

EXISTENCE THEOREMS FOR SADDLE POINTS OF SET-VALUED MAPS VIA NONLINEAR SCALARIZATION METHODS*

(非線形スカラー化手法を用いた集合値写像の鞍点の存在定理)

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Abstract

In the paper, we introduce five types of concepts for saddle points of set-valued maps and show existence theorems for these saddle points by using nonlinear scalarizing functions for sets introduced by Kuwano, Tanaka, and Yamada in 2009.

1 Introduction

Let X and Y be two real topological vector spaces, F a map on $X \times Y$. In real-valued case, $(x_0, y_0) \in X \times Y$ is a saddle point of F if

$$F(x_0, y) \leq F(x_0, y_0) \leq F(x, y_0)$$

for any $x \in X$ and $y \in Y$. In vector-valued case, a saddle point $(x_0, y_0) \in X \times Y$ with respect to partial ordering $\leq_C$ induced by a convex cone C is defined by

$$F(x, y_0) \not\leq_C F(x_0, y_0) \not\leq_C F(x_0, y)$$

for any $x \in X$ and $y \in Y$, and it is called C -saddle point of F . Many researchers have been investigated existence theorems for saddle points and C -saddle points. In [7] and [8], we consider five types of generalizations for C -saddle points and investigate sufficient conditions for the existence of these saddle points by using nonlinear scalarization methods for sets proposed in [4].

The aim of the paper is to introduce three types of existence theorems for cone saddle points of set-valued maps.

The organization of the paper is as follows. In Section 2, we review mathematical methodology proposed in [3] on comparison between two sets in an ordered vector space and some basic concepts of set-valued optimization. In Section 3, we consider two types of nonlinear scalarizing functions for sets proposed by the unified approach in [4], and

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investigate their properties. In Section 4, we introduce five types of concepts for cone saddle points of set-valued maps, and three types of existence theorems for these saddle points proved in [7, 8].

2 Mathematical Preliminaries

Throughout the paper, X and Y are two real topological vector spaces and C is a proper closed convex cone in Y (that is, $C \neq Y$, $C + C = C$ and $\lambda C \subset C$ for all $\lambda \geq 0$) with nonempty topological interior. We define a partial ordering $\leq_C$ on Y as follows:

$$x \leq_C y \quad \text{if} \quad y - x \in C \quad \text{for} \quad x, y \in Y.$$

Let F be a set-valued map from $S \subset X$ into 2^Y where $S := \{x \in X | F(x) \neq \emptyset\}$ and assume that S is a convex set. For $A \in 2^Y \setminus \{\emptyset\}$, we denote the topological interior of A by $\text{int}A$. Also, we denote the algebraic sum, algebraic difference of A and C by $A + C := \bigcup_{a \in A} (a + C)$, $A - C := \bigcup_{a \in A} (a - C)$, respectively. In addition, we denote the composite function of two functions f and g by $g \circ f$. When $x \leq_C y$ for $x, y \in Y$, we define the order interval between x and y by $[x, y] := \{z \in Y | x \leq_C z \text{ and } z \leq_C y\}$.

At first, we review some basic concepts of set-relation.

Definition 2.1. (See Ref. [3].) For any $A, B \in 2^Y \setminus \{\emptyset\}$ and convex cone C in Y , we write

$$\begin{aligned} A &\leq_C^{(1)} B \text{ by } A \subset \bigcap_{b \in B} (b - C), \text{ equivalently } B \subset \bigcap_{a \in A} (a + C), \\ A &\leq_C^{(2)} B \text{ by } A \cap \left(\bigcap_{b \in B} (b - C)\right) \neq \emptyset, \\ A &\leq_C^{(3)} B \text{ by } B \subset (A + C), \\ A &\leq_C^{(4)} B \text{ by } \left(\bigcap_{a \in A} (a + C)\right) \cap B \neq \emptyset, \\ A &\leq_C^{(5)} B \text{ by } A \subset (B - C), \\ A &\leq_C^{(6)} B \text{ by } A \cap (B - C) \neq \emptyset, \text{ equivalently } (A + C) \cap B \neq \emptyset. \end{aligned}$$

Proposition 2.1. (See [3].) For any $A, B \in 2^Y \setminus \{\emptyset\}$, the following statements hold:

$$\begin{aligned} A &\leq_C^{(1)} B \text{ implies } A \leq_C^{(2)} B, & A &\leq_C^{(1)} B \text{ implies } A \leq_C^{(4)} B, \\ A &\leq_C^{(2)} B \text{ implies } A \leq_C^{(3)} B, & A &\leq_C^{(4)} B \text{ implies } A \leq_C^{(5)} B, \\ A &\leq_C^{(3)} B \text{ implies } A \leq_C^{(6)} B, & A &\leq_C^{(5)} B \text{ implies } A \leq_C^{(6)} B. \end{aligned}$$

Proposition 2.2. (See [4].) For any $A, B \in 2^Y \setminus \{\emptyset\}$, the following statements hold:

- (i) For each $j = 1, \dots, 6$,
 $A \leq_C^{(j)} B$ implies $(A + y) \leq_C^{(j)} (B + y)$ for $y \in Y$, and
 $A \leq_C^{(j)} B$ implies $\alpha A \leq_C^{(j)} \alpha B$ for $\alpha \geq 0$.
- (ii) For each $j = 1, \dots, 5$, $\leq_C^{(j)}$ is transitive.
- (iii) For each $j = 3, 5, 6$, $\leq_C^{(j)}$ is reflexive.

From (b) and (c) of Proposition 2.2, $\leq_C^{(6)}$ is difficult to say as order. Hence, we consider mainly the cases of $j = 1, \dots, 5$ in the paper.

By using the set-relations defined in Definition 2.1, we consider the following five kinds

of set-valued optimization problems with $j = 1, \dots, 5$:

$$(j\text{-SVOP}) \begin{cases} j\text{-Optimize} & F(x) \\ \text{Subject to} & x \in S. \end{cases}$$

Then, we introduce some concepts of solutions for $(j\text{-SVOP})$. Let $x_0 \in S$. For each $j = 1, \dots, 5$, x_0 is a *minimal solution* of $(j\text{-SVOP})$ if for any $x \in S \setminus \{x_0\}$,

$$F(x) \leq_C^{(j)} F(x_0) \quad \text{implies} \quad F(x_0) \leq_C^{(j)} F(x); \quad (2.1)$$

and x_0 is a *maximal solution* of $(j\text{-SVOP})$ if for any $x \in S \setminus \{x_0\}$,

$$F(x_0) \leq_C^{(j)} F(x) \quad \text{implies} \quad F(x) \leq_C^{(j)} F(x_0). \quad (2.2)$$

If C is replaced by $\text{int}C$, then x_0 is a *weak minimal solution* (resp., *weak maximal solution*) of $(j\text{-SVOP})$. We denote the family of sets satisfying (2.1) (resp., (2.2)) by $\text{Min}_{(j)} F(S)$ (resp., $\text{Max}_{(j)} F(S)$) and the case of weak minimal (resp., weak maximal) by $\text{WMin}_{(j)} F(S)$ (resp., $\text{WMax}_{(j)} F(S)$) where $F(S) = \{F(x) | x \in S\}$. It is clear that if x_0 is a minimal (resp., maximal) solution of $(j\text{-SVOP})$ then x_0 is a weak minimal (resp., weak maximal) solution of $(j\text{-SVOP})$.

Let us recall some definitions of C -notions (see [2].) A subset A of Y is said to be C -convex (resp., C -closed) if $A + C$ is convex (resp., closed). Moreover, we say that F is C -notion on S if $F(x)$ has the property C -notion for every $x \in S$.

Next, we introduce several definitions of C -convexity and C -continuity for set-valued maps. These notions are used in Sections 3 and 4.

Definition 2.2. (See [4].) For each $j = 1, \dots, 5$,

- (i) F is called a *type (j) naturally quasi C -convex function* if for each $x, y \in S$ and $\lambda \in (0, 1)$, there exists $\mu \in [0, 1]$ such that

$$F(\lambda x + (1 - \lambda)y) \leq_C^{(j)} \mu F(x) + (1 - \mu)F(y).$$

- (ii) F is called a *type (j) naturally quasi C -concave function* if for each $x, y \in S$ and $\lambda \in (0, 1)$, there exists $\mu \in [0, 1]$ such that

$$\mu F(x) + (1 - \mu)F(y) \leq_C^{(j)} F(\lambda x + (1 - \lambda)y).$$

Definition 2.3. (See [8].) For each $j = 1, \dots, 5$,

- (i) F is called a *type (j) C -convexlike function* if for every $x, y \in S$ and $\lambda \in (0, 1)$, there exists $z \in S$ such that

$$F(z) \leq_C^{(j)} \lambda F(x) + (1 - \lambda)F(y).$$

- (ii) F is called a *type (j) C -concavelike function* if for every $x, y \in S$ and $\lambda \in (0, 1)$, there exists $z \in S$ such that

$$\lambda F(x) + (1 - \lambda)F(y) \leq_C^{(j)} F(z).$$

Definition 2.4. (See [2].) Let $x \in S$. Then,

- (i) F is called *C-lower continuous at x* if for every open set V with $F(x) \cap V \neq \emptyset$, there exists an open neighborhood U of x such that $F(y) \cap (V + C) \neq \emptyset$ for all $y \in U$. We shall say that F is *C-lower continuous on S* if it is *C-lower continuous at every point $x \in S$* ,
- (ii) F is called *C-upper continuous at x* if for every open set V with $F(x) \subset V$, there exists an open neighborhood U of x such that $F(y) \subset V + C$ for all $y \in U$. We shall say that F is *C-upper continuous on S* if it is *C-upper continuous at every point $x \in S$* .

3 Unified Types of Scalarizing Functions for Sets

In [4], we propose the following nonlinear scalarizing functions for sets: Let $V, V' \in 2^Y \setminus \{\emptyset\}$, and direction $k \in \text{int}C$. For each $j = 1, \dots, 5$, we define $I_{k,V'}^{(j)} : 2^Y \setminus \{\emptyset\} \rightarrow \mathbb{R} \cup \{\pm\infty\}$ by

$$I_{k,V'}^{(j)}(V) := \inf \left\{ t \in \mathbb{R} \mid V \leq_C^{(j)} (tk + V') \right\}.$$

In this section, we introduce some properties of these functions and several sufficient conditions for the existence of solutions of (j -SVOP).

Proposition 3.1. (See [6].) Let $A, B \in 2^Y \setminus \{\emptyset\}$. Then, the following statements hold:

- (i) If $A \leq_C^{(1)} B$, A is $(-C)$ -closed and B is C -closed then

$$I_{k,V'}^{(1)}(A) < I_{k,V'}^{(1)}(B).$$

- (ii) For each $j = 2, 3$, if $A \leq_{\text{int}C}^{(j)} B$ and B is C -closed then

$$I_{k,V'}^{(j)}(A) < I_{k,V'}^{(j)}(B).$$

- (iii) For each $j = 4, 5$, if $A \leq_{\text{int}C}^{(j)} B$ and A is $(-C)$ -closed then

$$I_{k,V'}^{(j)}(A) < I_{k,V'}^{(j)}(B).$$

Next, we introduce certain inherited properties on cone-convexity and cone-continuity of set-valued maps proved in [4, 5, 8, 10].

Lemma 3.1. (See [4, 5].) Let $k \in \text{int}C$ and $V' \in 2^Y \setminus \{\emptyset\}$. Then, the following statements hold:

- (i) For each $j = 1, 2, 3$, if F is type (j) naturally quasi C -convex, then $I_{k,V'}^{(j)} \circ F$ is quasi convex. Moreover, if F is type (j) naturally quasi C -concave, then $I_{k,V'}^{(j)} \circ F$ is quasi concave.
- (ii) For each $j = 4, 5$, if F is type (j) naturally quasi C -convex and V' is $(-C)$ -convex, then $I_{k,V'}^{(j)} \circ F$ is quasi convex. Moreover, if F is type (j) naturally quasi C -concave and V' is $(-C)$ -convex, then $I_{k,V'}^{(j)} \circ F$ is quasi concave.

Lemma 3.2. (See [8].) Let $k \in \text{int}C$ and $V' \in 2^Y \setminus \{\emptyset\}$. Then, the following statements hold:

- (i) For each $j = 1, 2, 3$, if F is type (j) C -convexlike and V' is C -convex, then $I_{k,V'}^{(j)} \circ F$ is convexlike.
- (ii) For each $j = 4, 5$, if F is type (j) C -convexlike and V' is $(-C)$ -convex, then $I_{k,V'}^{(j)} \circ F$ is convexlike.

Lemma 3.3. (See [8].) Let $k \in \text{int}C$ and $V' \in 2^Y \setminus \{\emptyset\}$. Then, the following statements hold:

- (i) For each $j = 1, 2, 3$, if F is type (j) C -concavelike and V' is C -convex, then $I_{k,V'}^{(j)} \circ F$ is concavelike.
- (ii) For each $j = 4, 5$, if F is type (j) C -concavelike and V' is $(-C)$ -convex, then $I_{k,V'}^{(j)} \circ F$ is concavelike.

Lemma 3.4. (See [10].) Let $k \in \text{int}C$ and $V' \in 2^Y \setminus \{\emptyset\}$. Then, the following statements hold:

- (i) For each $j = 1, 4, 5$, if F is C -lower continuous on S then $I_{k,V'}^{(j)} \circ F$ is lower semicontinuous on S . Moreover, if F is $(-C)$ -upper continuous on S then $I_{k,V'}^{(j)} \circ F$ is upper semicontinuous on S .
- (ii) For each $j = 2, 3$, if F is $(-C)$ -lower continuous on S then $I_{k,V'}^{(j)} \circ F$ is upper semicontinuous on S . Moreover, if F is C -upper continuous on S then $I_{k,V'}^{(j)} \circ F$ is lower semicontinuous on S .

Let $V' \in 2^Y \setminus \{\emptyset\}$ and direction $k \in \text{int}C$. To show sufficient conditions for the existence of solutions of (j-SVOP) by using properties of $I_{k,V'}^{(j)}$, we consider the following two kinds of scalar optimization problems:

$$\inf_{x \in S} (I_{k,V'}^{(j)} \circ F)(x) \quad \text{and} \quad \sup_{x \in S} (I_{k,V'}^{(j)} \circ F)(x).$$

Lemma 3.5. (See [7].) Assume that F is C -closed on S and $x_0 \in S$. Let $k \in \text{int}C$. For each $j = 1, 2, 3$, the following statements hold:

- (i) If x_0 is a solution of $\inf_{x \in S} (I_{k,V'}^{(j)} \circ F)(x)$, then x_0 is a weak minimal solution of (j-SVOP).
- (ii) If x_0 is a solution of $\sup_{x \in S} (I_{k,V'}^{(j)} \circ F)(x)$, then x_0 is a weak maximal solution of (j-SVOP).

Lemma 3.6. (See [7].) Assume that F is $(-C)$ -closed on S and $x_0 \in S$. Let $k \in \text{int}C$. For each $j = 4, 5$, the following statements hold:

- (i) If x_0 is a solution of $\inf_{x \in S} (I_{k,V'}^{(j)} \circ F)(x)$, then x_0 is a weak minimal solution of (j-SVOP).
- (ii) If x_0 is a solution of $\sup_{x \in S} (I_{k,V'}^{(j)} \circ F)(x)$, then x_0 is a weak maximal solution of (j-SVOP).

4 Existence Theorems for Saddle Points of Set-Valued Maps

At first, we introduce definitions of saddle points for set-valued maps proposed in [8]. For each $j = 1, \dots, 5$, if $(x_0, y_0) \in X \times Y$ satisfies the following properties:

- (i) $F(x, y_0) \leq_C^{(j)} F(x_0, y_0)$ implies $F(x_0, y_0) \leq_C^{(j)} F(x, y_0)$,
- (ii) $F(x_0, y_0) \leq_C^{(j)} F(x_0, y)$ implies $F(x_0, y) \leq_C^{(j)} F(x_0, y_0)$,

for any $x \in X$ and $y \in Y$, then we call it type (j) C -saddle point of F . It is equivalent to

$$F(x_0, y_0) \in \{\text{Min}_{(j)} F(X, y_0)\} \cap \{\text{Max}_{(j)} F(x_0, Y)\}.$$

If C is replaced by $\text{int}C$ then we call it type (j) weak C -saddle point of F .

In this section, we give three types of existence theorems for type (j) cone saddle points of set-valued maps. At first, we introduce the first existence theorems which are natural extensions of Sion's minimax theorem (see [9]).

Theorem 4.1. (See [7].) *Let X and Y be nonempty compact convex subsets of two real topological vector spaces, respectively, Z a real topological vector space with the partial ordering $\leq_C$, $k \in \text{int}C$, V' a nonempty subset of Z and $F : X \times Y \rightarrow 2^Z \setminus \{\emptyset\}$. Assume that F is C -closed and $(-C)$ -closed on $X \times Y$. If F satisfies the following conditions:*

- (i) $x \rightarrow F(x, y)$ is C -lower continuous and type (1) naturally quasi C -convex on X for every $y \in Y$,
- (ii) $x \rightarrow F(x, y)$ is $(-C)$ -upper continuous and type (1) naturally quasi C -concave on Y for every $x \in X$,

then F has at least one type (1)-weak saddle point.

Theorem 4.2. (See [7].) *Let X and Y be nonempty compact convex subsets of two real topological vector spaces, respectively, Z a real topological vector space with the partial ordering $\leq_C$, $k \in \text{int}C$, V' a nonempty subset of Z and $F : X \times Y \rightarrow 2^Z \setminus \{\emptyset\}$. Assume that F is C -closed on $X \times Y$. For each $j = 2, 3$, if F satisfies that*

- (i) $x \rightarrow F(x, y)$ is C -upper continuous and type (j) naturally quasi C -convex on X for every $y \in Y$,
- (ii) $x \rightarrow F(x, y)$ is $(-C)$ -lower continuous and type (j) naturally quasi C -concave on Y for every $x \in X$,

then F has at least one type (j) -weak saddle point.

Theorem 4.3. (See [7].) *Let X and Y be nonempty compact convex subsets of two real topological vector spaces, respectively, Z a real topological vector space with the partial ordering $\leq_C$, $k \in \text{int}C$, V' a nonempty subset of Z and $F : X \times Y \rightarrow 2^Z \setminus \{\emptyset\}$. Assume that F is $(-C)$ -closed on $X \times Y$ and V' is $(-C)$ -convex. For each $j = 4, 5$, if F satisfies that*

- (i) $x \rightarrow F(x, y)$ is C -lower continuous and type (j) naturally quasi C -convex on X for every $y \in Y$,
- (ii) $x \rightarrow F(x, y)$ is $(-C)$ -upper continuous and type (j) naturally quasi C -concave on Y for every $x \in X$,

then F has at least one type (j) -weak saddle point.

Next, we introduce the second existence theorems which are natural extensions of Fan type minimax theorem (see [1]).

Theorem 4.4. (See [8].) *Let X be a nonempty compact subset of real topological space, Y any space, Z a real topological vector space with the partial ordering $\leq_C$, $k \in \text{int}C$, V' a nonempty subset of Z and $F : X \times Y \rightarrow 2^Z \setminus \{\emptyset\}$. Assume that F is C -closed and $(-C)$ -closed on $X \times Y$. If F satisfies that*

- (i) $x \rightarrow F(x, y)$ is type (1) C -convexlike on X for every $y \in Y$,
- (ii) $x \rightarrow F(x, y)$ is $(-C)$ -upper continuous and type (1) C -concavelike on Y for every $x \in X$,

then F has at least one type (1)-weak saddle point.

Theorem 4.5. (See [8].) *Let X be a nonempty compact subset of real topological space, Y any space, Z a real topological vector space with the partial ordering $\leq_C$, $k \in \text{int}C$, V' a nonempty subset of Z and $F : X \times Y \rightarrow 2^Z \setminus \{\emptyset\}$. Assume that F is C -closed on $X \times Y$. For each $j = 2, 3$, if F satisfies that*

- (i) $x \rightarrow F(x, y)$ is type (j) C -convexlike on X for every $y \in Y$,
- (ii) $x \rightarrow F(x, y)$ is $(-C)$ -lower continuous and type (j) C -concavelike on Y for every $x \in X$,

then F has at least one type (j) -weak saddle point.

Theorem 4.6. (See [8].) *Let X be a nonempty compact subset of real topological space, Y any space, Z a real topological vector space with the partial ordering $\leq_C$, $k \in \text{int}C$, V' a nonempty subset of Z and $F : X \times Y \rightarrow 2^Z \setminus \{\emptyset\}$. Assume that F is $(-C)$ -closed on $X \times Y$. For each $j = 4, 5$, if F satisfies that*

- (i) $x \rightarrow F(x, y)$ is type (j) C -convexlike on X for every $y \in Y$,
- (ii) $x \rightarrow F(x, y)$ is $(-C)$ -upper continuous and type (j) C -concavelike on Y for every $x \in X$,

then F has at least one type (j) -weak saddle point.

Finally, we give the third existence theorems for type (j) cone saddle points of set-valued maps with separated form.

Theorem 4.7. (See [7].) *Let X and Y be nonempty compact subsets of two real topological spaces, respectively, Z a real ordered topological vector space with the partial ordering $\leq_C$, $k \in \text{int}C$, V' a nonempty subset of Z and $F : X \times Y \rightarrow 2^Z \setminus \{\emptyset\}$. If F satisfies that*

- (i) $F(x, y) := G_1(x) \cup G_2(y)$,
- (ii) G_1 is C -closed and C -lower continuous on X ,
- (iii) G_2 is $(-C)$ -closed and $(-C)$ -upper continuous on Y ,

where $G_1 : X \rightarrow 2^Z \setminus \{\emptyset\}$ and $G_2 : Y \rightarrow 2^Z \setminus \{\emptyset\}$, then F has at least one type (1) C -saddle point.

Theorem 4.8. (See [7].) *Let X and Y be nonempty compact subsets of two real topological spaces, respectively, Z a real topological vector space with the partial ordering $\leq_C$, $k \in \text{int}C$, V' a nonempty subset of Z and $F : X \times Y \rightarrow 2^Z \setminus \{\emptyset\}$. For each $j = 2, 3$, if F satisfies that*

- (i) $F(x, y) := G_1(x) \cup G_2(y)$,
- (ii) G_1 is C -closed and C -upper continuous on X ,
- (iii) G_2 is C -closed and $(-C)$ -lower continuous on Y ,

where $G_1 : X \rightarrow 2^Z \setminus \{\emptyset\}$ and $G_2 : Y \rightarrow 2^Z \setminus \{\emptyset\}$, then F has at least one type (j) weak C -saddle point.

Theorem 4.9. (See [7].) Let X and Y be nonempty compact subsets of two real topological spaces, respectively, Z a real topological vector space with the partial ordering $\leq_C$, $k \in \text{int}C$, V' a nonempty subset of Z and $F : X \times Y \rightarrow 2^Z \setminus \{\emptyset\}$. For each $j = 4, 5$, if F satisfies that

- (i) $F(x, y) := G_1(x) \cup G_2(y)$,
- (ii) G_1 is $(-C)$ -closed and C -lower continuous on X ,
- (iii) G_2 is $(-C)$ -closed and $(-C)$ -upper continuous on Y ,

where $G_1 : X \rightarrow 2^Z \setminus \{\emptyset\}$ and $G_2 : Y \rightarrow 2^Z \setminus \{\emptyset\}$, then F has at least one type (j) weak C -saddle point.

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