

CHARACTERIZATIONS OF NORMAL COVERS ON RECTANGULAR PRODUCTS AND INFINITE PRODUCTS

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1. INTRODUCTION

An open cover $\mathcal{O}$ of a topological space X is *normal* if there is a sequence $\{\mathcal{U}_n\}$ of open covers of X such that each $\mathcal{U}_n$ is a star-refinement of $\mathcal{U}_{n-1}$ for each $n \in \omega$, where $\mathcal{U}_0 = \mathcal{O}$.

We may well know the following characterization of normal covers of topological spaces. For example, it is seen in [AS, p.122], [Ho, Theorems 1.2 and 1.4] and [Mo, Theorem 1.2] etc.

Theorem 1.1 [Stone-Michael-Morita]. *Let X be a topological space and $\mathcal{O}$ an open cover of X . Then the following are equivalent.*

- (a) $\mathcal{O}$ is normal.
- (b) $\mathcal{O}$ has a σ -locally finite cozero refinement.
- (c) $\mathcal{O}$ has a σ -discrete cozero refinement.
- (d) $\mathcal{O}$ has a locally finite cozero refinement.
- (e) $\mathcal{O}$ has a locally finite, σ -discrete, cozero refinement which has a shrinking consisting of zero-sets.

Now, we recall that a product space $X \times Y$ is said to be *rectangular* if every finite cozero cover of $X \times Y$ has a σ -locally finite refinement consisting of cozero rectangles. This concept was introduced by Pasynkov [Pa] in dimension theory.

The following is easily seen by the definition (see [HM, Lemma 1]).

Fact 1.2. A product space $X \times Y$ is rectangular if and only if every normal cover of $X \times Y$ has a σ -locally finite refinement consisting of cozero rectangles.

In this report, for normal covers of rectangular products, we give some characterizations analogous to Stone-Michael-Morita's Theorem above in terms of refinements consisting of cozero rectangles. Next, we can apply these characterizations to the strong rectangularity and the base-paracompactness of rectangular products as well as in [Y2]. Finally, we also give the same kind of characterization of normal covers on infinite products of metrizable spaces.

These results for normal covers of rectangular products are included in the paper [Y3] with their complete proofs. But the paper does not refer to normal covers of infinite products stated in the last section.

Throughout this paper, all spaces are *topological spaces* without any separation axiom. However, paracompact spaces are assumed to be *Hausdorff*.

Let X be a space and $\mathcal{U}$ a cover of X . A cover $\mathcal{V}$ of X is called a *refinement* of $\mathcal{U}$ if each member of $\mathcal{V}$ is contained in some member of $\mathcal{U}$. A cover $\{W_U : U \in \mathcal{U}\}$ of X is called a *shrinking* of $\mathcal{U}$ if $\overline{W_U} \subset U$ for each $U \in \mathcal{U}$.

Let $X \times Y$ be a product space. A subset of the form $A \times B$ in $X \times Y$ is called a *rectangle*. For a subset R in $X \times Y$, R' and R'' denote the projections of R into X and Y , respectively. A rectangle $R = R' \times R''$ is called a *cozero rectangle* (*zero rectangle*) if R' and R'' are cozero-sets (zero-sets) in X and Y , respectively.

For the sake of convenience, a cover $\mathcal{G}$ of a product space $X \times Y$ is said to be *cozero rectangular* (*resp.*, *zero rectangular*, *rectangular*) if each member of $\mathcal{G}$ is a cozero rectangle (*resp.*, *zero rectangle*, *rectangle*) in $X \times Y$.

2. X -RECTANGULAR PRODUCTS

A product space $X \times Y$ is said to be *X -rectangular* [Oh] if every finite cozero cover $\mathcal{O}$ of $X \times Y$ has a cozero rectangular refinement $\mathcal{G}$ such that $\pi_X(\mathcal{G}) = \{G' : G \in \mathcal{G}\}$ is σ -locally finite in X .

Lemma 2.1. *A product space $X \times Y$ is X -rectangular if and only if every finite cozero cover $\mathcal{O}$ of $X \times Y$ has a cozero rectangular refinement $\mathcal{G}$ such that $\pi_X(\mathcal{G}) = \{G' : G \in \mathcal{G}\}$ is σ -discrete in X .*

Remark. X -rectangular products were originally defined for Tychonoff products in [Oh]. In the case of a Tychonoff product $X \times Y$, the proof of Lemma 3.1 can be obtained by a modification of that of [Oh, Theorem 2.2]. However, the assumption that Y is Tychonoff is necessary in his proof, because the Stone-Ćech compactification βY of Y has to be used there.

Theorem 2.2. *Let $X \times Y$ be an X -rectangular product and $\mathcal{O}$ an open cover of $X \times Y$. Then the following are equivalent.*

- (a) $\mathcal{O}$ is normal.
- (b) $\mathcal{O}$ has a σ -locally finite cozero rectangular refinement.
- (c) $\mathcal{O}$ has a σ -discrete cozero rectangular refinement.
- (d) $\mathcal{O}$ has a locally finite cozero rectangular refinement.
- (e) $\mathcal{O}$ has a locally finite, σ -discrete, cozero rectangular refinement which has a zero rectangular shrinking.

It is pointed out in the proof of [Ta, Theorem 1] that if X is a metric space, then the rectangularity of $X \times Y$ implies the X -rectangularity. So we have

Corollary 2.3. *Let X be a metric space. Let $X \times Y$ be a rectangular product and $\mathcal{O}$ an open cover of $X \times Y$. Then the following are equivalent.*

- (a) $\mathcal{O}$ is normal.
- (b) $\mathcal{O}$ has a σ -locally finite cozero rectangular refinement.
- (c) $\mathcal{O}$ has a σ -discrete cozero rectangular refinement.
- (d) $\mathcal{O}$ has a locally finite cozero rectangular refinement.
- (e) $\mathcal{O}$ has a locally finite, σ -discrete, cozero rectangular refinement which has a zero rectangular shrinking.

Remark. If a product space $X \times Y$ is not rectangular, there is a normal cover of $X \times Y$ which has no σ -locally finite cozero rectangular refinement (see Fact 1.2). In fact, there is a non-rectangular product with a metric factor (see [Pr], [Ta]). So we cannot exclude the assumption of rectangularity of $X \times Y$ in Corollary 2.3.

3. PRODUCTS WITH A σ -SPACE FACTOR

Recall that a regular T_1 -space X is a σ -space if there is a σ -discrete (closed) net of X .

Theorem 3.1. *Let X be a paracompact σ -space and Y a space. Let $\mathcal{O}$ be a normal cover of $X \times Y$. Then the following are equivalent.*

- (a) $\mathcal{O}$ has a σ -locally finite cozero rectangular refinement.
- (b) $\mathcal{O}$ has a σ -discrete cozero rectangular refinement.
- (c) $\mathcal{O}$ has a locally finite cozero rectangular refinement.
- (d) $\mathcal{O}$ has a locally finite, σ -discrete, cozero rectangular refinement which has a zero rectangular shrinking.

Theorem 3.1 immediately yields the following extension of Corollary 2.3.

Corollary 3.2. *Let X be a paracompact σ -space. Let $X \times Y$ be a rectangular product and $\mathcal{O}$ an open cover of $X \times Y$. Then the following are equivalent.*

- (a) $\mathcal{O}$ is normal.
- (b) $\mathcal{O}$ has a σ -locally finite cozero rectangular refinement.
- (c) $\mathcal{O}$ has a σ -discrete cozero rectangular refinement.
- (d) $\mathcal{O}$ has a locally finite cozero rectangular refinement.
- (e) $\mathcal{O}$ has a locally finite, σ -discrete, cozero rectangular refinement which has a zero rectangular shrinking.

Theorem 3.3. *If X is a paracompact Σ -space and Y is a paracompact P -space, then every open cover of $X \times Y$ has a locally finite, σ -discrete, cozero rectangular refinement which has a zero rectangular shrinking.*

Comparing Theorems 3.1 and 3.3, it is natural to raise the following question.

Question. Can “ σ -space” be replaced by “ Σ -space” in Theorem 3.1?

4. PRODUCTS WITH A FACTOR DEFINED BY TOPOLOGICAL GAMES

Telgársky [Te] introduced the topological game $G(\mathbf{DC}, X)$, where $\mathbf{DC}$ denotes the class of all spaces which have a discrete cover consisting of compact sets.

According to [GT], a function s from the family of all closed sets in X to itself is called a *winning strategy for Player I* in $G(\mathbf{DC}, X)$ if it satisfies

- (a) $s(F) \in \mathbf{DC}$ and $s(F) \subset F$ for each closed set F in X ,
- (b) if $\{F_n\}$ is a decreasing sequence of closed sets in X such that $s(F_n) \cap F_{n+1} = \emptyset$ for each $n \in \omega$, then $\bigcap_{n \in \omega} F_n = \emptyset$.

A space X is said to be **DC-like** if there is a winning strategy for Player I in $G(\mathbf{DC}, X)$.

Theorem 4.1. *Let X be a paracompact DC-like space and Y a space. Let $\mathcal{O}$ be a normal cover of $X \times Y$. Then the following are equivalent.*

- (a) $\mathcal{O}$ has a σ -locally finite cozero rectangular refinement.
- (b) $\mathcal{O}$ has a σ -discrete cozero rectangular refinement.
- (c) $\mathcal{O}$ has a locally finite cozero rectangular refinement.
- (d) $\mathcal{O}$ has a locally finite, σ -discrete, cozero rectangular refinement which has a zero rectangular shrinking.

If a Hausdorff space X is subparacompact and C -scattered or it has a σ -closure-preserving cover consisting of compact sets, then Player I has a winning strategy for $G(\text{DC}, X)$, that is, X is DC-like (see [Te, Theorems 9.7 and 14.7]). So the following is an immediate consequences of Theorem 4.2.

Corollary 4.2. *Suppose that a paracompact space X is C -scattered or has a σ -closure-preserving cover consisting of compact sets and Y is a space. Let $X \times Y$ be a rectangular product and $\mathcal{O}$ an open cover of $X \times Y$. Then the following are equivalent.*

- (a) $\mathcal{O}$ is normal.
- (b) $\mathcal{O}$ has a σ -locally finite cozero rectangular refinement.
- (c) $\mathcal{O}$ has a σ -discrete cozero rectangular refinement.
- (d) $\mathcal{O}$ has a locally finite cozero rectangular refinement.
- (e) $\mathcal{O}$ has a locally finite, σ -discrete, cozero rectangular refinement which has a zero rectangular shrinking.

5. APPLICATIONS TO RECTANGULAR PRODUCTS

A product space $X \times Y$ is called a *strongly rectangular* [Y1] if every finite cozero (or normal) cover of $X \times Y$ has a locally finite cozero rectangular refinement.

Theorem 2.2 immediately yields

Corollary 5.1. *If a product space $X \times Y$ is X -rectangular, then it is strongly rectangular.*

Theorem 4.1 immediately yields

Corollary 5.2. *Let X be a paracompact σ -space. Then $X \times Y$ is rectangular if and only if it is strongly rectangular for any space Y .*

Moreover, Corollary 4.2 immediately yields

Corollary 5.3. *Let X be a paracompact space which is C -scattered or has a σ -closure-preserving cover consisting of compact sets. Then $X \times Y$ is rectangular if and only if it is strongly rectangular for any space Y .*

A Hausdorff space X is said to be *base-paracompact* [Po] if there is a base $\mathcal{B}$ of X such that $|\mathcal{B}| = w(X)$ and every open cover of X has a locally finite refinement consisting of members of $\mathcal{B}$.

Proposition 5.4. *Let X and Y be base-paracompact spaces. Assume that every normal cover of $X \times Y$ has a locally finite cozero rectangular refinement which has a zero rectangular shrinking. If $X \times Y$ is paracompact, then it is base-paracompact.*

Theorem 3.1 and Proposition 5.4 immediately yield

Corollary 5.5. *Let X be a base-paracompact σ -space and Y a base-paracompact space. If $X \times Y$ is paracompact and rectangular, then it is base-paracompact.*

Zhong [Zh] actually proved that the product $X \times Y$ of a stratifiable space X and a paracompact space Y is rectangular if it is (countably) paracompact. So our Corollaries 5.2 and 5.5 are extensions of [Y2, Corollaries 4.2 and 4.4], respectively.

6. INFINITE PRODUCTS OF METRIZABLE SPACES

Let $X = \prod_{\lambda \in \Lambda} X_\lambda$ be a product space. A subset of the form $\prod_{\lambda \in \Lambda} Y_\lambda$ in X is called a *rectangle* if there is a finite subset θ of Λ such that $Y_\lambda = X_\lambda$ for each $\lambda \in \Lambda \setminus \theta$. Recall that a subset U of X is *R -distinguished* if $\pi_R^{-1}\pi_R(U) = U$ holds, where $R \subset \Lambda$ and π_R denotes the projection from X onto $\prod_{\lambda \in R} X_\lambda$. A subset V of X is called a *cylinder* (resp., *ω -cylinder*) if V is R -distinguished for some finite (resp., countable) $R \subset \Lambda$.

$$\text{rectangle} \implies \text{cylinder} \implies \omega\text{-cylinder}$$

A cover $\mathcal{G}$ of X is said to be *rectangular* (resp., *cylindrical*, *ω -cylindrical*) if each member of $\mathcal{G}$ is a rectangle (resp., cylinder, ω -cylinder) in X .

By Theorem 4.3, the following is easily seen by induction.

Lemma 6.1. *Let $X = \prod_{i \leq n} X_i$ be a finite product of paracompact Σ -spaces. Then every open cover of X has a locally finite, σ -discrete, cozero rectangular refinement which has a rectangular shrinking.*

For the countable product case, we have

Lemma 6.2. *Let $X = \prod_{n \in \omega} X_n$ be a countable product of metrizable spaces. Then every binary open cover of X has a locally finite, σ -discrete, open cylindrical refinement which has a cylindrical shrinking.*

For the uncountable product case, we can prove

Lemma 6.3. *Let $X = \prod_{\lambda \in \Lambda} X_\lambda$ be a product of metrizable spaces. Then every binary cover of X by open F_σ -sets has a locally finite, σ -discrete, open ω -cylindrical refinement which has a ω -cylindrical shrinking.*

A basic idea of the proof of Lemma 6.3 is due to Yamazaki's which is found in that of [Ya, Theorem 1.3].

Remark. Lemma 6.3 is an extension of [Kl, Theorem 1].

Finally, by Lemmas 6.1, 6.2 and 6.3, we can obtain the following result.

Theorem 6.4. *Let $X = \prod_{\lambda \in \Lambda} X_\lambda$ be a product of metrizable spaces and $\mathcal{G}$ an open cover of X . Then the following are equivalent.*

- (a) $\mathcal{G}$ is normal.
- (b) $\mathcal{G}$ has a σ -locally finite open rectangular refinement.
- (c) $\mathcal{G}$ has a σ -discrete open rectangular refinement.
- (d) $\mathcal{G}$ has a locally finite open rectangular refinement.
- (e) $\mathcal{G}$ has a locally finite, σ -discrete, open rectangular refinement which has a rectangular shrinking.

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