

Number of Proofs for Implicational Formulas

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An algorithm is shown which determines the number $0, 1, \dots, \infty$ of normal form proofs for implicational formulas. The number of proofs had not been studied well. Concerning to BCK-logic, it is proved by Komori and Hirokawa [3] that the number is identical to the number of BCK-minimal formulas of α . For general implicational formulas in intuitionistic logic, Ben-Yelles [1] showed an algorithm which enumerates all the normal form proofs for α when α has finitely many proofs. But we cannot use the algorithm to decide whether α has infinitely many proofs or not. We show a limit of proof search to decide whether α has infinitely many proofs.

Given an implicational formula α , we denote by $|\alpha|$ the number of occurrences of propositional variables and the implicational symbol ' $\rightarrow$ '. We consider proof figures in the intuitionistic logic in Natural Deduction System (NJ) [4]. We denote by $proof(\alpha)$ the set of normal form proofs of α . The cardinality of $proof(\alpha)$ is denoted by $\#proof(\alpha)$. The depth of a thread in a proof π is the number of minimum formula occurrences in the thread. The depth of π , denoted by $depth(\pi)$, is the maximal depth among all the threads in π . According to the formulae-as-types correspondence [2], a normal form proof π can be represented by a closed λ -term M in β -normal form. Then the $depth(\pi)$ is identical to the depth of Böhm-tree of M .

Theorem 1 *Given an implicational formula α ,*

$$\#proof(\alpha) = \infty$$

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iff there is a normal form proof $\pi \in \text{proof}(\alpha)$ such that

(1) $\text{depth}(\pi) \leq |\alpha| 2^{|\alpha|+1}$ and

(2) π contains a thread in which a formula ξ occurs twice as minimum formula occurrence.

$$\pi \left\{ \begin{array}{c} \vdots \\ \xi \\ \vdots \\ \xi \\ \vdots \\ \alpha \end{array} \right\} \pi_2$$

$\left. \begin{array}{c} \vdots \\ \xi \end{array} \right\} \pi_1$

Outline of proof. If-part is trivial. In fact, we can replace π_1 by π_2 . We can apply this rewriting successively. Thus we have $\# \text{proof}(\alpha) = \infty$. To prove only-if-part, assume that $\# \text{proof}(\alpha) = \infty$. Then there is a proof $\pi \in \text{proof}(\alpha)$ which contains a thread with depth $\geq 2d$, where $d = |\alpha| 2^{|\alpha|}$. Then the thread contains more than $2d$ minimum formula occurrences. Let ξ be an arbitrary minimum formula occurrence in the thread and $\{\delta_1, \dots, \delta_n\}$ the assumption set for the sub-proof for ξ . By the sub-formula property, all of $\xi, \delta_1, \dots, \delta_n$ are sub-formulas of α . So we have at most d such pairs $(\xi, \{\delta_1, \dots, \delta_n\})$. Since the depth of the thread is longer than $2d$, it contains *three* occurrences of the same minimum formula occurrence ξ with the same assumption set $\{\delta_1, \dots, \delta_n\}$. Let π_1, π_2 , and π_3 be sub-proof for such occurrences of ξ which π_i appears above π_{i+1} ($i = 1, 2$). Then we can replace π_2 by π_1 obtaining a smaller proof of α . We can apply this transformation until we obtain a proof of α with depth $\leq 2d$. ■

Theorem 2 *There is an algorithm which determines $\# \text{proof}(\alpha)$ for implicative formula α .*

Proof. Consider the set of normal form proofs of α with depth $\leq |\alpha| 2^{|\alpha|+1}$. Note that the set is finite. If this set contains some π which satisfies (2) of Theorem 1, then $\# \text{proof}(\alpha) = \infty$. Otherwise $\# \text{proof}(\alpha)$ is finite. ■

Theorem 1 without (1) is proved in Ben-Yelles [1]. Proof of Theorem 1 would remind some readers the similarity to the proof of *uvwx*-theorem and infinity test for context free languages. Further work shall be necessary on this similarity.

References

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