

Automorphisms on the ring of symmetric functions and stable and dual stable Grothendieck polynomials

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The stable Grothendieck polynomials G_λ and the dual stable Grothendieck polynomials g_λ are certain families of inhomogeneous symmetric functions parametrized by interger partitions λ . They are certain K -theoretic deformations of the Schur functions and dual to each other via the Hall inner product.

Historically the stable Grothendieck polynomials (parametrized by permutations) were introduced by Fomin and Kirillov [FK96] as a stable limit of the Grothendieck polynomials of Lascoux–Schützenberger [LS82]. In [Buc02] Buch gave a combinatorial formula for the stable Grothendieck polynomials G_λ for partitions using so-called set-valued tableaux, and showed that their span $\bigoplus_{\lambda \in \mathcal{P}} \mathbb{Z}G_\lambda$ is a bialgebra and its certain quotient ring is isomorphic to the K -theory of the Grassmannian $\text{Gr} = \text{Gr}(k, \mathbb{C}^n)$.

The dual stable Grothendieck polynomials g_λ were introduced by Lam and Pylyavskyy [LP07] as generating functions of reverse plane partitions, and shown to be the dual basis for G_λ via the Hall inner product. They also showed there that g_λ represent the K -homology classes of ideal sheaves of the boundaries of Schubert varieties in the Grassmannians.

In this article we give the following properties of g_λ and G_λ :

- (A) The linear map I given by

$$g_\lambda \mapsto \sum_{\mu \subset \lambda} g_\mu$$

is an algebra automorphism.

- (B) The Pieri formulas for G_λ (resp. g_λ) can be written as alternating sums of joins (resp. meets) of the leading terms (i.e. the terms appearing in the Pieri formula for the Schur functions s_λ).

In Section 2 we explain that the ring automorphism in (A) is written as both

- (a) the substitution $f(x) \mapsto f(1, x)$, (that is, $f(x_1, x_2, \dots) \mapsto f(1, x_1, x_2, \dots)$), and
(b) the map $H(1)^\perp$, where $H(1) = \sum_i h_i$,

where the linear map $F^\perp$ is the adjoint of the multiplication map $(F \cdot)$. The equivalence of two maps in (a) and (b) is previously known (more generally, $H(t)^\perp(f(x)) = f(t, x)$ where $H(t) = \sum_i t^i h_i$). The key observation to show $I(f(x)) = f(1, x)$ is that the substitution $f \mapsto f(1, 0, 0, \dots)$ maps $g_{\lambda/\mu}$ to 1 for any skew shape λ/μ ; then since I is a certain composition of this map and the coproduct on Λ it follows that $I = (f(x) \mapsto f(1, x))$.

In Section 3 we give an exposition for (B) without technical details of the proofs.

1 Stable and dual stable Grothendieck polynomials

For basic definitions for symmetric functions, see for instance [Mac95, Chapter I].

Let $\Lambda (= \Lambda(x) = \Lambda_K = \Lambda_K(x))$ be the ring of symmetric functions, namely the set of all symmetric formal power series of bounded degree in variable $x = (x_1, x_2, \dots)$ with coefficients in K . We omit the variable x when no confusion arise. Let $\hat{\Lambda}$ be its completion, consisting of all symmetric formal power series (with possibly unbounded degree). Let $\mathcal{P}$ be the set of partitions. The Schur functions s_λ ($\lambda \in \mathcal{P}$) are a family of

homogeneous symmetric functions satisfying $\Lambda = \bigoplus_{\lambda \in \mathcal{P}} K s_\lambda$ and $\widehat{\Lambda} = \prod_{\lambda \in \mathcal{P}} K s_\lambda$. The *Hall inner product* $(,)$ is a bilinear form on Λ for which $(s_\lambda, s_\mu) = \delta_{\lambda\mu}$. This is naturally extended to $(,): \widehat{\Lambda} \times \Lambda \rightarrow K$.

In [Buc02, Theorem 3.1] Buch gave a combinatorial description of the *stable Grothendieck polynomial* G_λ as a generating function of so-called *set-valued tableaux*. We do not review the detail here and just recall some of its properties: $G_\lambda \in \widehat{\Lambda}$ (although $G_\lambda \notin \Lambda$ if $\lambda \neq \emptyset$), G_λ is an infinite linear combination of $\{s_\mu\}_{\mu \in \mathcal{P}}$ whose lowest degree component is s_λ . Hence $\widehat{\Lambda} = \prod_{\lambda \in \mathcal{P}} K G_\lambda$, i.e. every element in $\widehat{\Lambda}$ is uniquely written as an infinite linear combination of G_λ . Moreover the span $\bigoplus_\lambda K G_\lambda (\subset \widehat{\Lambda})$ is a bialgebra, in particular the expansion of the product

$$G_\mu G_\nu = \sum_\lambda c_{\mu\nu}^\lambda G_\lambda$$

and the coproduct

$$\Delta(G_\lambda) = \sum_{\mu, \nu} d_{\mu\nu}^\lambda G_\mu \otimes G_\nu$$

are finite.

Next we recall the dual stable Grothendieck polynomial $g_{\lambda/\mu}$. For a skew shape λ/μ , a *reverse plane partition* of shape λ/μ is a filling of the boxes in λ/μ with positive integers such that the numbers are weakly increasing in every row and column.

Definition 1.1 ([LP07]). For a skew shape λ/μ , the *dual stable Grothendieck polynomial* $g_{\lambda/\mu}$ is defined by

$$g_{\lambda/\mu} = \sum_T x^T, \quad (1)$$

summed over reverse plane partitions T of shape λ/μ , where $x^T = \prod_i x_i^{T(i)}$ where $T(i)$ is the number of columns of T that contain i .

When $\mu = \emptyset$ we write $g_\lambda = g_{\lambda/\emptyset}$. It is shown in [LP07] that $g_{\lambda/\mu} \in \Lambda$ and g_λ has the highest degree component s_λ and forms a basis of Λ that is dual to G_λ via the Hall inner product:

$$(G_\lambda, g_\mu) = \delta_{\lambda\mu}. \quad (2)$$

Hence the product (resp. coproduct) structure constants for $\{G_\lambda\}$ coincide with the coproduct (resp. product) structure constants for $\{g_\lambda\}$:

$$g_\mu g_\nu = \sum_\lambda d_{\mu\nu}^\lambda g_\lambda \quad \text{and} \quad \Delta(g_\lambda) = \sum_{\mu, \nu} c_{\mu\nu}^\lambda g_\mu \otimes g_\nu.$$

2 On the automorphism

2.1 Hopf structure of Λ

The ring Λ is a self-dual Hopf algebra with a coproduct $\Delta: \Lambda = \Lambda(x) \rightarrow \Lambda(x, y) \hookrightarrow \Lambda(x) \otimes \Lambda(y); f(x) \mapsto f(x, y)$, a counit $\epsilon: \Lambda \rightarrow K; f \mapsto f(0, 0, \dots)$, i.e. $\epsilon(s_\lambda) = \delta_{\lambda\emptyset}$, and an antipode $S: \Lambda \rightarrow \Lambda; s_\lambda \mapsto (-1)^{|\lambda|} s_{\lambda'}$. Here λ' denotes the transpose of $\lambda \in \mathcal{P}$.

For $F \in \widehat{\Lambda}$, we have linear maps

- $(F, -): \Lambda \rightarrow K; f \mapsto (F, f)$, and
- $F^\perp: \Lambda \rightarrow \Lambda; f \mapsto \sum (F, f_1) f_2$

where we put $\Delta(f) = \sum f_1 \otimes f_2$ for $f \in \Lambda$ by the Sweedler notation. It is known that the multiplication map $(F \cdot)$ and the map $F^\perp$ are adjoint, i.e. $(FG, f) = (G, F^\perp(f))$ for $\forall F, G \in \widehat{\Lambda}$ and $\forall f \in \Lambda$.

Note that

$$F^\perp = ((F, -) \otimes \text{id}) \circ \Delta = (\text{id} \otimes (F, -)) \circ \Delta \quad (3)$$

where the second equality is by cocommutativity. We also have

$$(F, -) = \epsilon \circ F^\perp \quad (4)$$

since $\epsilon \circ F^\perp = \epsilon \circ ((F, -) \otimes \text{id}) \circ \Delta = ((F, -) \otimes \epsilon) \circ \Delta = (F, -) * \epsilon = (F, -)$. The following lemma is standard:

Lemma 2.1. For $F, G \in \widehat{\Lambda}$,

- (1) $(FG, -) = (F, -) * (G, -)$ where $*$ denotes the convolution product on $\text{Hom}(\Lambda, K)$.
- (2) $(FG)^\perp = G^\perp \circ F^\perp = (F^\perp \circ G^\perp)$.

2.2 The maps $H(t)^\perp$ and $E(t)^\perp$

There are well-known generating functions

$$H(t) = \sum_{i \geq 0} t^i h_i, \quad E(t) = \sum_{i \geq 0} t^i e_i$$

where $t \in K$ (hence $H(t), E(t) \in \widehat{\Lambda}$). Let

$$H^\perp(t) := H(t)^\perp = \sum_{i \geq 0} t^i h_i^\perp, \quad E^\perp(t) := E(t)^\perp = \sum_{i \geq 0} t^i e_i^\perp.$$

It is known (see [Mac95, Chapter 1.5, Example 29]) that

$$H^\perp(t), E^\perp(t): \Lambda \longrightarrow \Lambda \text{ are ring automorphisms,} \quad (5)$$

$$H^\perp(t)(f(x_1, x_2, \dots)) = f(t, x_1, x_2, \dots) \quad \text{for } f \in \Lambda. \quad (6)$$

The proof of (5) was as follows: for $F \in \widehat{\Lambda}$, we can see that the map $F^\perp: \Lambda \longrightarrow \Lambda$ is an algebra automorphism if and only if $F(x, y) = F(x)F(y)$ and $F(0) = 1$, and it is easy to see that $H(t)$ and $E(t)$ satisfy them.

To show (6), it then suffices to show it when $f = h_n$, which is straightforward.

From (5), (6) and (4) we have

$$(H(t), -), (E(t), -): \Lambda \longrightarrow K \text{ are ring homomorphisms,} \quad (7)$$

$$(H(t), f) = f(t, 0, 0, \dots). \quad (8)$$

Since $H(t)E(-t) = 1$, by Lemma 2.1 and the fact that the counit is the identity with respect to the convolution product we have

Lemma 2.2. (1) $(H(t), -) * (E(-t), -) = \epsilon$, where $\epsilon: \Lambda \longrightarrow K$ is the counit.

- (2) $H(t)^\perp \circ E(-t)^\perp = \text{id}_\Lambda$.

2.3 Descriptions of $H(t)$, $(H(t), -)$ and $H(t)^\perp$

Let $c(\lambda/\mu)$ denote the number of columns in the skew shape λ/μ .

Proposition 2.3. $(H(t), g_{\lambda/\mu}) = t^{c(\lambda/\mu)}$ for any skew shape λ/μ .

Proof. By (8) we have $(H(t), g_{\lambda/\mu}) = g_{\lambda/\mu}(t, 0, 0, \dots)$. By (1), it is the generating function of reverse plane partitions on λ/μ filled with one alphabet 1. Clearly there is exactly one such filling, whose weight is $x_1^{c(\lambda/\mu)}$. Hence $g_{\lambda/\mu}(t, 0, 0, \dots) = t^{c(\lambda/\mu)}$. $\square$

Next we give another description of the map $I: g_\lambda \mapsto \sum_{\mu \subset \lambda} g_\mu$.

For a skew shape λ/μ and a totally ordered set X called *alphabets* (most commonly $\{1, 2, 3, \dots\}$), we shall denote by $\text{RPP}(\lambda/\mu, X)$ the set of reverse plane partition of shape λ/μ where each box is filled with an element of X . The expression (1) of $g_{\lambda/\mu}$ as a generating function of reverse plane partitions implies

$$\Delta(g_{\lambda/\mu}) = \sum_{\mu \subset \nu \subset \lambda} g_{\lambda/\nu} \otimes g_{\nu/\mu}, \quad (9)$$

since we have a natural bijection between $\text{RPP}(\lambda/\mu, \{1, 2, \dots, 1', 2', \dots\})$ and $\bigsqcup_{\mu \subset \nu \subset \lambda} \text{RPP}(\nu/\mu, \{1, 2, \dots\}) \times \text{RPP}(\lambda/\nu, \{1', 2', \dots\})$ where $1 < 2 < \dots < 1' < 2' < \dots$.

By (3) and Proposition 2.3, we apply $(H(t), -) \otimes \text{id}$ and $\text{id} \otimes (H(t), -)$ to (9) and obtain

Proposition 2.4. *The algebra automorphism $H(t)^\perp: \Lambda \rightarrow \Lambda$ satisfies*

$$H(t)^\perp(g_{\lambda/\mu}) = \sum_{\mu \subset \nu \subset \lambda} t^{c(\lambda/\nu)} g_{\nu/\mu} = \sum_{\mu \subset \nu \subset \lambda} t^{c(\nu/\mu)} g_{\lambda/\nu} \quad (10)$$

for any $\mu \subset \lambda$.

In particular, setting $\mu = \emptyset$ and $t = 1$ in (10), for any $\lambda \in \mathcal{P}$ we have

$$H^\perp(1)(g_\lambda) = \sum_{\nu \subset \lambda} g_\nu,$$

hence

$$I = H^\perp(1) = (f(x) \mapsto f(1, x)). \quad (11)$$

In particular (11) recovers that $I: \Lambda \rightarrow \Lambda$ is a ring automorphism. Moreover, (10) and (11) imply

$$I(g_{\lambda/\mu}) = \sum_{\mu \subset \nu \subset \lambda} g_{\nu/\mu} = \sum_{\mu \subset \nu \subset \lambda} g_{\lambda/\nu}. \quad (12)$$

2.3.1 Dual map

Next we recall that $H^\perp(t): \Lambda \rightarrow \Lambda$ and $(H(t), \cdot): \widehat{\Lambda} \rightarrow \widehat{\Lambda}$ are adjoint. By (2) and $H(t)^\perp(g_\mu) = \sum_{\lambda \subset \mu} t^{c(\mu/\lambda)} g_\lambda$ (by setting $\mu = \emptyset$ in (10)) we have

$$H(t)G_\lambda = \sum_{\lambda \subset \mu} t^{c(\mu/\lambda)} G_\mu. \quad (13)$$

Setting $\lambda = \emptyset$ in (13) we get $H(t) = \sum_{\lambda \in \mathcal{P}} t^{c(\lambda)} G_\lambda$, and by plugging it into (13) we have

$$\left(\sum_{\mu \in \mathcal{P}} t^{c(\mu)} G_\mu \right) G_\lambda = \sum_{\lambda \subset \mu} t^{c(\mu/\lambda)} G_\mu. \quad (14)$$

Remark 2.5. Since $I = H^\perp(1)$ it follows that $I^* = (H(1), \cdot) = ((\sum_\lambda G_\lambda) \cdot)$, and (14) specializes to

$$\left(I^*(G_\lambda) = \right) \left(\sum_{\mu \in \mathcal{P}} G_\mu \right) G_\lambda = \sum_{\lambda \subset \mu} G_\mu \quad (15)$$

which appeared in [Buc02, Section 8].

2.4 Description of $E(t)$, $(E(t), -)$ and $E(t)^\perp$

In this section we give descriptions using G_λ and g_λ for the element $E(t)$ and maps $(E(t), -)$ and $E^\perp(t)$. Note that by $I = H^\perp(1)$ and $I^* = (H(1)\cdot)$ it follows that $I^{-1} = E^\perp(-1)$ and $(I^*)^{-1} = (E(-1)\cdot)$.

By a tour-de-force combinatorial argument we can prove

Proposition 2.6. *The ring homomorphism $(E(t), -): \Lambda \longrightarrow K$ satisfies*

$$(E(t), g_{\lambda/\mu}) = \begin{cases} t^{c(\lambda/\mu)}(t+1)^{|\lambda/\mu|-c(\lambda/\mu)} & \text{if } \lambda/\mu \text{ is a vertical strip,} \\ 0 & \text{otherwise} \end{cases}$$

for any skew shape λ/μ . In particular, for any $\lambda \in \mathcal{P}$,

$$(E(t), g_\lambda) = \begin{cases} 1 & \text{if } \lambda = \emptyset, \\ t(t+1)^{n-1} & \text{if } \lambda = (1^n) \ (n \geq 1), \\ 0 & \text{otherwise.} \end{cases}$$

Later We give a sketch of the proof of Proposition 2.6, and beforehand give as its corollaries descriptions for $E(t)$ and $E(t)^\perp$.

Proposition 2.7. *The ring automorphism $E(t)^\perp: \Lambda \longrightarrow \Lambda$ satisfies*

$$\begin{aligned} E(t)^\perp(g_{\lambda/\mu}) &= \sum_{\substack{\mu \subset \nu \subset \lambda \\ \lambda/\nu: \text{ vertical strip}}} t^{c(\lambda/\nu)}(t+1)^{|\lambda/\nu|-c(\lambda/\nu)} g_{\nu/\mu} \\ &= \sum_{\substack{\mu \subset \nu \subset \lambda \\ \nu/\mu: \text{ vertical strip}}} t^{c(\nu/\mu)}(t+1)^{|\nu/\mu|-c(\nu/\mu)} g_{\lambda/\nu} \end{aligned}$$

for any skew shape λ/μ . In particular, for any $\lambda \in \mathcal{P}$,

$$\begin{aligned} E(t)^\perp(g_\lambda) &= \sum_{\substack{\nu \subset \lambda \\ \lambda/\nu: \text{ vertical strip}}} t^{c(\lambda/\nu)}(t+1)^{|\lambda/\nu|-c(\lambda/\nu)} g_\nu \\ &= \begin{cases} g_\lambda + \sum_{k=1}^{l(\lambda)} t(t+1)^{k-1} g_{\lambda/(1^k)} & \text{if } \lambda \neq \emptyset, \\ g_\emptyset & \text{if } \lambda = \emptyset. \end{cases} \end{aligned} \tag{16}$$

Proof. Proved similarly to Proposition 2.4, with Proposition 2.6 in hand. $\square$

Now we have a description of $E(-1)^\perp = I^{-1}$ by setting $t = -1$ in the proposition above.

Corollary 2.8. *The ring automorphism $E(-1)^\perp = I^{-1}: \Lambda \longrightarrow \Lambda$ satisfies*

$$I^{-1}(g_{\lambda/\mu}) = \sum_{\substack{\mu \subset \nu \subset \lambda \\ \lambda/\nu: \text{ rook strip}}} (-1)^{|\lambda/\nu|} g_{\nu/\mu} = \sum_{\substack{\mu \subset \nu \subset \lambda \\ \nu/\mu: \text{ rook strip}}} (-1)^{|\nu/\mu|} g_{\lambda/\mu}.$$

In particular, when $\mu = \emptyset$ we have

$$I^{-1}(g_\lambda) = \sum_{\lambda/\nu: \text{ rook strip}} (-1)^{|\lambda/\nu|} g_\nu = \begin{cases} g_\lambda - g_{\lambda/(1)} & \text{if } \lambda \neq \emptyset, \\ 1 & \text{if } \lambda = \emptyset. \end{cases} \tag{17}$$

Since $E^\perp(t)$ and $(E(t)\cdot)$ are adjoint, by (16) and (2) we have the following:

Proposition 2.9. *The element $E(t) = \sum_{i \geq 0} t^i e_i \in \widehat{\Lambda}$ satisfies*

$$E(t)G_\lambda = \sum_{\mu/\lambda: \text{ vertical strip}} t^{c(\mu/\lambda)} (t+1)^{|\mu/\lambda| - c(\mu/\lambda)} G_\mu. \quad (18)$$

In particular, setting $\lambda = \emptyset$ we have

$$E(t) = 1 + \sum_{n \geq 1} t(t+1)^{n-1} G_{(1^n)},$$

and hence

$$\left(1 + \sum_{n \geq 1} t(t+1)^{n-1} G_{(1^n)}\right) G_\lambda = \sum_{\mu/\lambda: \text{ vertical strip}} t^{c(\mu/\lambda)} (t+1)^{|\mu/\lambda| - c(\mu/\lambda)} G_\mu. \quad (19)$$

2.5 Sketch of the proof of Proposition 2.6

We recall the *incidence algebras* (see [Sta12, Chapter 3.6] for details). Let $\text{Int}(\mathcal{P}) = \{(\mu, \lambda) \in \mathcal{P} \times \mathcal{P} \mid \mu \subset \lambda\}$, consisting of all comparable (ordered) pairs in $\mathcal{P}$ (or equivalently all skew shapes, by identifying (μ, λ) with λ/μ). The *incidence algebra* $I(\mathcal{P}) = I(\mathcal{P}, K)$ is the algebra of all functions $f: \text{Int}(\mathcal{P}) \rightarrow K$ where multiplication is defined by the convolution

$$(fg)(\mu, \lambda) = \sum_{\mu \subset \nu \subset \lambda} f(\mu, \nu) g(\nu, \lambda). \quad (20)$$

Then $I(\mathcal{P}, K)$ is an associative algebra with two-sided identity $\delta := ((\mu, \lambda) \mapsto \delta_{\mu\lambda})$.

A linear function $f: \Lambda \rightarrow K$ can be considered as an element of $I(\mathcal{P}, K)$ by setting $f(\mu, \lambda) = f(g_{\lambda/\mu})$. Then the convolution product $*$ on $\text{Hom}(\Lambda, K)$ coincides with the multiplication on $I(\mathcal{P})$ due to (9), i.e. this inclusion $\text{Hom}(\Lambda, K) \rightarrow I(\mathcal{P})$ is as algebras. Note that the counit $\epsilon \in \text{Hom}(\Lambda, K)$ is mapped to $\delta \in I(\mathcal{P})$.

Define $i_t, j_t \in I(\mathcal{P})$ by

$$i_t(\mu, \lambda) = t^{c(\lambda/\mu)}$$

and

$$j_t(\mu, \lambda) = \begin{cases} (-1)^{|\lambda/\mu|} t^{c(\lambda/\mu)} (t-1)^{|\lambda/\mu| - c(\lambda/\mu)} & \text{if } \lambda/\mu \text{ is a vertical strip,} \\ 0 & \text{otherwise.} \end{cases}$$

By Proposition 2.3 $(H(t), -) \in \text{Hom}(\Lambda, K)$ corresponds to $i_t \in I(\mathcal{P})$. Since $(H(t), -) * (E(-t), -) = \epsilon$, it suffices to show that $i_t j_t = \delta$ in order to prove that $(E(-t), -)$ corresponds to j_t , whence Proposition 2.6 follows by replacing t with $-t$.

By the definitions of i_t and j_t and (20)

$$(i_t j_t)(\mu, \lambda) = \sum_{\substack{\mu \subset \nu \subset \lambda \\ \lambda/\nu: \text{ vertical strip}}} t^{c(\nu/\mu)} (-1)^{|\lambda/\nu|} t^{c(\lambda/\nu)} (t-1)^{|\lambda/\nu| - c(\lambda/\nu)}. \quad (21)$$

Now it suffices to show that the value of the right-hand side of (21) is $\delta_{\mu\lambda}$, which is not hard.

3 On the Pieri rules for G_λ and g_λ

The (row) Pieri formula for G_λ was given by Lenart [Len00, Theorem 3.2]: for any partition $\lambda \in \mathcal{P}$ and integer $a \geq 0$,

$$G_{(a)} G_\lambda = \sum_{\mu/\lambda: \text{ horizontal strip}} (-1)^{|\mu/\lambda| - a} \binom{r(\mu/\lambda) - 1}{|\mu/\lambda| - a} G_\mu, \quad (22)$$

where $r(\mu/\lambda)$ denotes the number of the rows in the skew shape μ/λ . Namely,

$$c_{(a),\lambda}^{\mu} = (-1)^{|\mu/\lambda|-a} \binom{r(\mu/\lambda)-1}{|\mu/\lambda|-a}.$$

Subsequently, the (row) Pieri formula for g_{λ} is given in [Buc02, Corollary 7.1] (as a formula for $d_{\lambda,(a)}^{\mu}$, the coproduct structure constants for G_{λ}):

$$g_{(a)}g_{\lambda} = \sum_{\mu/\lambda: \text{horizontal strip}} (-1)^{a-|\mu/\lambda|} \binom{r(\lambda/\bar{\mu})}{a-|\mu/\lambda|} g_{\mu}, \quad (23)$$

where $\bar{\mu} = (\mu_2, \mu_3, \dots)$. Namely,

$$d_{(a),\lambda}^{\mu} = (-1)^{a-|\mu/\lambda|} \binom{r(\lambda/\bar{\mu})}{a-|\mu/\lambda|}.$$

Example 3.1. For $\lambda = (2, 1)$ and $a = 2$,

$$\begin{aligned} G_{(2)}G_{\lambda} &= G_{\begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \end{array}} + G_{\begin{array}{|c|c|} \hline \square & \square \\ \hline \end{array}} + G_{\begin{array}{|c|} \hline \square \\ \hline \end{array}} + G_{\begin{array}{|c|c|} \hline \square & \square \\ \hline \end{array}} - G_{\begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \end{array}} - G_{\begin{array}{|c|c|} \hline \square & \square \\ \hline \end{array}} - 2G_{\begin{array}{|c|} \hline \square \\ \hline \end{array}} + G_{\begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \end{array}}, \\ g_{(2)}g_{\lambda} &= g_{\begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \end{array}} + g_{\begin{array}{|c|c|} \hline \square & \square \\ \hline \end{array}} + g_{\begin{array}{|c|} \hline \square \\ \hline \end{array}} + g_{\begin{array}{|c|c|} \hline \square & \square \\ \hline \end{array}} - 2g_{\begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \end{array}} - g_{\begin{array}{|c|c|} \hline \square & \square \\ \hline \end{array}} - g_{\begin{array}{|c|} \hline \square \\ \hline \end{array}} + g_{\begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \end{array}}. \end{aligned}$$

By the example above we can observe

$$\sum_{\nu \subset \mu} c_{(a),\lambda}^{\nu} = 1 \quad (24)$$

for each μ such that μ/λ is a horizontal strip of size $\geq a$, and

$$\sum_{\nu \supset \mu} d_{(a),\lambda}^{\nu} = 1 \quad (25)$$

for each μ such that μ/λ is a horizontal strip of size $\leq a$.

(24) and (25) can be shown through a tour de force argument, which we omit here.

Letting $\tilde{G}_{\kappa} = \sum_{\kappa \subset \eta} G_{\eta}$ and $\tilde{g}_{\kappa} = \sum_{\eta \subset \kappa} g_{\eta}$, we see (24) and (25) are equivalent to

$$\sum_{\mu} c_{(a),\lambda}^{\mu} \tilde{G}_{\mu} = \sum_{\mu \supset \exists (\text{h.s./}\lambda \text{ of size } a)} G_{\mu}, \quad (26)$$

$$\sum_{\mu} d_{(a),\lambda}^{\mu} \tilde{g}_{\mu} = \sum_{\mu \subset \exists (\text{h.s./}\lambda \text{ of size } a)} g_{\mu}. \quad (27)$$

Since $H(1)G_{\lambda} = \tilde{G}_{\lambda}$ and $H(1)^{\perp}(g_{\lambda}) = \tilde{g}_{\lambda}$ (shown in Section 2),

$$\text{since } G_{(a)}G_{\lambda} = \sum_{\mu} c_{(a),\lambda}^{\mu} G_{\mu} \quad \text{we have} \quad G_{(a)}\tilde{G}_{\lambda} = \sum_{\mu} c_{(a),\lambda}^{\mu} \tilde{G}_{\mu}, \quad (28)$$

$$\text{since } g_{(a)}g_{\lambda} = \sum_{\mu} d_{(a),\lambda}^{\mu} g_{\mu} \quad \text{we have} \quad \tilde{g}_{(a)}\tilde{g}_{\lambda} = \sum_{\mu} d_{(a),\lambda}^{\mu} \tilde{g}_{\mu}. \quad (29)$$

Let $\lambda^{(1)}, \lambda^{(2)}, \dots$ be the list of all horizontal strips over λ of size a . Combining (26) and (28), we have

Proposition 3.2. *We have*

$$G_{(a)}\tilde{G}_{\lambda} = \sum_{\mu \supset \lambda^{(i)} \text{ for } \exists i} G_{\mu} \quad (30)$$

$$= \sum_i \tilde{G}_{\lambda^{(i)}} - \sum_{i < j} \tilde{G}_{\lambda^{(i)} \cup \lambda^{(j)}} + \sum_{i < j < k} \tilde{G}_{\lambda^{(i)} \cup \lambda^{(j)} \cup \lambda^{(k)}} - \dots, \quad (31)$$

and

$$G_{(a)}G_\lambda = \sum_i G_{\lambda^{(i)}} - \sum_{i < j} G_{\lambda^{(i)} \cup \lambda^{(j)}} + \sum_{i < j < k} G_{\lambda^{(i)} \cup \lambda^{(j)} \cup \lambda^{(k)}} - \cdots \quad (32)$$

Note that the right-hand sides of (30) and (31) are equal by the Inclusion-Exclusion Principle, and the equivalence of (31) and (32) follows from that $H(1)G_\lambda = \tilde{G}_\lambda$.

Similarly, by (27) and (29) we have

Proposition 3.3. *We have*

$$\tilde{g}_{(a)}\tilde{g}_\lambda = \sum_{\mu \subset \lambda^{(i)} \text{ for } \exists i} g_\mu \quad (33)$$

$$= \sum_i \tilde{g}_{\mu^{(i)}} - \sum_{i < j} \tilde{g}_{\mu^{(i)} \cap \mu^{(j)}} + \sum_{i < j < k} \tilde{g}_{\mu^{(i)} \cap \mu^{(j)} \cap \mu^{(k)}} - \cdots, \quad (34)$$

and

$$g_{(a)}g_\lambda = \sum_i g_{\lambda^{(i)}} - \sum_{i < j} g_{\lambda^{(i)} \cap \lambda^{(j)}} + \sum_{i < j < k} g_{\lambda^{(i)} \cap \lambda^{(j)} \cap \lambda^{(k)}} - \cdots. \quad (35)$$

Similarly, the right-hand sides of (33) and (34) are equal by the Inclusion-Exclusion Principle, and the equivalence of (34) and (35) follows from that $H(1)^\perp: g_\lambda \mapsto \tilde{g}_\lambda$ is an algebra morphism.

References

- [Buc02] Anders Skovsted Buch, *A Littlewood-Richardson rule for the K-theory of Grassmannians*, Acta Math. **189** (2002), no. 1, 37–78. MR1946917
- [FK96] Sergey Fomin and Anatol N. Kirillov, *The Yang-Baxter equation, symmetric functions, and Schubert polynomials*, Proceedings of the 5th Conference on Formal Power Series and Algebraic Combinatorics (Florence, 1993), 1996, pp. 123–143. MR1394950
- [Len00] Cristian Lenart, *Combinatorial aspects of the K-theory of Grassmannians*, Ann. Comb. **4** (2000), no. 1, 67–82. MR1763950
- [LP07] Thomas Lam and Pavlo Pylyavskyy, *Combinatorial Hopf algebras and K-homology of Grassmannians*, Int. Math. Res. Not. IMRN **24** (2007), Art. ID rnm125, 48. MR2377012
- [LS82] Alain Lascoux and Marcel-Paul Schützenberger, *Structure de Hopf de l’anneau de cohomologie et de l’anneau de Grothendieck d’une variété de drapeaux*, C. R. Acad. Sci. Paris Sér. I Math. **295** (1982), no. 11, 629–633. MR686357
- [Mac95] Ian G. Macdonald, *Symmetric functions and Hall polynomials*, Second, Oxford Mathematical Monographs, The Clarendon Press, Oxford University Press, New York, 1995. With contributions by A. Zelevinsky, Oxford Science Publications.
- [Sta12] Richard P. Stanley, *Enumerative combinatorics. Volume 1*, Second, Cambridge Studies in Advanced Mathematics, vol. 49, Cambridge University Press, Cambridge, 2012. MR2868112

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