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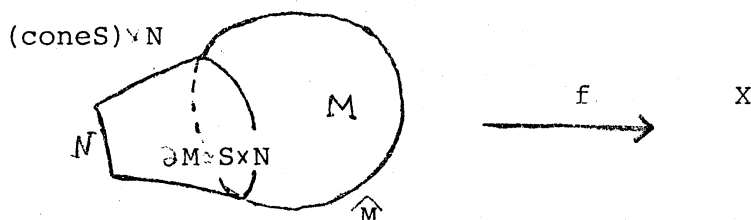
On relations between the Brown-Peterson cohomology theory and the ordinary mod p cohomology theory

Nobuaki Yagita

Musashi Institute of Technology

§1. Introduction. We recall the bordism theory with singularities. Let S be a close (weakly complex) manifold. An S -manifold (manifold with singularities of type S ,) means

$\hat{M} = M \cup (\text{cone } S) \times N$ where M is an open manifold with $\partial M \cong S \times N$ and N is a close manifold.



The bordism group $MU(S)_*(X)$ is a group of bordism classes of maps $f : M \rightarrow X$.

Define a bordism operation Q_S by

$$Q_S[\hat{M}, f] = [N, f|_N].$$

We will show that this operation exhibits relationship between $BP_*(X)$ and $H_*(X; \mathbb{Z}_p)$.

§2. $BP(S)_*(X)$. A main reference of this section is [1][6]

Let $MU_* \cong \mathbb{Z}[x_1, \dots]$ and let $x_{p^i-1} = v_i$ (, where we take v_i as a Milnor manifold, i.e., $c_{4p^i-1}(v_i) = p \mod p^2$). For each sequence $S = (p_1, \dots)$, p_i ; close manifold, Sullivan [5] also defined $MU(S)_*(X)$. The most important tool of this theory is the following Sullivan exact sequence.

$$\begin{array}{ccc}
 MU(S)_*(X) & \xrightarrow{P} & MU(S,P)_*(X) \\
 \nwarrow \delta & & \nwarrow i \\
 & MU(S,P)_*(X) &
 \end{array}$$

From this exact sequence, if S is regular, then

$$MU(S)_*(S^0) \cong MU^*/(S) = MU / \text{ideal}(P_1, \dots).$$

Hence $MU(\dots, x_i, \dots i=p^j-1)_*(X)_{(p)} \cong BP_*(X)$.

By Quillen's splitting theorem, we can prove

$$MU(p, v_{i_1}, \dots)_*(X) \cong MU_{(p)}^* \otimes_{BP_*} BP(p, v_{i_1}, \dots)_*(X).$$

In particular,

$$\begin{aligned}
 MU(p, v_1, v_2, \dots)_*(X) &\cong MU_{(p)}^* \otimes_{BP_*} BP(p, v_1, v_2, \dots)_*(X) \\
 &\cong MU_{(p)}^* \otimes_{BP_*} H_*^*(X; Z_p) \quad (\text{since } BP(p, v_1, \dots) = Z_p).
 \end{aligned}$$

This fact shows that $H_*(X; Z_p)$ is essentially the bordism theory of type $(p, v_1, v_2, \dots)$. (Of course, $H_*(X; Z_p)$ is the bordism theory of type $(p, x_1, x_2, \dots)$ but the above fact shows singularities of x_i , $i \neq p^j-1$ do not appear.)

§3. Milnor operations. Taking Spanier-Whitehead dual, the bordism operation induces cohomology operation. (Quillen's geometric picture arguments [4], also induce cohomology operation.)

Recall the Milnor operation Q_i , namely, $Q_0 =$ the Bockstein operation and $Q_{i+1} = \phi^p Q_i - Q_i \phi^p$.

Theorem 1. [6] In $H^*(X; Z_p) = BP(p, v_1, \dots)^*(X)$, the cobordism operation induced from a Milnor manifold is the Milnor operation, i.e., $Q_{v_i} = Q_i$.

Proof. Since Q_{v_i} is defined from taking the boundaries, Q_{v_i} is a derivation. Recall that the product of Lens spaces

$L_p^m \times \dots \times L_p^m$ is a retract of the Eilenberg MacLane spectrum KZ_p .

Hence we have only to prove $Q_{v_i} = Q_i$ in $H^*(L_p; \mathbb{Z}_p)$.

It is well known

$$H^*(L_p; \mathbb{Z}_p) \simeq \mathbb{Z}_p[x]/(x^m) \otimes \Delta(\alpha), \quad Q_i \alpha = x^{p^i} \text{ and } Q_i x = 0.$$

By the Gysin sequence, it is also well known

$$BP^*(L_p) \simeq BP^*[x]/([p], (x^m))$$

$$\text{where } [p] = c_1(\xi_1 \otimes \dots \otimes \xi_1) = px + a_1 x^2 + \dots = px + v_1 x^{p-1} + \dots,$$

$$a_{p^i-1} = v_i \pmod{(p, \dots, v_{i-1})}.$$

In $BP^*(L_p)$, that $[p]=0$ means there is a manifold Y such that

$$\begin{aligned} \partial Y &= px \vee a_1 x^2 \vee \dots \vee a_{p^i-1} x^{p^i} \vee \dots \\ &= px \vee (v_1 x^{p-1} \vee v_1 w_1) \vee (v_2 x^{p-2} \vee v_2 w_2) \vee \dots \end{aligned}$$

where $w_i \in BP^+ \cdot BP^*(L_p)$.

Attach the cones.

$$\hat{Y} = Y \cup_i \text{cone}_{v_i} x (x^{p^i} \vee w_i).$$

Then we have

$$Q_{v_i} \hat{Y} = x^{p^i} \vee w_i.$$

In $H^*(X; \mathbb{Z}_p)$, $BP^+ BP^*(L_p) = 0$ and hence $Q_{v_i} \hat{Y} = x^{p^i}$. In particular, $Q_{p\text{-points}} \hat{Y} = x$. It is immediate seen that $Q_{p\text{-points}} =$ the Bockstein Q_0 . Since there is only one α such that $Q_0 \alpha = x$, we have $\hat{Y} = \alpha$. Hence $Q_{v_i} \alpha = Q_{v_i} \hat{Y} = x^{p^i} = Q_i \alpha$.

Since x is a closed manifold and x has no singularities, $Q_i x = 0$.

Hence we have the theorem.

q.e.d.

§4. Relations $BP^*(X)$ to $H^*(X; \mathbb{Z}_p)$.

Theorem 2. If $pb_0 + v_1 b_1 + \dots + v_n b_n = 0$ in $BP^*(X)$, then there is $y \in H^*(X; \mathbb{Z}_p)$, such that $Q_j(y) = i(b_j)$ where $i : BP \rightarrow K\mathbb{Z}_p$ is the natural inclusion map (Thom map).

Proof. Because there is Y such that

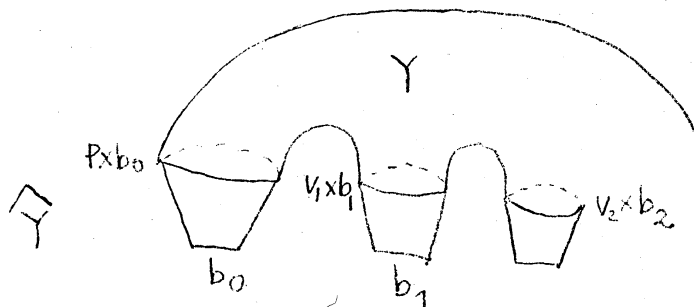
$$\partial Y = p \times b_0 \vee v_1 \times b_1 \vee \dots \vee v_n \times b_n.$$

Construct $\hat{Y}$ as in the proof

of Theorem 1. Then

$$Q_{v_i} \hat{Y} = b_i.$$

q.e.d.



Example 1. Let X be a finite H -space. By [2], there is a system of even dimensional indecomposable elements in $H(X; \mathbb{Z}_p)$

$$(y_k, \dots, y_1), |y_i| = (p^{k+1} - 1) / (p - 1) - p^i, \quad (*)$$

such that there is $i_{\ell, \ell'}$ with $y_{\ell} = Q_{\ell'} i_{\ell, \ell'} (1 \leq \ell \neq \ell' \leq k)$.

Conjecture 1. If there is a finite H -space of type $(*)$, then there are y_{ℓ}' such that $i(y_{\ell}') = y_{\ell}$ and

$$v_h y_{\ell}' + v_{\ell} y_h' = 0 \quad \text{in } BP^*(X).$$

This conjecture is true for Lie groups, for example, we have

$$(1) \quad H^*(F_4; \mathbb{Z}_3) \simeq \mathbb{Z}_3[x_8] / (x_8^3) \otimes \Lambda(x_3, \dots), Q_1 x_3 = 0.$$

$$\text{In } BP^*(F_4), v_1 x_8' = 0.$$

$$(2) \quad H^*(E_8; \mathbb{Z}_3) \simeq \mathbb{Z}_3[x_8, x_{20}] / (x_8^3, x_{20}^3) \otimes \Lambda(x_3, x_7, x_{15}, \dots).$$

$$\text{In } BP^*(E_8), v_1 x_8' + v_2 x_{20}' = 0, v_1 x_{20}' = 0.$$

Example 2. Let K be the Eilenberg MacLane space $K(\mathbb{Z}, 3)$.

The mod p cohomology ring is

$$H^*(K; \mathbb{Z}_p) \simeq \mathbb{Z}_p[\delta \otimes \tau, \dots] \otimes \Lambda(\tau, \tau^p, \tau^{p^2}, \dots)$$

For simplicity of notations, let denote $\mathbb{P}^{p^n-1} \otimes \tau = c_n$, $\delta c_n = b_n$.

Then $Q_m \tau = b_m$, $Q_m c_n = (b_{n-m})^{p^m}$ for $n > m \geq 0$, $Q_m c_n = (b_{m-n})^{p^n}$ for $m > n \geq 0$,

$$Q_m c_m = 0 \text{ and } Q_m b_n = 0.$$

Conjecture 2. There are b_j' in $BP^*(K)$ such that $i(b_j') = b_j$ and

$$v_1 b_1' + v_2 b_2' + \dots = 0.$$

Proposition 1. In $BP^*(K^{2p^4+2p^3}; Z_p) = P(1)^*(K^{2p^4+2p^3})$, there are b'_j for $1 \leq j \leq 4$ such that $i(b'_j) = b_j$ and $\text{mod } (p, v_1, \dots)^2$

$$v_1 b'_1 + \dots + v_4 b'_4 = 0$$

$$v_1 b'_3 p + v_2 b'_2 p^2 + v_3 b'_1 p^3 = 0, \quad v_1 b'_2 p + v_2 b'_1 p^2 = 0, \quad v_1 b'_1 p^3 = 0.$$

Proof. We will prove only the first relation. We notice that $|b_i| = 2(p^n+1)$ and $|c_i| = 2(p^n+1)-1$. The degrees of the differentials of Atiyah-Hirzebruch spectral sequence $P(1)^{E_r}$ which converges to $P(1)^*(X)$ are $4m-1$. Hence we can prove $b_1, \dots, b_4$ are permanent in $P(1)^{E_r}$. Since $d_{2p-1} = v_1 b_1$, we have $v_1 b_1 = 0$ in $P(1)^E$. This means $v_1 b_1 \in F^s$ (the associated filtration), $s > |b_1|$. But $4m$ -dimensional elements are generated, as an $P(1)^*$ -module, by $b_1^{\otimes 1} \dots b_4^{\otimes 4}$.

Therefore there is a relation

$$v_1 b_1 + \dots = 0 \text{ in } P(1)^*(K^{2p^4+2p^3}).$$

The fact that there is only one γ with $Q_1 \gamma = b_1$ implies $i(b_j) = Q_j \gamma$.

q.e.d.

§5. Relations between A-H spectral sequence and the Sullivan exact sequence.

Lemma. Let $wx=0$ in $P(1)^*(X) = BP^*(X; Z_p) = BP(p)^*(X)$ for $0 \neq w \in P(1)^*$, and let $i(x) \neq 0$ in $H^*(X; Z_p)$. From Sullivan exact sequence there is $y \in BP(p, w)^*(X)$. Then in A-H spectral sequence $P(1)^{E_r}$,

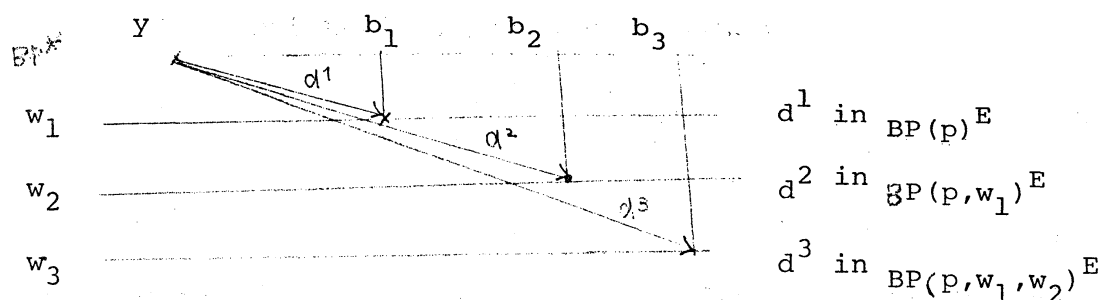
$$d_r(y') = \lambda wx,$$

where $\lambda \neq 0 \in Z_p$, $y' \in P(1)^{E_r^{*,*}}$ corresponds to y .

Theorem 3. Let $(w_1, \dots, w_s) = J_s |w_i| > |w_{i+1}|$, regular sequence in $P(1)^* = BP^*/p$. Let $b_j \in P(1)^*$, $0 \neq i(b_j)$ in $H^*(X; Z_p)$. Suppose $w_1 b_1 + \dots + w_s b_s = 0$ in $P(1)^*(X)$. Then there is $y \in P(1)^{E_2^{*,*}}$ such that

$$d_{r_t}(y) = \lambda_t w_t i(b_t) \quad \text{in } BP(p, J_{t-1})^{E_{r_t}^{*,*}}, \quad 0 < t \leq s$$

where $\lambda_t \neq 0$ in Z_p .



When we study relations in $BP^*(X)$ with decomposable elements of BP^* , Theorem 3 is useful. For example, there is a relation in $BP(p)^*(K^{2p^4+2p^3})$

$$v_1^p b_2 + v_2 b_1^p + v_3 b_2^p + v_4 b_3^p = 0 \mod (p, v_1, v_2, \dots)^2 - \{v_1^2\}.$$

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