

BIFURCATION AND SYMMETRY BREAKING FOR BREZIS-NIRENBERG PROBLEM ON $\mathbb{S}^n$

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1. INTRODUCTION

This note presents a brief review of the progressing article [20]. We consider the Brezis-Nirenberg problem on thin annuli in a n -dimensional standard sphere $\mathbb{S}^n$ as an extension of work of Gladiali-Grossi-Pacella-Srikanth [8] that considers the problem on expanding annuli in $\mathbb{R}^n$. We note a similar extension has done by Morabito [15] dealing the problem on expanding annuli in the space which includes a n -dimensional hyperbolic space $\mathbb{H}^n$ as a typical case.

Let $\Delta_{\mathbb{S}^n}$ be a Laplace-Beltrami operator on n -dimensional ($n \geq 2$) sphere $\mathbb{S}^n = \{X = (X_1, \dots, X_n, X_{n+1}) \in \mathbb{R}^{n+1} : |X| = 1\}$. Further, let $p > 1$ and $\Omega_{\theta_1, \theta_2} = \{X \in \mathbb{S}^n : \cos \theta_1 < X_{n+1} < \cos \theta_2\}$, $\theta_1, \theta_2 \in (0, \pi)$ be a thin annulus on $\mathbb{S}^n$. We consider following Brezis-Nirenberg problem on $\Omega_{\theta_1, \theta_2}$:

$$(1) \quad \begin{cases} \Delta_{\mathbb{S}^n} U + \lambda U + U^p = 0, & U > 0 \quad \text{in } \Omega_{\theta_1, \theta_2}, \\ U = 0 & \text{on } \partial\Omega_{\theta_1, \theta_2}, \end{cases}$$

Let λ_1 be the first eigenvalue of $-\Delta_{\mathbb{S}^n}$ on $\Omega_{\theta_1, \theta_2}$ and assume $\lambda < \lambda_1$. Moreover let $P : \mathbb{S}^n \setminus \{(0, \dots, 0, -1)\} \rightarrow \mathbb{R}^n$ be a stereographic projection defined by

$$(2) \quad P(X_1, \dots, X_n, X_{n+1}) = \frac{1}{X_{n+1} + 1} (X_1, \dots, X_n), \quad X \in \mathbb{S}^n \setminus \{(0, \dots, 0, -1)\}.$$

We define $A_{R, \epsilon} = P(\Omega_{\theta_1, \theta_2})$, concretely

$$(3) \quad R - \epsilon = \tan \frac{\theta_2}{2}, R + \epsilon = \tan \frac{\theta_1}{2} \quad \text{and} \quad A_{R, \epsilon} = \{x \in \mathbb{R}^n : R - \epsilon < |x| < R + \epsilon\}.$$

We note on $A_{R,\epsilon}$, Riemannian metric $g = \sum_{i,j} g_{ij} dx_i \otimes dx_j$ is induced, where $g_{ij} = 4(1 + |x|^2)^{-2} \delta_{ij}$. Hence (1) is expressed with this metric as:

$$(4) \quad \begin{cases} \Delta w + \frac{n(n-2) + 4\lambda}{(1 + |x|^2)^2} w + 4(1 + |x|^2)^{\frac{(n-2)p - (n+2)}{2}} w^p = 0, & w > 0 \quad \text{in } A_{R,\epsilon}, \\ w = 0 & \text{on } \partial A_{R,\epsilon}, \end{cases}$$

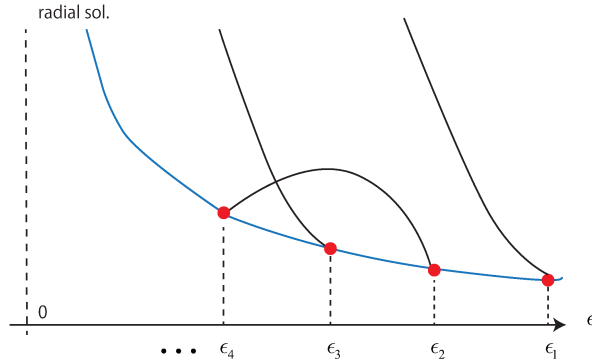
here w satisfies

$$(5) \quad U(P^{-1}x) = (1 + |x|^2)^{\frac{n-2}{2}} w(x) \quad \text{for } x \in \overline{A_{R,\epsilon}}.$$

In the next section, regarding ϵ a parameter of $A_{R,\epsilon}$, we obtain results for the existence of bifurcation solutions of (4) from radially symmetric positive solution.

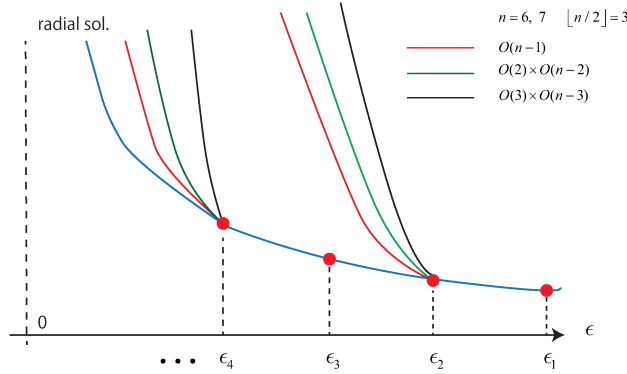
2. MAIN RESULTS

Theorem 1. *Assume $n \geq 2$, $\lambda < \lambda_1$, $p > 1$. Then there exists $\bar{k} \geq 0$, such that for $k \geq \bar{k}$, we have an unique ϵ_k and at $\epsilon = \epsilon_k$, non-radially symmetric positive bifurcation solution from radially symmetric positive solution of (4) exists. Especially, this bifurcation solution has $O(n-1)$ group invariant symmetry. We note ϵ_k satisfies $\lim_{k \rightarrow \infty} \epsilon_k = 0$.*



Under the same assumptions with Theorem 1 except that k is even number, we obtain the multiple existence result of bifurcation solutions.

Theorem 2. *Assume $n \geq 2$, $\lambda < \lambda_1$, $p > 1$. Then there exists $\bar{k} \geq 0$, such that for $k \geq \bar{k}$ and k is even number, we have an unique ϵ_k and at $\epsilon = \epsilon_k$, $\lfloor n/2 \rfloor$ non-radially symmetric positive bifurcation solutions from radially symmetric positive solution of (4) exists. Especially these bifurcation solutions have $O(h) \times O(n-h)$ ($1 \leq h \leq \lfloor n/2 \rfloor$) group invariant symmetry respectively. We note ϵ_k satisfies $\lim_{k \rightarrow \infty} \epsilon_k = 0$.*



For the proof of Theorems 1 and 2, uniqueness of positive radial solution of (4) plays an important role (we rely the detail, how uniqueness of positive radial solution of (4) is used on [20]). Using the uniqueness result and Leray-Schauder degree argument we obtain Theorems 1 and 2.

3. UNIQUENESS OF THE RADIAL POSITIVE SOLUTION OF THE EQUATION (4)

We note positive radial solution of (4) satisfies

$$(6) \quad \begin{cases} w_{rrr} + \frac{n-1}{r} w_r + \frac{n(n-2)+4\lambda}{(1+r^2)^2} w + 4(1+r^2)^{\frac{(n-2)p-(n+2)}{2}} w^p = 0, & w > 0, \quad r \in (R-\epsilon, R+\epsilon) \\ w(R \pm \epsilon) = 0. \end{cases}$$

By applying theorems 13 and 14 of [19], we obtain following theorem.

Theorem A . Let $\lambda_{n,p} = (6 + (6 - 4n)p) / ((p + 3)(p - 1))$ and θ_λ be a unique $\theta \in (0, \pi/2)$ satisfying $G(\tan(\theta/2)) = 0$ where

$$G(r) = Ar^4 + Br^2 + A,$$

and

$$\begin{aligned} A &= (n + 2 - (n - 2)p)((n - 2)p + n - 4) \\ &= (p + 3)[3n^2 - 6n - (n^2 - 4n + 4)p] - 8(n - 1)^2, \\ B &= (p + 3)[-6n^2 + 12n + (2n^2 + 4\lambda - 4)p + 2\lambda p^2 - 6\lambda - 12] + 16(n - 1)^2. \end{aligned}$$

Moreover, assume $\epsilon > 0$ be small enough. Then, the equation (6) has a unique positive solution for $1 < p$ and $\lambda \in (-\infty, \lambda_{1,\epsilon})$ except for the following four cases.

- (i) $n \geq 3$, $1 < p \leq (n + 2)/(n - 2)$, $\lambda \in (-\infty, \lambda_{n,p})$ and $1 \in (R - \epsilon, R + \epsilon)$.
- (ii) $n \geq 3$, $p > (n + 2)/(n - 2)$, $\lambda \in (-\infty, \lambda_{n,p}]$ and $1 \in (R - \epsilon, R + \epsilon)$.
- (iii) $n = 2$, $1 < p$, $\lambda \in (-\infty, -2/(p + 3))$ and $1 \in (R - \epsilon, R + \epsilon)$.

- (iv) $n \geq 3$, $p > (n+2)/(n-2)$, $\lambda \in (\lambda_{n,p}, \lambda_{1,\epsilon})$ and $\tan(\theta_\lambda/2) \in (R-\epsilon, R+\epsilon)$ or $\tan((\pi - \theta_\lambda)/2) \in (R-\epsilon, R+\epsilon)$.

On the other hand, applying Theorem 2.21 and Theorem 2.24 of Ni-Nussbaum [17], we have the following result.

Theorem B . *Let $\lambda \geq -n(n-2)/4$, $n \geq 2$ and*

$$(7) \quad \left(\frac{R+\epsilon}{R-\epsilon} \right) \leq \begin{cases} (n-1)^{\frac{1}{n-2}}, & n \geq 3 \\ e, & n = 2. \end{cases}$$

Then equation (6) has a unique positive solution for $1 < p$ and $\lambda \in (-\infty, \lambda_{1,\epsilon})$.

Remark 1. *Since we assume $\epsilon > 0$ is small, we can remove the assumption (7). Nevertheless, even combining Theorem A and Theorem B, following three cases remain that do not guarantee the uniqueness of the positive solution of (6) (note that in the case $p > (n+2)/(n-2)$, $\lambda_{n,p} > -n(n-2)/4$ holds).*

(I) $n \geq 3$, $1 < p \leq (n+2)/(n-2)$, $\lambda \in (-\infty, \lambda_{n,p})$ and $1 \in (R-\epsilon, R+\epsilon)$.

(II) $n \geq 3$, $p > (n+2)/(n-2)$, $\lambda \in (-\infty, -n(n-2)/4)$ and $1 \in (R-\epsilon, R+\epsilon)$.

(III) $n = 2$, $1 < p$, $\lambda \in (-\infty, -2/(p+3))$ and $1 \in (R-\epsilon, R+\epsilon)$.

We also note that in the case of Brezis-Nirenberg problem on thin annulus with Dirichlet boundary condition in $\mathbb{R}^n$, uniqueness of positive radial solution without any restriction can be obtained through Theorem 7 of [19] and Theorem 2.21 of [17]. This difference with $\mathbb{S}^n$ case motivates the study of progressing article [20].

Assume p satisfies $1 < p$. We consider (6) in somewhat generalized form:

$$(8) \quad \begin{cases} u_{rr} + \frac{f_r(r)}{f(r)} u_r - g(r)u + h(r)u^p = 0, & u > 0, \quad r \in (R', R''), \\ u(R') = 0, \quad u(R'') = 0, \end{cases}$$

where $-\infty < R' < R''$, $f \in C^1([R', R''])$ and f is positive and non-decreasing on (R', R'') , $g \in C^1((R', R'')) \cap C([R', R''])$, $h \in C^1((R', R'')) \cap C([R', R''])$ and h is positive on $[R', R'']$. In the case $R'' = \infty$, $u(R'') = 0$ means $\lim_{r \rightarrow \infty} u(r) = 0$.

We consider the uniqueness of positive solution of (8). We can show the following lemma.

Lemma 1. *Let u_1 and u_2 be solutions of (8) satisfying $u_{1,r}(R') > u_{2,r}(R')$. Then it holds that*

$$(9) \quad \frac{d}{dr} \left(\frac{u_1(r)}{u_2(r)} \right) > 0, \quad r \in (R', R).$$

3.1. An application to equation (6).

First, we introduce an auxiliary function φ as

$$(10) \quad \begin{cases} (r^{n-1}\varphi_r(r))_r + \frac{n(n-2)+4\lambda}{(1+r^2)^2} r^{n-1}\varphi(r) = 0 \text{ in } r \in (R - \epsilon_0, R + \epsilon_0) \\ \varphi(R - \epsilon_0) = 1, \varphi_r(R - \epsilon_0) = 1 \\ \varphi \text{ is monotone increasing on } r \in (R - \epsilon_0, R + \epsilon_0), \end{cases}$$

where ϵ_0 is a small positive number. We note if $\epsilon_0 > 0$ is sufficiently small, φ satisfying the above monotone property clearly exists. Here we put

$$(11) \quad w(r) = \varphi(r)u(r)$$

then (6) with $0 < \epsilon < \epsilon_0$ can be rewritten as

$$(12) \quad \begin{cases} (r^{n-1}\varphi(r)^2 u_r(r))_r + 4r^{n-1} (1 + r^2)^{\frac{(n-2)p-(n+2)}{2}} \varphi(r)^{p+1} u(r)^p = 0, & r \in (R - \epsilon, R + \epsilon), \\ u(r) > 0, & r \in (R - \epsilon, R + \epsilon), \\ u(R \pm \epsilon) = 0. \end{cases}$$

Hence putting $R' = R - \epsilon, R'' = R + \epsilon$ and

$$(13) \quad \begin{cases} f(r) = r^{n-1}\varphi(r)^2 \\ g(r) \equiv 0 \\ h(r) = 4(1 + r^2)^{\frac{(n-2)p-(n+2)}{2}} \varphi(r)^{p-1}, \end{cases}$$

we see that equation (12) takes the form of (8).

Remark 2. We note above f, g, h satisfies the properties assumed in (8). Especially, non-decreasing property of $f(r)$, $r \in (R', R'')$ holds.

Now we introduce Pohožaev function.

Definition 1. For positive solutions u of (8) with f, g, h as (13) and a, b, c of class $C^1[R - \epsilon, R + \epsilon]$ functions, we define Pohožaev function $J(r; u)$ as

$$(14) \quad J(r; u) = \frac{1}{2}a(r)u_r(r)^2 + b(r)u_r(r)u(r) + \frac{1}{2}c(r)u(r)^2 + \frac{1}{p+1}a(r)h(r)u(r)^{p+1}.$$

For such $J(r; u)$, we obtain by direct computation that

$$\frac{d}{dr}J(r; u) = A(r)u_r(r)^2 + B(r)u_r(r)u(r) + G(r)u(r)^2 + H(r)u(r)^{p+1},$$

where

$$(15) \quad \begin{cases} A(r) = \frac{1}{2}a_r(r) - \frac{f_r(r)}{f(r)}a(r) + b(r) \\ B(r) = b_r(r) - \frac{f_r(r)}{f(r)}b(r) + c(r) \\ G(r) = \frac{1}{2}c_r(r) \\ H(r) = -b(r)h(r) + \frac{1}{p+1}(a(r)h(r))_r. \end{cases}$$

Here we define $F_1(r)$ and $F_2(r)$ as

$$(16) \quad F_1(r) = \int_R^r \frac{dt}{f(t)} \quad \text{and} \quad F_2(r) = \int_R^r \frac{F_1(t)}{f(t)} dt,$$

respectively. We put

$$(17) \quad \begin{cases} c(r) &= -1 \\ b(r) &= c_1 f(r) + f(r)F_1(r) \\ a(r) &= c_2 f(r)^2 - 2c_1 f(r)^2 F_1(r) - 2f(r)^2 F_2(r), \end{cases}$$

where c_1 and c_2 are arbitrary real constant. Then we can easily see that $A(r) \equiv B(r) \equiv G(r) \equiv 0$ on $[R - \epsilon, R + \epsilon]$ and hence

$$(18) \quad \frac{dJ(r; u)}{dr} = H(r)u(r)^{p+1}.$$

Now, we fix the constant c_1 and c_2 as

$$(19) \quad \begin{cases} c_1(\epsilon) = \frac{F_2(R - \epsilon) - F_2(R + \epsilon)}{F_1(R + \epsilon) - F_1(R - \epsilon)} \\ c_2(\epsilon) = \frac{2(F_1(R + \epsilon)F_2(R - \epsilon) - F_1(R - \epsilon)F_2(R + \epsilon))}{F_1(R + \epsilon) - F_1(R - \epsilon)}. \end{cases}$$

Then we can see that

$$(20) \quad a(R \pm \epsilon) = 0$$

holds.

Remark 3. In [18, 19], $a(r), b(r)$ and $c(r)$ are taken to satisfy $A(r) \equiv B(r) \equiv H(r) \equiv 0$. Hence, in [18, 19], Pohožaev function satisfies

$$\frac{dJ(r; u)}{dr} = G(r)u(r)^2.$$

Next, we show that $a(r)$ and $b(r)$ are of order $O(\epsilon)$.

Lemma 2. Let $a(r)$ and $b(r)$ be as (17), further $c_1(\epsilon)$ and $c_2(\epsilon)$ be as (19). Then, it holds that

$$|a(r)| \leq C_1 \epsilon, \quad |b(r)| \leq C_2 \epsilon, \quad r \in (R - \epsilon, R + \epsilon)$$

where C_1 and C_2 are positive constants independent of ϵ .

Using this lemma, we can show the monotone property of H .

Lemma 3. *For sufficiently small $\epsilon > 0$, $H(r)$ is monotone decreasing on $(R - \epsilon, R + \epsilon)$ and it holds that $H(R - \epsilon) > 0$, $H(R + \epsilon) < 0$.*

Using these lemmas, we can prove the uniqueness of the positive solution of (12) without assuming (I)-(III) of Remark 1.

Theorem 3. *Let $\epsilon > 0$ be sufficiently small. Then, the equation (6) has a unique positive solution for $1 < p$ and $\lambda \in (-\infty, \lambda_{1,\epsilon})$.*

REFERENCES

- [1] C. Bandle and R. Benguria, *The Brezis-Nirenberg Problem on S^3* , J. Differential Equations **178** (2002), 264–279.
- [2] H. Bandle and L. A. Peletier, *Elliptic Equations with critical exponent on spherical caps of S^3* , Journal D'Analysis mathématique **98** (2006), 279–316.
- [3] T. Bartsch, M. Calpp, M. Grossi, and F. Pacella, *Asymptotically radial solutions in expanding annular domains*, Math. Ann. **352** (2012), no. 2, 485–515.
- [4] M. Bonforte, F. Gazzola, G. Grillo, and J. L. Vázquez, *Classification of radial solutions to the Emden-Fowler equation on the hyperbolic space*, Calc. Var. Partial Differential Equations **46** (2013), no. 1-2, 375–401.
- [5] R. Brown, *A Topological Introduction to Nonlinear Analysis (2nd. Ed.)*, Birkhauser, Boston, 2004.
- [6] C. V. Coffman, *A nonlinear boundary value problem with many positive solutions*, J. Differential Equations **54** (1984), 429–437.
- [7] P. Felmer, S. Martínez, and K. Tanaka, *Uniqueness of radially symmetric positive solutions for $-\Delta u + u = u^p$ in an annulus*, J. Differential Equations **245** (2008), 1198–1209.
- [8] F. Gladiali, M. Grossi, F. Pacella, and P. N. Srikanth, *Bifurcation and symmetry breaking for a class of semilinear elliptic equations in an annulus*, Calc. Var. **40** (2011), 295–317.
- [9] C. Bandle and Y. Kabeya, *On the positive, “radial” solutions of a semilinear elliptic equation in $\mathbb{H}^N$* , Adv. Nonlinear Anal. **1** (2012), no. 1, 1–25.
- [10] Y. Kabeya and K. Tanaka, *Uniqueness of positive radial solutions of semilinear elliptic equations in $\mathbb{R}^N$ and Séré’s non-degeneracy condition*, Comm. Partial Differential Equations **24** (1999), 563–598.
- [11] R. Kajikiya, *Multiple positive solutions of the Emden-Fowler equation in hollow thin symmetric domains*, Calc. Var. **52** (2015), 681–704.
- [12] A. Kosaka, *Emden equation involving the critical Sobolev exponent with the third-kind boundary condition in S^3* , Kodai Math. J. **35** (2012), no. 3, 613–628.
- [13] Y. Y. Li, *Existence of many positive solutions of semilinear elliptic equations on annulus*, J. Differential Equations **83** (1990), 348–367.
- [14] G. Mancini and K. Sandeep, *On a semilinear elliptic equation in $\mathbb{H}^n$* , Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) **7** (2008), no. 4, 635–671.
- [15] F. Morabito, *Radial and non-radial solutions to an elliptic problem on annular domains in Riemannian manifolds with radial symmetry*, Calc. Var. **258** (2015), 1461–1493.
- [16] R. D. Nassbaum, *The fixed point index for local condensing maps*, Ann. Mat. Pura. Appl **89** (1971), 217–258.

- [17] W.-M. Ni and R. D. Nussbaum, *Uniqueness and nonuniqueness for positive radial solutions of $\Delta u + f(u, r) = 0$* , Commun. Pure and Appl. Math. **38** (1985), 67–108.
- [18] N. Shioji and K. Watanabe, *A generalized Pohožaev identity and uniqueness of positive radial solutions of $\delta u + g(r)u + h(r)u^p = 0$* , J. Differential Equations **255** (2013), 4448–4475.
- [19] ———, *Uniqueness and nondegeneracy of positive radial solutions of $\operatorname{div}(\rho \nabla u) + \rho(-gu + hu^p) = 0$* , Calc. Var. **55** (2016), 0–0.
- [20] ———, *Bifurcation and symmetry breaking for Brezis-Nirenberg problem on $\mathbb{S}^n$* , preprint.
- [21] J. Smoller and A. Wasserman, *Bifurcation and symmetry-breaking*, Invent. Math. **100** (1990), 63–95.