

Palais-Smale Condition for Some Semilinear Parabolic Equations

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1 Introduction

In this paper we are concerned with the following mixed problem to semilinear parabolic equation:

$$u_t(t, x) - \Delta u(t, x) = |u(t, x)|^{p-1}u(t, x), \quad (t, x) \in (0, T) \times \Omega, \quad (1)$$

$$u(0, x) = u_0(x), \quad x \in \Omega, \quad (2)$$

$$u|_{\partial\Omega} = 0, \quad t \in (0, T). \quad (3)$$

Here, $1 < p \leq \frac{N+2}{N-2}$, $\Omega \subset R^N$ ($N \geq 3$) is a bounded domain with smooth boundary $\partial\Omega$. In the case when $1 < p < \frac{N+2}{N-2}$, of course, we can treat the low dimensional case $N = 1, 2$, but for simplicity we restrict our attention to the above mentioned case. For large initial data u_0 in some sense, it is well-known that the solution $u(t, x)$ to the problem (1)-(3) blows up in a finite time (see Ikehata-Suzuki[7], Ishii[9], Levine[10], Ôtani[11], Tsutsumi[16], and Payne-Sattinger[12]), meanwhile for small initial data, exponentially decaying solutions are obtained (see [7] and the references therein). In this paper, we have much interest in solutions to (1)-(3) which neither blowup nor decay. In that occasion, we proceed our argument based on the following local well-posedness theorem due to [7] (see also, Hoshino-Yamada[5]). In the following, $\|\cdot\|_q$ ($1 \leq q \leq \infty$) means the usual (real) $L^q(\Omega)$ -norm.

Proposition 1.1 *For each $u_0 \in H_0^1(\Omega)$, there exists a number $T_m > 0$ such that the problem (1.1)-(1.3) has a unique solution $u \in C([0, T_m); H_0^1(\Omega))$ which becomes classical on $(0, T_m)$. Furthermore, if $T_m < +\infty$, then*

$$\lim_{t \uparrow T_m} \|u(t, \cdot)\|_\infty = +\infty,$$

and in particular, in the case when $1 < p < \frac{N+2}{N-2}$ one also has

$$\lim_{t \uparrow T_m} \|\nabla u(t, \cdot)\|_2 = +\infty.$$

Set

$$X = H_0^1(\Omega),$$

$$J(u) = \frac{1}{2} \|\nabla u\|_2^2 - \frac{1}{p+1} \|u\|_{p+1}^{p+1},$$

$$\begin{aligned}
I(u) &= \|\nabla u\|_2^2 - \|u\|_{p+1}^{p+1}, \\
\mathcal{N} &= \{v \in X \setminus \{0\} \mid I(v) = 0\}, \\
d_p &= \inf_{v \in \mathcal{N}} J(v) = \inf_{\lambda \geq 0} \{\sup J(\lambda v) \mid v \in X \setminus \{0\}\}.
\end{aligned}$$

It is easy to show that the potential depth d_p (see Sattinger[13]) satisfies $d_p > 0$ because of the Sobolev continuous embedding $X \hookrightarrow L^{p+1}(\Omega)$ ($1 < p \leq \frac{N+2}{N-2}$). The stable and unstable sets are defined as usual:

$$\begin{aligned}
W &= \{u \in X \mid J(u) < d_p, I(u) > 0\} \cup \{0\}, \\
V &= \{u \in X \mid J(u) < d_p, I(u) < 0\}.
\end{aligned}$$

Furthermore, for later use we define the following notations.

$$\begin{aligned}
E &= \{u \in X \mid -\Delta u = |u|^{p-1}u \text{ in } \Omega, u|_{\partial\Omega} = 0\}, \\
E^* &= \{u \in D^{1,2}(R^N) \mid -\Delta u = |u|^{p-1}u \text{ in } R^N\}, \\
E_+^* &= \{u \in E^* \mid u \geq 0 \text{ in } R^N\}, \\
J_*(u) &= \frac{1}{2} \int_{R^N} |\nabla u(x)|^2 dx - \frac{1}{p+1} \int_{R^N} |u(x)|^{p+1} dx.
\end{aligned}$$

Here $D^{1,2}(R^N)$ denotes the closure of $C_0^\infty(R^N)$ with respect to the norm $\|\nabla u\|_{L^2(R^N)}$. In particular, in the case when $p = \frac{N+2}{N-2}$, because of the Sobolev embedding $S\|u\|_{L^{p+1}(R^N)} \leq \|\nabla u\|_{L^2(R^N)}$ for $u \in D^{1,2}(R^N)$, one also has

$$d^* = \inf_{\lambda \geq 0} \{\sup J_*(\lambda v) \mid v \in D^{1,2}(R^N) \setminus \{0\}\} = \frac{1}{N} S^N > 0.$$

Note that $d^* = d_p$ with $p = \frac{N+2}{N-2}$.

Remark 1.1 In the case when $p = \frac{N+2}{N-2}$, it is well-known (Struwe[14]) that the family $\{u_\varepsilon^*(x)\}$ such as

$$u_\varepsilon^*(x) = \frac{[N(N-2)\varepsilon^2]^{\frac{N-2}{4}}}{[\varepsilon^2 + |x|^2]^{\frac{N-2}{2}}}, \quad \varepsilon > 0$$

satisfies

$$-\Delta u = |u|^{p-1}u \text{ in } R^N,$$

so that $E_+^* \setminus \{0\} \neq \emptyset$.

By the way, quite recently, in [7] the following result has been shown with regard to the singularity of a global solution to the problem (1)-(3) under the assumptions below: let $u(t, x)$ be a solution to (1.1)-(1.3) as in Proposition 1.1. Furthermore, one assumes that

$$(A.1) \quad u_0 \geq 0.$$

$$(A.2) \quad p = \frac{N+2}{N-2}.$$

$$(A.3) \quad \Omega = \{x \in R^N \mid |x| < 1\}.$$

(A.4) $u(t, x) = u(t, |x|)$, $u_r(t, r) < 0$ on $0 < r \leq 1$ with $r = |x|$.

Finally, assume $T_m = +\infty$. For $1 < p \leq \frac{N+2}{N-2}$ set

$$C_0 = \frac{2(p+1)}{p-1} \lim_{t \rightarrow +\infty} J(u(t, \cdot)). \quad (4)$$

Note that $C_0 \geq 0$ if $T_m = +\infty$ (see [10]). Then, their results read as follows.

Theorem 1.1 ([7]) *Assume (A.1)-(A.4). Let $u(t, x)$ be a solution to (1)-(3) on $[0, T_m)$ as in Proposition 1.1. Suppose $T_m = +\infty$ and $C_0 > 0$. Then, there exists a sequence $\{t_n\}$ with $t_n \rightarrow +\infty$ as $n \rightarrow +\infty$ such that*

- (1) $|\nabla u(t_n, x)|^2 \rightarrow C_0 \delta_0$ (weakly-*) in $C_0(\Omega)^*$,
- (2) $u(t_n, x)^{p+1} \rightarrow C_0 \delta_0$ (weakly-*) in $C_0(\Omega)^*$,

as $n \rightarrow +\infty$. Here, δ_0 means the usual Dirac measure having a unit mass at the origin.

Since $C_0 > 0$ if and only if $u(t, \cdot) \notin (W \cup V)$ for all $t \geq 0$, their theorem states that a global orbit $u(t, \cdot)$ which neither decay nor blowup (if this kind of solution can be constructed!) have a strong singularity at the origin. In connection with this result, we have just noticed that for such a sequence $\{t_n\}$ constructed in Theorem 1.1 above, $\{u(t_n, \cdot)\}$ becomes a Palais-Smale sequence so that the global compactness result due to Struwe[15] can be applied to this functional sequence. Our first result reads as follows:

Theorem 1.2 *Let $\{u(t_n, \cdot)\}$ be a sequence as in Theorem 1.1. Under the same assumptions as in Theorem 1.1, there exist an integer $k \in N$, a sequence of radii $\{R_n^i\}$ with $\lim_{n \rightarrow +\infty} R_n^i = +\infty$, a sequence $\{x_n^i\} \in \Omega$, and $u^i \in E_+^* \setminus \{0\}$ ($1 \leq i \leq k$) such that (taking a subsequence)*

$$\begin{aligned} \lim_{n \rightarrow +\infty} \left\| \nabla(u(t_n, \cdot) - \sum_{i=1}^k u_n^i) \right\|_{L^2(\mathbb{R}^N)} &= 0, \\ \lim_{t \rightarrow +\infty} J(u(t, \cdot)) &= \lim_{n \rightarrow +\infty} J(u(t_n, \cdot)) = kd^*, \\ \lim_{n \rightarrow +\infty} \left\| \nabla u(t_n, \cdot) \right\|_2^2 &= \sum_{i=1}^k \left\| \nabla u^i \right\|_{L^2(\mathbb{R}^N)}^2 = kS^N, \end{aligned}$$

where

$$u_n^i(x) = (R_n^i)^{\frac{N-2}{2}} u^i(R_n^i(x - x_n^i)) \quad (1 \leq i \leq k), \quad n = 1, 2, \dots$$

Remark 1.2 *By considering scaling and translation, one finds that the compactness of $\{u(t_n, \cdot)\}$ destroyed in Theorem 1.1 is restored once more. On the other hand, for the proof of this Theorem, we have to notice the following fact (see [14]) that each u^i is of the form $u^i(x) = u_\varepsilon^*(x)$ (see Remark 1.1) with some ε and satisfies $J_*(u^i) = d^*$ (least energy level).*

Remark 1.3 *Under the assumptions $\Omega = \text{star-shaped}$ and $u_0(x) \geq 0$, one can get the quite same results as in the radial case above. In the case when u_0 changes sign, however, even if Ω is star-shaped, one needs a few modifications of the results above (see [14]).*

The next result is concerned with the case when $1 < p < \frac{N+2}{N-2}$. It seems not to be known that any global solutions to (1)-(3) naturally contain a subsequence which is relatively compact in X in the subcritical case. Our second result reads as follows:

Theorem 1.3 *Let $1 < p < \frac{N+2}{N-2}$ and $u(t, x)$ be a solution on $[0, T_m)$ as in Proposition 1.1. If $T_m = +\infty$, then there exists a sequence $\{t_n\}$ with $t_n \rightarrow +\infty$ as $n \rightarrow +\infty$ such that $\{u(t_n, \cdot)\}$ becomes relatively compact in X so that there exists an element $u_\infty \in E$ such that $u(t_n, \cdot) \rightarrow u_\infty$ in X as $n \rightarrow +\infty$ along a subsequence.*

Remark 1.4 *In Theorem 1.3, if, in particular, $C_0 > 0$, then one has $u_\infty \in E \setminus \{0\}$. Furthermore, the construction of such a sequence $\{t_n\}$ is in the same way as in Theorem 1.2.*

2 Palais-Smale sequence

In this section, reviewing some results concerning Theorem 1.1 due to [7] we shall construct some Palais-Smale sequences of a global solution to the problem (1)-(3).

First, suppose $1 < p \leq \frac{N+2}{N-2}$ and $T_m = +\infty$ in Proposition 1.1. Since its solution satisfies the energy identity:

$$J(u(t, \cdot)) + \int_0^t \|u_t(s, \cdot)\|_2^2 ds = J(u_0) \quad (5)$$

for all $t \geq 0$, this implies that the function $t \mapsto J(u(t, \cdot))$ is monotone decreasing so that $C_0 \geq 0$ (see (4)) is meaningful. Letting $t \rightarrow +\infty$ in (5), the improper integral $\int_0^\infty \|u_t(s, \cdot)\|_2^2 ds$ is finite determined. Therefore, there exists a sequence $\{t_n\}$ with $t_n \rightarrow +\infty$ as $n \rightarrow +\infty$ such that

$$\lim_{n \rightarrow +\infty} \|u_t(t_n, \cdot)\|_2^2 = 0.$$

Note that this sequence $\{t_n\}$ coincides with the one in Theorem 1.1.

Next, multiplying the both sides of (1) by $u(t, x)$ and integrating it over Ω , we have

$$(u_t(t, \cdot), u(t, \cdot)) = -I(u(t, \cdot)), \quad (6)$$

where $(f, g) = \int_\Omega f(x)g(x)dx$. Because of [2], it holds true that $\|u(t, \cdot)\|_2 \leq C$ for all $t \geq 0$ with some constant $C > 0$. Therefore, one has

$$|I(u(t_n, \cdot))| \leq C \|u_t(t_n, \cdot)\|_2$$

for all $n \in N$. Letting $n \rightarrow +\infty$, it follows that

$$\lim_{n \rightarrow +\infty} I(u(t_n, \cdot)) = 0. \quad (7)$$

On the other hand, the identity holds good:

$$J(u) = \frac{p-1}{2(p+1)} \|\nabla u\|_2^2 + \frac{1}{p+1} I(u). \quad (8)$$

So, from (8) with $u = u(t_n, \cdot)$ and (6)-(7) we find that

Lemma 2.1 *Let $u(t, \cdot)$ be as in Proposition 1.1. If $T_m = +\infty$, then there exists a sequence $\{t_n\}$ with $t_n \rightarrow +\infty$ as $n \rightarrow \infty$ such that*

$$\begin{aligned}\lim_{n \rightarrow +\infty} \|u_t(t_n, \cdot)\|_2 &= 0, \\ \lim_{n \rightarrow +\infty} \|\nabla u(t_n, \cdot)\|_2^2 &= C_0, \\ \lim_{n \rightarrow +\infty} \|u(t_n, \cdot)\|_{p+1}^{p+1} &= C_0.\end{aligned}$$

From this lemma, one obtains the next ones:

Lemma 2.2 *Let $u(t, x)$ be a local solution constructed in Proposition 1.1. If $T_m = +\infty$, then there exists a Palais-Smale sequence to the problem (1)-(3).*

Proof. Let $\{t_n\}$ be as in Lemma 2.1. Then, it follows that

$$J(u_0) \geq J(u(t_n, \cdot)) \rightarrow \frac{p-1}{2(p+1)} C_0 \geq 0 \text{ as } n \rightarrow +\infty. \quad (9)$$

Furthermore, for such sequence, since $J \in C^1(X, R)$, by equation (1) we have

$$J'(u(t_n, \cdot))[v] = -(u_t(t_n, \cdot), v)$$

for each $v \in X$, where $J'(u) \in X^*$ means the usual Fréchet-derivative of J at $u \in X$. By this equality and the Schwarz inequality together with the Poincaré inequality one gets:

$$|J'(u(t_n, \cdot))[v]| \leq C_1 \|u_t(t_n, \cdot)\|_2 \|\nabla v\|_2$$

which implies

$$\|J'(u(t_n, \cdot))\|_{H^{-1}(\Omega)} \rightarrow 0 (n \rightarrow +\infty), \quad (10)$$

where $C_1 > 0$ is a Poincaré constant. We find that $\{u(t_n, \cdot)\}$ becomes a Palais-Smale sequence because of (9) and (10). ■

In particular, in the case when $p \in (1, \frac{N+2}{N-2})$ one gets the following compactness result. For the detailed proof, see the forthcoming paper [8].

Lemma 2.3 *Suppose $p \in (1, \frac{N+2}{N-2})$. Let $u(t, x)$ be a global (i.e., $T_m = +\infty$) solution to (1)-(3) as in Proposition 1.1. Then, the sequence $\{u(t_n, \cdot)\}$ constructed in Lemma 2.1 becomes relatively compact in X .*

Now, we are in a position to prove Theorems 1.2 and 1.3.

Proof of Theorem 1.2. This result is a direct consequence of [14] (Theorem 3.1, p.184) and Lemma 2.2 and so, we shall omit the details. But, since $\Omega = \text{ball}$, note that $E = \{0\}$ holds true in the present case. ■

Proof of Theorem 1.3. The first half is a direct consequence of Lemma 2.3. In order to prove $u_\infty \in E$, note that the following estimates are proven:

$$\|f(u) - f(v)\|_{1+\frac{1}{p}} \leq p(\|u\|_{p+1} + \|v\|_{p+1})^{p-1} \|u - v\|_{p+1}$$

for all $u, v \in L^{p+1}(\Omega)$, and

$$|(f(u(t_n, \cdot)) - f(u_\infty), \phi)| \leq \|f(u(t_n, \cdot)) - f(u_\infty)\|_{1+\frac{1}{p}} \|\phi\|_{p+1}$$

for each $\phi \in C_0^\infty(\Omega)$, where $\{u(t_n, \cdot)\}$ is a sequence constructed in the first half. By combining these estimates with Lemma 2.1 and the Sobolev embedding $X \hookrightarrow L^{p+1}(\Omega)$, one obtains the desired statement. ■

From Lemma 2.1 one has a result reviewed from the view point of the Palais-Smale condition.

Corollary 2.1 *Let $1 < p \leq \frac{N+2}{N-2}$ and $u(t, x)$ be a global solution constructed in Proposition 1.1, i.e., $T_m = +\infty$. If $C_0 = 0$, then the sequence $\{u(t_n, \cdot)\}$ stated in Lemma 2.1 becomes relatively compact, and in fact, $u(t, \cdot) \rightarrow 0$ in X as $t \rightarrow +\infty$.*

From Theorem 1.1 and Corollary 2.1 with $p = \frac{N+2}{N-2}$, one can say that it depends on the least energy level $\frac{p-1}{2(p+1)}C_0$ whether the Palais-Smale condition holds good or not to the sequence $\{u(t_n, \cdot)\}$ as in Lemma 2.1.

Finally in this section, we shall apply Theorem 1.3 and Lemma 2.2 for the finite time blowup problem concerning (1)-(3). First, as a consequence of [14] one obtains the following lemma.

Lemma 2.4 *Let Ω be a bounded smooth domain and $p = \frac{N+2}{N-2}$. Then, for all $v \in E$, one has $J(v) \in \{0\} \cup (d^*, +\infty)$, and also, for each $w \in E^* \setminus \{0\}$, one has $J_*(w) \in \{d^*\} \cup (2d^*, +\infty)$.*

The following result gives a kind of alternative proof of [11] concerning blowup problem.

Proposition 2.1 *Let $1 < p \leq \frac{N+2}{N-2}$ and $u(t, x)$ be a local solution of (1)-(3) on $[0, T_m)$ constructed in Proposition 1.1. If $u(t_0, \cdot) \in V$ for some $t_0 \in [0, T_m)$, then $T_m < +\infty$.*

Proof. First, we shall deal with the case when $1 < p < \frac{N+2}{N-2}$. Suppose $T_m = +\infty$. Then, it follows from Theorem 1.3 that there exist a Palais-Smale sequence $\{u(t_n, \cdot)\}$ to the problem (1)-(3) and $u_\infty \in E$ such that $u(t_n, \cdot) \rightarrow u_\infty$ in X along a subsequence. On the other hand, it is well-known (see [6]) that $u(t, \cdot) \in V$ for all $t \in [t_0, \infty)$. Since W is a neighbourhood of 0 in X , if $u_\infty = 0$, then $u(t_m, \cdot) \in W$ holds with some t_m and this contradicts the fact that $W \cap V = \emptyset$. Thus, $u_\infty \in E \setminus \{0\}$. Because of the monotone decreasingness of a function $t \mapsto J(u(t, \cdot))$, one obtains $J(u(t_n, \cdot)) \geq J(u_\infty) \geq d_p$ which contradicts $u(t_n, \cdot) \in V$ with large t_n .

Next, we are concerned with the critical case $p = \frac{N+2}{N-2}$. Suppose $T_m = +\infty$. Obviously, $C_0 > 0$ holds true. Then, from Lemma 2.2 and Theorem 3.1 of [14], p.184 that there exist a Palais-Smale sequence $\{u(t_n, \cdot)\}$, $k \in \mathbb{N}$, $u^0 \in E$, and $u^i \in E^* \setminus \{0\}$ ($1 \leq i \leq k$) such that

$$\lim_{n \rightarrow +\infty} J(u(t_n, \cdot)) = \lim_{t \rightarrow +\infty} J(u(t, \cdot)) = J(u^0) + \sum_{i=1}^k J_*(u^i).$$

By Lemma 2.4 and the monotone decreasingness of a function $t \mapsto J(u(t, \cdot))$, one finds that

$$J(u(t, \cdot)) \geq d^*$$

for all $t \geq 0$. This contradicts also $u(t, \cdot) \in V$ for all $t \geq t_0$. ■

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