_i non positive) the symbol $\sigma(a_1(x, D)a_2(x, D))(x, \xi)$ of the product of $a_1(x, D)$ and $a_2(x, D)$ can be written as follows,

$$(3.2) \quad \sigma(a_1(x, D)a_2(x, D))(x, \xi) = (a_1 \circ a_2)(x, \xi)$$

and we have

$$(3.3) \quad a_1(x, D)(a_2(x, D)u)(x) = (a_1 \circ a_2)(x, D)u(x)$$

for $u \in L^2(R^n)$, where $a_1 \circ a_2$ is defined by (2.9). Next we shall show that (3.2) and (3.3) are valid for any ρ_i, δ_i . To do so, we need some preparations. Let $a \in A_{\rho, \delta}^\kappa$ and $u \in H_\rho^\kappa$. Then we can define $a(x, D)u(x)$ which belongs to $\hat{H}_\delta^\kappa$. In fact, put $\tilde{a}(z, \eta) = e^{-\delta \langle x \rangle^\kappa + \rho \langle \xi \rangle^\kappa} a(x, \xi)$. Then $\tilde{a}(z, \xi) \in A_{0,0}^\kappa$. Noting that $e^{\rho \langle \xi \rangle^\kappa} \hat{u}(\xi)$ we can define

$$(3.4) \quad e^{-\delta \langle x \rangle^\kappa} a(x, D)u(x) = \int e^{ix\xi} \tilde{a}(x, \xi) e^{\rho \langle \xi \rangle^\kappa} \hat{u}(\xi) \bar{d}\xi,$$

which is in L^2 , that is, $a(x, D)u \in \hat{H}_\delta^\kappa$. For $\epsilon > 0$ we denote $\chi_\epsilon(x) = e^{-\epsilon \langle x \rangle^2}$ and $\chi_\epsilon(D) = e^{-\epsilon \langle D \rangle^2}$.

Lemma 3.1. (i) Let $a \in A_{\rho, \delta}^\kappa (\rho, \delta \in R), u \in L^2$ and $\epsilon_0 > 0$ chosen in Proposition 2.3. Then for any $\epsilon > 0$

$$(3.5) \quad a(x, D)(\chi_\epsilon(D)\chi_\epsilon(x)u)(x) = (a(x, \xi)\chi_\epsilon(\xi)) \circ \chi_\epsilon(x)(x, D)u(x)$$

and

$$(3.6) \quad (a\chi_\epsilon(\xi)) \circ \chi_\epsilon(x) \in A_{\rho-\epsilon_0, \delta-\epsilon_0}^\kappa.$$

(ii) Let $u \in L^2$ and $\epsilon_0 > 0$ chosen in Proposition 2.3. Then there is $\epsilon_1 > 0$ such that for any $\epsilon > 0$

$$(3.7) \quad e^{-\rho \langle D \rangle^\kappa} (e^{-\delta \langle x \rangle^\kappa} \chi_\epsilon(x) \chi_\epsilon(D) u)(x) = a_\epsilon(x, D) u(x),$$

where

$$(3.8) \quad a_\epsilon(x, \xi) = e^{-\rho \langle \xi \rangle^\kappa} \circ (e^{-\delta \langle x \rangle^\kappa} \chi_\epsilon(x) \chi_\epsilon(\xi)) \in A_{-\rho-\epsilon_0, -\delta-\epsilon_0}^\kappa,$$

for $|\rho| \leq \epsilon_0$ and $\rho < \epsilon_1$. We can prove the following lemma by use of Lemma 3.1.

Lemma 3.2. Let $u \in H_{\rho, \delta}^\kappa$ and $|\rho|, |\delta| \leq \epsilon_0/2$ (ϵ_0 is given in Proposition 2.3). Then for any $\epsilon > 0$ there is $u_\epsilon \in H_{\epsilon_0/2, \epsilon_0/2}^\kappa$ such that

$$(3.9) \quad \|u - u_\epsilon\|_{H_{\rho, \delta}^\kappa} < \epsilon.$$

Lemma 3.3. Let $a \in A_{\rho, \delta}^\kappa$, $0 < \epsilon'_0, \tilde{\epsilon}_0 \leq \epsilon_0$ (ϵ_0 is given in Proposition 2.3) and $u \in H_{\epsilon'_0, \tilde{\epsilon}_0}^\kappa$. Then there is $\epsilon_2 > 0$ independent of a, ρ and δ such that $a(x, D)u(x)$ belongs to $H_{\epsilon'_0-\rho, \tilde{\epsilon}_0-\delta}^\kappa$ if $0 < \epsilon'_0 - \rho \leq \min\{\epsilon_0, \epsilon_2\}$ and $0 < \tilde{\epsilon}_0 - \delta \leq \epsilon_0$.

Lemma 3.4. Let $a_i \in A_{\rho_i, \delta_i}^\kappa$ ($i = 1, 2$) and $u \in H_{\epsilon'_0, \tilde{\epsilon}_0}^\kappa$ ($\epsilon'_0, \tilde{\epsilon}_0 > 0$). Then if $|\rho_1| \leq \epsilon_0, |\delta_2| \leq \epsilon_0, 0 < \epsilon'_0 - \rho_2 \leq \epsilon_0 \min\{1, \rho_{a_2}\}, 0 < \tilde{\epsilon}_0 - \delta_2 \leq \epsilon_0, 0 < \epsilon'_0 - \rho_2 - \rho_1 \leq \epsilon_0 \min\{1, \rho_{a_1}\}$ and $0 < \tilde{\epsilon}_0 - \delta_2 - \delta_1 \leq \epsilon_0$ are valid (ϵ_0 is given in Proposition 2.3), we have

$$(3.10) \quad a_1(x, D)(a_2(x, D)u)(x) = (a_1 \circ a_2)(x, D)u(x),$$

which is in $H_{\epsilon'_0-\rho_1-\rho_2, \tilde{\epsilon}_0-\delta_1-\delta_2}^\kappa$.

Let $a \in A_{\rho, \delta}^\kappa$ ($|\rho|, |\delta| \leq \epsilon_0/4$), $u \in H_{\epsilon_0/2, \epsilon_0/2}^\kappa$ and $|\rho_1|, |\delta_1| < \epsilon_0/4$. Put $w = e^{\delta_1 \langle x \rangle^\kappa} e^{\rho_1 \langle D \rangle^\kappa} u$, which is in $H_{\epsilon_0/2-\rho_1, \epsilon_0/2-\delta_1}^\kappa$. Since we can write $u = e^{-\rho_1 \langle D \rangle^\kappa} (e^{-\delta_1 \langle x \rangle^\kappa} w)$, we get by use of Lemma 3.4 with $\epsilon'_0 = \epsilon_0/2 - \rho_1, \tilde{\epsilon}_0 = \epsilon_0/2 - \delta_1, a_1 = a(x, \xi) e^{-\rho_1 \langle \xi \rangle^\kappa}$ and $a_2 = e^{-\delta_1 \langle x \rangle^\kappa} e^{\rho_1 \langle D \rangle^\kappa}, \epsilon_{a_2} = 1$,

$$a(x, D)u(x) = a(x, D)(e^{-\rho_1 \langle D \rangle^\kappa} (e^{-\delta_1 \langle x \rangle^\kappa} w)) = ((a(x, \xi) e^{-\rho_1 \langle \xi \rangle^\kappa}) \circ e^{-\delta_1 \langle x \rangle^\kappa})(x, D)w(x).$$

Noting that $a_1(x, \xi) := (e^{(\delta_1-\delta) \langle x \rangle^\kappa} e^{\rho_1 \langle D \rangle^\kappa}) \circ (a(x, \xi) e^{-\rho_1 \langle \xi \rangle^\kappa}) \circ e^{-\delta_1 \langle x \rangle^\kappa} \in A_{0,0}^\kappa$, we obtain

$$(3.11) \quad \|au\|_{H_{\rho_1-\rho, \delta_1-\delta}^\kappa} = \|a_1(x, D)w\|_{L^2} \leq C\|w\|_{L^2} = C\|u\|_{H_{\rho_1, \delta_1}^\kappa}$$

for any $u \in H_{\epsilon_0/2, \epsilon_0/2}^\kappa$. Since $H_{\epsilon_0/2, \epsilon_0/2}^\kappa$ is dense in $H_{\rho_1, \delta_1}^\kappa$ from Lemma 3.2, we get the following theorem.

Theorem 3.5 Let $a \in A_{\rho, \delta}^\kappa$ ($|\rho|, |\delta| \leq \epsilon_0/4$), $|\rho_1|, |\delta_1| < \epsilon_0/4$, where ϵ_0 are given in Proposition 2.3. Then $a(x, D)$ maps from $H_{\rho_1, \delta_1}^\kappa$ to $H_{\rho_1-\rho, \delta_1-\delta}^\kappa$ and satisfies the following inequality

$$(3.12) \quad \|au\|_{H_{\rho_1-\rho, \delta_1-\delta}^\kappa} \leq C\|u\|_{H_{\rho_1, \delta_1}^\kappa}$$

for any $u \in H_{\rho_1, \delta_1}^\kappa$.

For $a \in A_{\rho, \delta}^\kappa$, we define

$$(3.13) \quad a^t(x, \xi) = os - \int \int e^{iy\eta} a(x+y, \xi+\eta) dy d\bar{\eta},$$

and $a^*(x, \xi) = a^t(\bar{x}, \xi)$. Then we can prove the following lemma, by the same way as that of the proof (i) of Proposition 2.3.

Lemma 3.6. *Let $a \in A_{\rho, \delta}^\kappa$ and $|\rho|, |\delta| \leq \epsilon_0$. Then $a^t(x, \xi)$ defined in (2.29) belongs to $A_{\rho, \delta}^\kappa$. Moreover it holds*

$$(3.14), \quad \begin{aligned} (a^t(x, D)u, \varphi)_{L^2} &= (u, a(x, D)\varphi)_{L^2}, \\ (a^*(x, D)u, \varphi)_{L^2} &= (u, a(x, D)\varphi)_{L^2}, \end{aligned}$$

for any $u, \varphi \in H_{\epsilon_0}^\kappa$.

The relation (3.14) and the inequality (3.12) yield

$$|(a^t u, \varphi)| \leq \|u\|_{H_{\rho-\rho_1, \delta-\delta_1}^\kappa} \|\bar{a}\varphi\|_{H_{\rho_1-\rho, \delta_1-\delta}^\kappa} \leq C \|u\|_{H_{\rho-\rho_1, \delta-\delta_1}^\kappa} \|\varphi\|_{H_{\rho_1, \delta_1}^\kappa},$$

if $|\rho|, |\delta| \leq \epsilon_0/4$ and $|\rho_1|, |\delta_1| < \epsilon_0/4$. Therefore taking account that $H_{\epsilon_0/2, \epsilon_0/2}^\kappa$ is dense in $H_{\rho_1, \delta_1}^\kappa$, we get from (3.14)

$$(3.15) \quad \|a^t u\|_{H_{-\rho_1, -\delta_1}^\kappa} \leq C \|u\|_{H_{\rho-\rho_1, \delta-\delta_1}^\kappa},$$

for any $u \in H_{\rho_1, \delta_1}^\kappa$. Thus we get the following proposition.

Proposition 3.7. *Let $a \in A_{\rho, \delta}^\kappa$ and $|\rho|, |\delta| \leq \epsilon_0/4$ and $|\rho_1|, |\delta_1| < \epsilon_0/4$. Then the pseudodifferential operators $a^t(x, D)$ and $a^*(x, D)$ satisfy (3.15).*

Noting that $(e^{\delta\langle x \rangle^\kappa} e^{\rho\langle D \rangle^\kappa})^t = e^{\rho\langle D \rangle^\kappa} e^{\delta\langle x \rangle^\kappa}$, we have for $u \in H_{\rho, \delta}^\kappa$

$$\begin{aligned} e^{\rho\langle D \rangle^\kappa} e^{\delta\langle x \rangle^\kappa} u(x) &= (e^{\delta\langle x \rangle^\kappa} e^{\rho\langle D \rangle^\kappa})^t (e^{-\rho\langle D \rangle^\kappa} e^{-\delta\langle x \rangle^\kappa} e^{\delta\langle x \rangle^\kappa} e^{\rho\langle D \rangle^\kappa} u)(x) \\ &= (e^{\delta\langle x \rangle^\kappa} e^{\rho\langle D \rangle^\kappa})^t \circ (e^{-\delta\langle x \rangle^\kappa} e^{-\rho\langle D \rangle^\kappa})^t e^{\delta\langle x \rangle^\kappa} e^{\rho\langle D \rangle^\kappa} u(x). \end{aligned}$$

Moreover we can see from Proposition 2.3 and Lemma 2.9 that $(e^{\delta\langle x \rangle^\kappa} e^{\rho\langle \xi \rangle^\kappa})^t \circ (e^{-\delta\langle x \rangle^\kappa} e^{-\rho\langle \xi \rangle^\kappa})^t$ is in $A_{0,0}^\kappa$. Hence we obtain the fact below.

Proposition 3.8. *Let $|\rho|, |\delta| \leq \epsilon_0/4$. Then u belongs to $H_{\rho, \delta}^\kappa$ if and only if $u \in \tilde{H}_{\rho, \delta}^\kappa$.*

The following result on the multiple symbols of pseudodifferential operators is a special case of Lemma 2.2 of Chapter 7 in Kumanogo's book [12].

Lemma 3.9. *Let $r_j(x, \zeta) \in A_{0,0}^\kappa$ ($j = 1, 2, \dots, v$) and put*

$$q_v(x, D) = r_1(x, D)r_2(x, D) \cdots r_v(x, D).$$

Then the symbol $q_v(x, \zeta)$ belongs to $A_{0,0}^\kappa$ and satisfies

$$(3.16) \quad |q_v^{(\alpha)}(x, \zeta)| \leq C^v \prod_{j=1}^v C_{r_j} \bar{\epsilon}_v^{|\alpha+\beta|} |\alpha+\beta|!,$$

for $(x, \zeta) \in R^{2n}$, $\alpha, \beta \in N^n$, where C is independent of v and $\bar{\epsilon}_v = \min\{\epsilon_{r_j}/4\}$.

We can prove easily the following lemma as a corollary of Lemma 3.9, by using the Neumann series method.

Lemma 3.10. Let $r(x, \xi)$ be in $A_{0,0}^\kappa$. If $C_r > 0$ is sufficiently small, then there is the inverse $(I + r(x, D))^{-1}$ which is a pseudodifferential operator with its symbol contained in $A_{0,0}^\kappa$.

Lemma 3.11. Let $j(x, \xi) \in \gamma^d S(\varepsilon_1, g_1)$. Then if $\varepsilon_1 > 0$ is small enough, there are $k_1(x, \xi) \in \gamma^d S(\varepsilon_1 < x >^{-1} < \xi >^{-1}, g_1)$, $\varepsilon_0 > 0$ independent of ε_1 and $r_\infty(x, \xi) \in A_{-\varepsilon_0, -\varepsilon_0}^{1/d}$ such that $(I + j(x, D))^{-1} = k(x, D) + k_1(x, D) + r_\infty(x, D)$, where $k(x, \xi) = (1 + j(x, \xi))^{-1}$.

4 Fourier Integral Operators

For $\vartheta \in AS(\rho_\vartheta < \xi > + \delta_\vartheta < x >, g)(\rho_\vartheta, \delta_\vartheta \geq 0)$, where $d\kappa \leq 1$, we denote

$$\phi(x, \xi) = x\xi - i\vartheta(x, \xi).$$

For $a \in A_{0,0}^\kappa$ we define a Fourier integral operator with a phase function $\phi(x, \xi)$ as follows,

$$(4.1) \quad a_\phi(x, D)u(x) = \int_{R^n} e^{i\phi(x, \xi)} a(x, \xi) \hat{u}(\xi) \bar{d}\xi,$$

for $u \in H_{\varepsilon_0, \varepsilon_0}$. Putting $p(x, \xi) = a(x, \xi)e^{\vartheta(x, \xi)}$, we can see $p(x, \xi) \in A_{\rho_\vartheta, \delta_\vartheta}^\kappa$. Therefore we can regard $a_\phi(x, D)$ as a pseudo differential operator with its symbol $p = ae^\vartheta$ defined in §2 and consequently it follows from Theorem 3.5 that $a_\phi(x, D)$ acts continuously from $H_{\rho, \delta}^\kappa$ to $H_{\rho-\rho_\vartheta, \delta-\delta_\vartheta}^\kappa$. However in order to construct the inverse operator of $p(x, D)$ it is better to regard $p(x, D)$ as a Fourier integral operator. In particular for $a = 1$ we denote

$$(4.2) \quad I_\phi(x, D)u(x) = \int e^{i\phi(x, \xi)} \hat{u}(\xi) \bar{d}\xi,$$

$$(4.3) \quad I_\phi^R(x, D)v(x) = \int e^{ix\xi} \bar{d}\xi \int e^{i\phi(y, \xi)} v(y) dy.$$

Theorem 4.1. Let $a \in \gamma^d S(\langle x \rangle^m \langle \xi \rangle^\ell, g_1)$, $\vartheta \in \gamma^d S(\rho_\vartheta \langle \xi \rangle^\kappa + \delta_\vartheta \langle x \rangle^\kappa, g_1)$ and $\phi = x\xi - i\vartheta(x, \xi)$. Assume $d\kappa \leq 1$. Then if $\rho_\vartheta, \delta_\vartheta$ are sufficiently small, $\tilde{a}(x, D) = I_\phi(x, D)a(x, D)I_\phi^{-1}$ and $\tilde{a}'(x, D) = I_\phi(x, D)^{-1}a(x, D)I_\phi(x, D)$ are pseudodifferential operators of which symbols are given by

$$(4.4) \quad \tilde{a}(x, \xi) = p(x, \xi) + r(x, \xi),$$

$$(4.5) \quad a'(x, \xi) = p'(x, \xi) + r'(x, \xi),$$

where

$$(4.6) \quad p(x, \xi) - a(x - i\nabla_\xi \vartheta(x, \Phi), \xi + i\nabla_x \vartheta(x, \Phi)) \in \gamma^d S(\langle x \rangle^{m-1} \langle \xi \rangle^{\ell-1}, g_1),$$

$$(4.7) \quad \tilde{p}'(x, \xi) - a(x + i\nabla_\xi \vartheta(\Phi', \xi), \xi - i\nabla_x \vartheta(\Phi', \xi)) \in \gamma^d S(\langle x \rangle^{m-1} \langle \xi \rangle^{\ell-1}, g_1),$$

where $\Phi = \Phi(x, x, \xi)$ and $\Phi' = \Phi'(x, \xi, \xi)$ are given by (4.6) and (4.19) respectively and r, r' belong to $A_{-\varepsilon_0, -\varepsilon_0}^\kappa$ for an $\varepsilon_0 > 0$ independent of ρ_ϑ .

This theorem is proved in [10] in the case of $d = \kappa = 1$. We can prove it similar way as that of [10].

Next we consider a phase function $\vartheta \in \gamma^d S(\langle x \rangle^\sigma \langle \xi \rangle^\delta, g_1)$. When $\sigma + \delta = \kappa = 1/d < 1$ or $\sigma + \delta = 1$ and $d = \min(\delta^{-1}, \sigma^{-1})$, Theorem 4.1 holds also, that is, we can prove Theorem 4.6 below. So far we consider

only $d, \sigma, \delta, \kappa$ above. We note that $d > 1$.

Lemma 4.2. *Let $a(x, \xi) \in \gamma^d S(< x >^m < \xi >^\ell, g_1)$ and $\vartheta \in \gamma^d S(\rho_\vartheta < \xi >^\delta < x >^\sigma, g_1) (\rho_\vartheta \geq 0)$. Put $\phi = x\xi - i\vartheta(x, \xi)$ and $\tilde{a}(x, D) = a_\phi(x, D)I_{-\phi}^R(x, D)$. If ρ_ϑ is sufficiently small, then $\tilde{a}(x, \xi)$ belongs to $S(< x >^m < \xi >^\ell, g)$ and moreover satisfies*

$$(4.8) \quad \tilde{a}(x, \xi) = \tilde{p}(x, \xi) + r(x, \xi),$$

for $x, \xi \in \mathbb{R}^n$, and

$$(4.9) \quad \begin{aligned} \tilde{p}(x, \xi) - \sum_{|\gamma| < N} \gamma!^{-1} D_y^\gamma \partial_\eta^\gamma \{a(x, \Phi(x, y, \eta)) J(x, y, \eta)\}_{y=x, \eta=\xi} \\ \in \gamma^d S(C^{1+N} N!^d < x >^{m-N} < \xi >^{\ell-N}, g_1) \end{aligned}$$

for any N , where $\Phi(x, y, \xi)$ is a solution of the following equation,

$$(4.10) \quad \Phi(x, y, \xi) - i\tilde{\nabla}_x \vartheta(x, y, \Phi(x, y, \xi)) = \xi,$$

$$(4.11) \quad \tilde{\nabla}_x \vartheta(x, y, \xi) = \int_0^1 \nabla \vartheta(y + t(x - y), \xi) dt,$$

$J(x, y, \xi) = \frac{D\Phi(x, y, \xi)}{D\xi}$ is the Jacobian of Φ , $r(x, \xi) \in A_{-\varepsilon_0, -\varepsilon_0}^{1/d}$, and $C > 0, \varepsilon_0 > 0$ are independent of ρ_ϑ .

Lemma 4.3. *Let $a(x, \xi)$ and ϑ be satisfied with the same condition as one of Lemma 4.2. For $\phi = x\xi - i\vartheta(x, \xi)$ put $a'(x, \xi) = I_{-\phi}^R(x, D)a_\phi(x, D)$. Then if ρ_ϑ and δ_ϑ are sufficiently small, $a'(x, \xi)$ belongs to $S(< x >^m < \xi >^\ell, g)$ and moreover satisfies*

$$(4.12) \quad a'(x, \xi) = p'(x, \xi) + r'(x, \xi),$$

$$(4.13) \quad \begin{aligned} p'(x, \xi) - \sum_{|\gamma| < N} \gamma^{-1} D_y^\gamma \partial_\eta^\gamma \{a(\Phi'(y, \xi, \eta), \xi) J'(y, \xi, \eta)\}_{y=x, \eta=\xi} \\ \in \gamma^d S(C^{1+N} N!^d < x >^{m-N} < \xi >^{\ell-N}, g_1), \end{aligned}$$

for any non negative integer N , where $\Phi'(y, \xi, \eta)$ is a solution of the equation

$$(4.14) \quad \Phi'(y, \xi, \eta) - i\tilde{\nabla}_\xi \vartheta(\Phi'(y, \xi, \eta), \xi, \eta) = y,$$

$$(4.15) \quad \tilde{\nabla}_\xi \vartheta(y, \xi, \eta) = \int_0^1 \nabla_\xi \vartheta(y, \eta + t(\xi - \eta)) dt,$$

and $J'(y, \xi, \eta) = \frac{D\Phi'(y, \xi, \eta)}{Dy}$, and $r'(x, \xi) \in A_{-\varepsilon_0, -\varepsilon_0}^{1/d}$ ($\varepsilon_0 > 0$ is independent of ρ_ϑ).

Lemma 4.4. *Let $\vartheta(x, \xi) \in \gamma^d S(\rho_\vartheta \langle x \rangle^\sigma \langle \xi \rangle^\delta, g_1)$. If ρ_ϑ and δ_ϑ are sufficiently small, there is the inverse of $I_\phi(x, D)$, which maps continuously from H_{ρ_1, δ_1} to $H_{\rho_1 - \rho_\vartheta, \delta_1 - \delta_\vartheta}$ for $|\rho_1|, |\delta_1|$ small enough and satisfies*

$$(4.16) \quad \begin{aligned} I_\phi(x, D)^{-1} &= I_{-\phi}^R(x, D)(I + j(x, D))^{-1} = (I + j'(x, D))^{-1} I_{-\phi}^R(x, D) \\ &= I_{-\phi}^R(x, D)(k(x, D) + k_1(x, D) + r(x, D)) = (k'(x, \xi) + k'_1(x, D) + r'(x, D)) I_{-\phi}^R(x, D), \end{aligned}$$

where $j(x, \xi) = J(x, 0, \xi) - 1 + r_1(x, \xi)$, $j'(x, \xi) = J'(x, \xi, 0) - 1 + r_2(x, \xi)$, $k(x, \xi) = J(x, 0, \xi)^{-1}$, $k'(x, \xi) = J'(x, \xi, 0)^{-1}$ and $k_1, k'_1 \in \gamma^d S(< x >^{-1} < \xi >^{-1}, g_1)$ and $r, r' \in A_{-\varepsilon_0, -\varepsilon_0}^{1/d}$.

Lemma 4.5. *Let $a(x, \xi)$ and ϑ be satisfied with the same condition as one of Lemma 3.3. Let $\phi = x\xi - i\vartheta$. Then we have*

$$(4.17) \quad \sigma(I_\phi(x, D)a(x, D))(x, \xi) = I_\phi \circ a(x, \xi) = e^{\vartheta(x, \xi)}(q(x, \xi) + r(x, \xi)),$$

$$(4.18) \quad \sigma(a(x, D)I_\phi(x, D))(x, \xi) = a \circ I_\phi(x, \xi) = e^{\vartheta(x, \xi)}(q'(x, \xi) + r'(x, \xi)),$$

where r, r' is in $A_{-\varepsilon_0, -\varepsilon_0}^{1/d}$, if ρ_ϑ is sufficiently small, and q, q' satisfies

$$(4.19) \quad q(x, \xi) - \sum_{|\gamma| < N} \gamma!^{-1} D_y^\delta \partial_\eta^\gamma \{a(x + y - i\tilde{\nabla}_\xi \vartheta(x, \xi, \eta), \xi)\}_{y=\eta=0} \\ \in \gamma^{1/d} S(C^{1+N} N!^d < x >^{m-N} < \xi >^{\ell-N}, g_1),$$

$$(4.20) \quad q'(x, \xi) - \sum_{|\gamma| < N} \gamma^{-1} D_y^\gamma \partial_\eta^\gamma \{a(x, \xi + \eta - i\tilde{\nabla}_x \vartheta(x, y, \xi))\}_{y=\eta=0} \\ \in \gamma^d S(C^{1+N} N!^d < x >^{m-N} < \xi >^{\ell-N}, g_1),$$

for any positive integer N , and $C > 0$ and $\varepsilon_0 > 0$ are independent of ρ_ϑ , where $\tilde{\nabla}_\xi \vartheta(x, \xi, \eta) = \int_0^1 \nabla_\xi \vartheta(x, \xi + t\eta) dt$ and $\tilde{\nabla}_x \vartheta(x, y, \xi) = \int_0^1 \nabla_x \vartheta(x + ty, \xi) dt$.

Summing up Lemma 4.2-Lemma 4.5, we obtain the following theorem.

Theorem 4.6. *Let $a \in \gamma^d S(< x >^m < \xi >^\ell, g_1)$, $\vartheta \in \gamma^d S(\rho_\vartheta < \xi >^\delta < x >^\sigma, g_1)$ and $\phi = x\xi - i\vartheta(x, \xi)$. Assume that $\sigma + \delta = \kappa = 1/d < 1$ or $\sigma + \delta = \kappa = 1$, $d = \min(\delta^{-1}, \sigma^{-1})$. Then if $\rho_\vartheta, \delta_\vartheta$ are sufficiently small, $\tilde{a}(x, D) = I_\phi(x, D)a(x, D)I_\phi^{-1}$ and $\tilde{a}'(x, D) = I_\phi(x, D)^{-1}a(x, D)I_\phi(x, D)$ are pseudodifferential operators of which symbols are given by*

$$(4.21) \quad \tilde{a}(x, \xi) = p(x, \xi) + r(x, \xi),$$

$$(4.22) \quad \tilde{a}'(x, \xi) = p'(x, \xi) + r'(x, \xi),$$

where

$$(4.23) \quad p(x, \xi) - a(x - i\nabla_\xi \vartheta(x, \Phi), \xi + i\nabla_x \vartheta(x, \Phi)) \in \gamma^d S(< x >^{m-1} < \xi >^{\ell-1}, g_1),$$

$$(4.24) \quad \tilde{p}'(x, \xi) - a(x + i\nabla_\xi \vartheta(\Phi', \xi), \xi - i\nabla_x \vartheta(\Phi', \xi)) \in \gamma^d S(< x >^{m-1} < \xi >^{\ell-1}, g_1),$$

where $\Phi = \Phi(x, x, \xi)$ and $\Phi' = \Phi'(x, \xi, \xi)$ are given by (4.6) and (4.10) respectively and r, r' belong to $A_{-\varepsilon_0, -\varepsilon_0}^{1/d}$ for an $\varepsilon_0 > 0$ independent of ρ_ϑ .

5 Criterion to L^2 -well posed Cauchy problem

For $T > 0$ let consider the following Cauchy problem,

$$(5.1) \quad \partial_t u(t, x) - i\Delta u(t, x) - b(t, x, D)u(t, x) = 0,$$

$$(5.2) \quad u(0, x) = u_0(x),$$

for $(t, x) \in (0, T) \times R^n$. We assume that $b(t, x, \xi)$ is in $C^0([0, T]; S_{1,0}^1)$. Moreover we suppose that there are $C \in R, K > 0$ such that

$$(5.3) \quad \text{Re} b(t, x, \xi) \leq C,$$

for $x, \xi \in R^n$ with $|x|, |\xi| \geq K$ and $t \in [0, T]$. Then we can prove the following theorem by use of the same method as that of [3] and [7].

Theorem 5.1. *Assume that the above conditions (4.3)-(4.5) are valid. For any $u_0 \in L^2$ and $f \in C^0([0, T]; L^2)$ there exists a unique solution $u \in C^0([0, T]; L^2) \cap C^1([0, T]; H^{-2})$ of the Cauchy problem (5.1)-(5.2).*

6 Proof of Theorem

Assume that $u(t, x)$ satisfies (1)-(2) in the introduction. Put $v(t, x) = e^{\rho t \langle D \rangle^\kappa} u(t, x)$. Then v satisfies the following Cauchy problem,

$$(6.1) \quad \frac{\partial}{\partial t} v(t, x) = (i\Delta + c(t, x, D))v(t, x),$$

$$(6.2) \quad v(0, x) = u_0(x),$$

where

$$(6.3) \quad \begin{aligned} c(t, x, D) &= \rho \langle D \rangle^\kappa + e^{\rho \langle D \rangle^\kappa} b(t, x, D) e^{-\rho \langle D \rangle^\kappa} \\ &= \rho \langle D \rangle^\kappa + b(t, x, D) + b_1(t, x, D) + r_2(t, x, D), \end{aligned}$$

where $b_1(x, \xi) \in \gamma^d S(< \xi > < x >^{-1}, g_1)$, $r_1(t, z, \zeta) \in A_{-\varepsilon_0 + c\rho_0 T, -\varepsilon_0}^\kappa$ from Theorem 4.1. Oncemore we change the unknown function v to w as follows,

$$(6.4) \quad w(t, x) = I_\phi(x, D)v(t, x),$$

where $\phi = x\xi - i\varepsilon\vartheta(t, x, \xi)$ and ϑ is given by

$$\begin{aligned} \vartheta(t, x, \xi) &= \vartheta_0(x, \xi) \phi_0\left(\frac{\langle x \rangle}{M \langle \xi \rangle}\right) + t \langle \xi \rangle^{\sigma+\delta} (1 - \phi_0\left(\frac{\langle x \rangle}{M \langle \xi \rangle}\right)), \\ \vartheta_0(x, \xi) &= \frac{x \cdot \xi}{\langle x \rangle^{1-\sigma} \langle \xi \rangle^{1-\delta} \varepsilon_1} \phi_0\left(\frac{x \cdot \xi}{\langle x \rangle \langle \xi \rangle \varepsilon_1}\right) + \langle \xi \rangle^{\delta-\sigma} f(|x \cdot \xi|) [\phi_+\left(\frac{x \cdot \xi}{\langle x \rangle \langle \xi \rangle \varepsilon_1}\right) - \phi_-\left(\frac{x \cdot \xi}{\langle x \rangle \langle \xi \rangle \varepsilon_1}\right)], \\ f(t) &= \int_0^t (1+s^2)^{\frac{\sigma-1}{2}} ds, \end{aligned}$$

and $\phi_\pm(t) = \chi(\pm t)$, $\phi_0(t) = 1 - \phi_+(t) - \phi_-(t)$ and $\chi(t) \in \gamma^d(R)$ such that $\chi(t) = 1$ for $t \geq 1$, $\chi(t) = 0$ for $t \leq 1/2$, $\chi'(t) \geq 0$ and $0 \leq \chi(t) \leq 1$. Then we can see that $\vartheta(t, x, \xi)$ belongs to $\gamma^d S(\langle x \rangle^\sigma \langle \xi \rangle^\delta, g_1)$ and that there are $\varepsilon_1 > 0, M > 0, K > 0, c_0 > 0$ such that ϑ satisfies

$$(6.5) \quad (\partial_t + \xi \cdot \nabla_x) \vartheta(t, x, \xi) \geq c_0 (\langle \xi \rangle^{2\delta} \langle x \rangle^{2\sigma-2} + \langle \xi \rangle^{\sigma+\delta} + \langle \xi \rangle \langle x \rangle^{\sigma+\delta-1}) - c_1,$$

for $x, \xi \in R^n$ with $|x|, |\xi| \geq K, |t| \leq T$.

It follows from Lemma 4.4 that if $|\varepsilon|$ is sufficiently small, we have the inverse $I_\phi(x, D)^{-1}$. Therefore we get the following Cauchy problem of w from (6.1)-(6.2),

$$(6.6) \quad \frac{\partial}{\partial t} w(t, x) = (\partial_t I_\phi) I_\phi(x, D)^{-1} w(t, x) + I_\phi(i\Delta + c(t, x, D)) I_\phi(x, D)^{-1} w(t, x),$$

$$(6.7) \quad w(0, x) = I_\phi(x, D) u_0(x).$$

Since $\vartheta(t, x, \xi) \in \gamma^d S(\langle x \rangle^\sigma \langle \xi \rangle^\delta, g_1)$, it follows from (4.10) that $\nabla_\xi \vartheta(x, \Phi(x, \xi)) \in \gamma^d S(\langle x \rangle^\sigma \langle \xi \rangle^{\delta-1}, g_1)$, $\nabla_x \vartheta(x, \Phi(x, \xi)) \in \gamma^d S(\langle x \rangle^{\sigma-1} \langle \xi \rangle^\delta, g_1)$, and $\Phi(x, \xi) - \xi \in \gamma^d S(\langle x \rangle^{\sigma-1} \langle \xi \rangle^\delta, g_1)$. Hence we have from (4.16) in Theorem 4.6 and Proposition 2.3

$$(6.8) \quad \begin{aligned} \sigma(I_\phi \Delta I_\phi^{-1})(x, \xi) &= -|\xi + i\varepsilon \nabla_x \vartheta(x, \Phi)|^2 + a_1(x, D) + r_2(x, \xi), \\ &= -(|\xi|^2 + |\nabla_x \vartheta(t, x, \xi)|^2 + 2i\varepsilon \xi \cdot \nabla_x \vartheta(t, x, \xi)) + a'_1(x, \xi) + r_2(x, D) \end{aligned}$$

where $a_1 \in S(\langle \xi \rangle^{\sigma-1} \langle x \rangle^{-1}, g)$, $a'_1 \in S(\langle x \rangle^{2\sigma-2} \langle \xi \rangle^{2\delta} + \langle \xi \rangle \langle x \rangle^{-1}, g_1)$ and $r_2 \in A_{-\varepsilon_0 + |c|\varepsilon|, -\varepsilon_0 + |c|\varepsilon|}^{1/d}$ for some $c > 0$ (independent of ε). Here we choose ε such that r_2 belongs to $S(1, g)$. Thus we obtain the equation of w from (6.6)-(6.7),

$$(5.10) \quad \frac{\partial w}{\partial t} = (i\Delta + \rho \langle D \rangle^{\sigma+\delta} + b(t, x, D) + \varepsilon(\partial_t + \xi \cdot \nabla_x) \vartheta)(t, x, D) w(t, x) + r_3(t, x, D) w(t, x),$$

$$(5.11) \quad w(0) = I_{\phi(0)}(x, D) u_0(x),$$

where $r_4 \in S(\langle \xi \rangle^{2\delta} \langle x \rangle^{2\sigma-2} + \langle \xi \rangle \langle x \rangle^{-1}, g_1)$. Moreover taking account of the assumptions (5) in the introduction and (6.5) we can choose conveniently $K > 0$, ε and ρ such that we have

$$\rho p(x, \xi) + \text{Re} b(t, x, \xi) - \varepsilon H_a \theta(x, \xi) + \text{Re} r_4(t, x, \xi) \leq 0,$$

for $x, \xi \in R^n$ with $|x|, |\xi| \geq K$, where $K > 0$ is sufficiently large. Therefore we can solve the Cauchy problem (6.6)-(6.7) by use of Theorem 5.1, since $w(0) = I_{\phi(0)} u_0$ belongs to L^2 , and consequently we get the solution $u = e^{-\rho t \langle D \rangle^\kappa} I_\phi(x, D)^{-1} w(t, x) = e(t, x, D)^{-1} I_{\phi(t)}(x, D)^{-1} I_\phi(x, D)^{-1} w$, which satisfies (6) from Lemma 1.2. This completes the proof of Theorem.

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