AUTOMORPHISMS OF $K3$ SURFACES IN POSITIVE CHARACTERISTIC

JONGHAE KEUM (JOINT WITH IGOR DOLGACHEV)

The main topic of this talk consists of extending the known results about automorphisms of complex algebraic $K3$ surfaces to the case when the ground field is an algebraically closed field of positive characteristic. We review first what is known in the case when the ground field is $\mathbf{C}$.

1. Automorphisms of complex projective K3 surfaces

Recall that an algebraic $K3$ surface over an algebraically closed field k is a smooth projective surface X over k such that the canonical class K_X is zero and the first Betti number is zero. When $k = \mathbf{C}$ all K3 surfaces are diffeomorphic. The fundamental group is trivial and the second cohomology group $H^2(X, \mathbf{Z})$ is a free group of rank 22 equipped with an integral symmetric bilinear form defined by the cup product. The lattice $H^2(X, \mathbf{Z})$ is even, unimodular and of signature (3,19), hence isomorphic to $U \oplus U \oplus U \oplus E_8(-1) \oplus E_8(-1)$, where $E_8(-1)$ is the lattice defined by the negative of Cartan matrix of simple root system of type E_8 and U the standard hyperbolic plane. The subgroup S_X of $H^2(X, \mathbf{Z})$ generated by algebraic 2-cycles is isomorphic to the Picard group $\mathrm{Pic}(X)$ of line bundles on X. As a sublattice of $H^2(X, \mathbf{Z})$ it is an even, not necessarily unimodular lattice of signature $(1, r-1)$. It is called the Picard lattice, or Neron -Severi lattice of X. The Picard lattice is an important invariant of a K3 surface. The deficiency from the unimodularity is measured by the discriminant group $D(S_X)$ which is a finite abelian group equipped with a quadratic form with values in $\mathbf{Q}/2\mathbf{Z}$. It is more or less known which lattices can occur as the Picard lattice of a $K3$ surface.

Let $\mathrm{Aut}(X)$ be the group of regular automorphisms of X (or, equivalently, holomorphic automorphisms, or birational automorphisms). The main tool in describing the structure of $\mathrm{Aut}(X)$ is the celebrated Global Torelli Theorem due to I. Piatetsky-Shapiro and I. Shafarevich [PS]. According to this theorem, an isometry $\sigma : H^2(X, \mathbf{Z}) \to H^2(X', \mathbf{Z})$ between the cohomology lattices of two K3 surfaces X and X' is induced by an isomorphism $f : X' \to X$, i.e. $\sigma = f^*$, if and only if σ sends an ample divisor of X to an ample divisor of X' and the linear extension of σ over $\mathbf{C}$ sends the 1-dimensional space of holomorphic 2-forms $\Omega^2(X) \subset H^2(X, \mathbf{C})$ on X to $\Omega^2(X')$ on X'.

The group $\mathrm{Aut}(X)$ has a natural representation in the orthogonal groups $O(H^2(X, \mathbf{Z}))$ and $O(S_X)$. The cohomology class $[C]$ of a smooth rational curve C on X is a vector in S_X of norm $([C], [C]) = -2$. This follows from the adjunction formula because $K_X = 0$. This implies that $[C]$ is a (-2)-root. Let W_X be the subgroup of $O(S_X)$

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generated by reflections in the classes of smooth rational curves (or, equivalently, all (-2)-roots). It is called (-2)-refection subgroup of $O(S_X)$.

Theorem 1.1. *Let $\rho : Aut(X) \to O(S_X)$, $f \to f^*$, be the representation of $Aut(X)$ on the lattice of algebraic cycles. Let $Aut(X)^*$ be the image of ρ. Then $Aut(X)^* \cap W_X = 1$ and $G = Aut(X)^* \cdot W_X$ is of finite index in $O(S_X)$ and is equal to the semi direct product of $Aut(X)^*$ and W_X.*

The kernel of the representation ρ is contained in a group of projective automorphisms of X with respect to any projective embedding of X. It is known that X has no holomorphic vector fields, so this group must be finite. Thus Theorem 1 gives a criterion for finiteness of Aut(X) in terms of the lattice S_X: the group Aut(X) is finite if and only if the subgroup W_X is of finite index in $O(S_X)$. All abstract even lattices M of signature $(1, r-1)$ with the property that the subgroup generated by reflections in (-2)-roots is of finite index in $O(M)$ have been classified by V. Nikulin and E. Vinberg. All of them can be realized as the Picard lattice of some K3 surface. We call them 2-reflexive lattices. This gives, in principle, a classification of all $K3$ surfaces with finite automorphism group.

Let us describe more precise results about automorphisms of complex projective K3 surfaces. The automorphism group Aut(X) acts on the 1-dimensional space of holomorphic 2-forms $\Omega^2(X)$ on X. Let $\chi : Aut(X) \to \mathbf{C}^*$ be the corresponding character. The image of χ is a finite cyclic group.(It may be infinite if X is non-projective.) All possible such finite cyclic groups were described by S. Kondo [Ko1] and S. Vorontsov [V].

Theorem 1.2. *The possible order of $\chi(Aut(X))$ is a divisor of one of the numbers* $66, 44, 42, 36, 28,$ *or* 12, *or a power of prime $p^k \leq 66$, where $p \leq 19$ and $k \leq 4$.*

An automorphism g is called symplectic if $\chi(g) = 1$. The name is explained by the fact that a non-zero holomorphic 2-form on X defines a symplectic structure on X, so a symplectic automorphism preserves a symplectic structure. The possible order n of a symplectic automorphism and the number f of its fixed points is given by V. Nikulin [N].

Theorem 1.3. *The possible pairs (n, f) are as follows.*

$$(n, f) = (2,8), (3,6), (4,4), (5,4), (6,2), (7,3), (8,2).$$

More generally, a subgroup $G \subset$ Aut(X) is called symplectic if $G \subset$ Ker(χ). The classification of abelian groups of symplectic automorphisms was first given by Nikulin. It was later extended to not necessarily abelian groups by S. Mukai [M].

Theorem 1.4. *Let G be a symplectic subgroup of $Aut(X)$ which is not contained in any other symplectic subgroup. Then G is isomorphic to one of the* 11 *groups*

$$PSL(2, \mathbf{F}_7), A_6, S_5, M_{20}, F_{384}, A_{4,4}, T_{192}, H_{192}, N_{72}, M_9, T_{48}.$$

Each case is supported by an example.

It is an amazing discovery of S. Mukai that each of the 11 groups can be realized as a subgroup of a sporadic simple group, the Mathieu group M_{23}. Among all subgroups of M_{23} these subgroups can be characterized by the property that they have ≥ 5 orbits in the natural action of M_{23} on the projective line $\mathbf{P}^1(\mathbf{F}_{23})$. Recall that M_{23} is realized as a stabilizer of a point in the Mathieu group M_{24} which is defined as a certain group acting quintuply transitively on the set $\mathbf{P}^1(\mathbf{F}_{23})$.

Later on, another proof of Theorem 1.4 was given by S. Kondo who found another remarkable connections with rank 24 unimodular negative definite lattices called Niemeier lattices [Ko2].

2. Automorphisms of K3 surfaces in positive characteristic

Now let X be a K3 surface over an algebraically closed field k of characteristic $p > 0$. The Picard lattice S_X (= Neron -Severi lattice) of X is defined to be the group of all divisors modulo numerical equivalence, which with the intersection pairing becomes a lattice. The main difficulty in extending the previous results to the case of positive characteristic is the absence of the Global Torelli Theorem. However, Theorem 1.1 admits a generalization to the case of supersingular K3 surfaces (A. Ogus [O]). Recall that in the complex case, Hodge theory gives that r =rankPic$(X)\leq h^{1,1}(X) = 20$. The absence of the Hodge decomposition gives only $r \leq 22$. In the extreme case $r = 22$ the surface X is called supersingular. Also note that $r = 21$ does not occur.

In any characteristic, it is known that $Aut(X)^* \cap W_X = 1$ but it is not known that the semi direct product $Aut(X)^* \cdot W_X$ is of finite index in $O(S_X)$. Still we have

Theorem 2.1. *If $S_X = \mathrm{Pic}(X)$ is 2-reflexive, i.e. W_X is of finite index in $O(S_X)$, then $Aut(X)$ is finite.*

The automorphism group Aut(X) acts on the 1-dimensional space of regular 2-forms $\Omega^2(X)$ on X. Let $\chi : Aut(X) \to k^*$ be the corresponding character. Little is known about the image of χ.

An automorphism g is called symplectic if $\chi(g) = 1$. Next we consider symplectic automorphisms of X of finite order. If the order n is coprime to the characteristic p, then we have the same result as in the complex case.

Theorem 2.2. *Let g be a symplectic automorphism of finite order n. If $(n,p) = 1$, then g has only finitely many fixed points and the possible pairs (n, f) are the same as in Theorem 1.3, that is,*

$$(n, f) = (2,8), (3,6), (4,4), (5,4), (6,2), (7,3), (8,2).$$

Proof. At a fixed point of g, g is linearizable because $(n,p) = 1$. This implies that the quotient surface $Y = X/g$ has at worst A_m-singularities and its minimal resolution is a $K3$ surface. So, Nikulin's argument [N] for the complex $K3$ surfaces works except the following 2 cases.

Case 1: $n = 11$ and X/g has 2 A_{10}-singularities.

Case 2: $n = 15$ and X/g has 3 singularities of type A_{14}, A_4 and A_2.

(When the ground field is $\mathbf{C}$, both cases are ruled out easily by the fact that the Picard number of $K3$ surfaces cannot exceed 20.)

In both cases, let $\tilde{Y}$ be the minimal resolution of $Y = X/g$. Then $\tilde{Y}$ is a supersingular $K3$ surface. The Picard lattice $S_{\tilde{Y}}$ contains the sublattice of rank 20 coming from the resolution of singularities, $A_{10} \oplus A_{10}$ in Case 1 and $A_{14} \oplus A_4 \oplus A_2$ in Case 2. The discriminant group of the sublattice of rank 20 is an abelian group of length 2, non-p-elementary. On the other hand, the discriminant group of $S_{\tilde{Y}}$

is a p-elementary group of length 2σ, where σ is the Artin invariant of $\tilde{Y}$. This implies that $\sigma = 1$ and hence $S_{\tilde{Y}}$ is a p-elementary even lattice of rank 22, signature $(1, 21)$ and length 2. Now some calculation shows that such a lattice cannot contain $A_{10} \oplus A_{10}$ unless $p = 11$, and $A_{14} \oplus A_4 \oplus A_2$ unless $p = 3$ or $p = 5$. □

The situation is completely different in the case when the order $n = p^k$. In this case, g is automatically symplectic, and is called wild. We have the following result (I. Dolgachev and J.Keum [DK]).

Theorem 2.3. *Let g be an automorphism of order p and let X^g be the set of fixed points of g. Then one of the following cases occur :*
(1) X^g is finite and consists of 0, 1 or 2 points.
(2) X^g is a divisor such that the Kodaira dimension of the pair (X, X^g) is equal to 0. In this case X^g is a nodal cycle, i.e. the union of smooth rational curves.
(3) X^g is a divisor such that the Kodaira dimension of the pair (X, X^g) is equal to 1. In this case $p \le 11$ and there exists a divisor D with support on X^g such that the linear system $|D|$ defines an elliptic or quasi-elliptic fibration $\phi : X \to \mathbf{P}^1$.
(4) X^g is a divisor such that the Kodaira dimension of the pair (X, X^g) is equal to 2. In this case X^g is equal to the support of some nef and big divisor D. Take D minimal with this property. Let $d = \dim H^0(X, \mathcal{O}_X(D - X^g))$ and $N = \frac{1}{2}D^2 + 1$. Then $p(N - d - 1) \le 2N - 2$.

In the complex case, the quotient of the surface by a finite group of symplectic automorphisms has only rational double points and is birationally isomorphic to a $K3$ surface. This is no longer true in positive characteristic [DK].

Theorem 2.4. *Let g be as in Theorem 2.3. If $|X^g| = 2$, $X/(g)$ is birationally a $K3$ surface. If $|X^g| = 1$, $X/(g)$ is birationally either a K3 surface or a rational surface. The K3 case can occur only if $p \le 5$. If $\mathrm{Dim} X^g > 0$, the quotient $X/(g)$ is always a rational surface.*

The following is the main result of this talk, which says that there is no wild p-cyclic action on a $K3$ surface if $p > 11$.

Theorem 2.5. *If X admits an automorphism g of order p, then $p \le 11$.*

Proof. We give a sketch of proof. According to Theorem 2.3-2.4 and some detailed analysis in [Dolgachev-Keum], we are reduced to prove the bound in the following two cases :

Case 1 : $|X^g| = 1$ and $X/(g)$ is a rational surface with an elliptic Gorenstein singularitiy.

Case 2 : X^g is a nodal cycle and $X'/(g)$ is a rational surface with an elliptic Gorenstein singularitiy, where X' is the orbifold $K3$ surface obtained by contracting X^g to a rational double point.

We claim that in both cases X admits a g-stable elliptic fibration such that X^g is contained in a fibre.

For simplicity, let's consider the first case. The second case can be handled similarly.

Note that the quotient surface $Y = X/(g)$ has trivial canonical divisor. Let

$$\sigma : \tilde{Y} \to Y$$

be the minimal resolution. Then

$$K_{\tilde{Y}} = -\Delta,$$

where Δ is an effective divisor whose reduced divisor is the exceptional set of σ. There is a birational morphism

$$\phi : \tilde{Y} \to Z$$

onto a rational surface Z with $K_Z^2 = 0$. More precisely, ϕ is a blow up of points on simple components of a member F of the anti-canonical system $|-K_Z|$. Then

$$\phi^* F = \Delta + \sum a_i E_i$$

for some positive integers a_i, where $\cup E_i$ is the exceptional set of ϕ. Note that the intersection number

$$(\phi^* F)\Delta_i = 0$$

for every component Δ_i of Δ. Denote by $\overline{E_i}$ the image in Y of E_i, and consider the effective Weil divisor $D := \sum a_i \overline{E_i}$ on Y. Calculating Mumford's intersection number on a normal surface, we have

$$\sigma^* D = \phi^* F,$$

and hence $D^2 := (\sigma^* D)^2 = F^2 = 0$. Since Y is $\mathbf{Q}$-factorial, the pull back $\pi^* D$, where $\pi : X \to Y = X/g$ is the quotient morphism, is an effective integral divisor of self intersection 0. Let A be an effective integral divisor of self intersection 0, proportional to $\pi^* D$ and not divisible in $\mathrm{Pic}(X)$. Then A is the sum of a fibre C of an elliptic fibration and possibly, sections, $S_1, S_2, ..., S_t$. Here we assume $p > 11$. Let $A = P + N$ be the Zariski decomposition, i.e., P is numerically effective, $PN_i = 0$ for each component of N, and the intersection matrix $(N_i N_j)$ is negative definite. Since $g^*(\pi^* D) = \pi^* D$ (equal as divisors), $g^* A = A$ as divisors. The uniqueness of the Zariski decomposition implies that $g^* P = P$ as divisors. Note that in our case $P = C + \sum S_i/2$. So $g^* C = C$, i.e., the elliptic pencil $|C|$ is g-stable. Now it is easy to see that the fibre C contains X^g. This proves the claim.

A similar argument as in [Section 5, DK] shows that X admits an automorphism g of order p, the characteristic, and a g-stable elliptic fibration only if $p \leq 11$. This contradicts to the assumption $p > 11$. □

The main observation of Mukai in proving Theorem 1.4 is that the number of fixed points of a symplectic automorphism of order n is equal to

$$\epsilon(n) = 24(n \prod_{p|n} (1 + \frac{1}{p}))^{-1}.$$

A Mathieu representation of a group G is a 24-dimensional representation with character $\chi(g) = \epsilon(\mathrm{ord}(g))$. The natural action of a finite group G of symplectic automorphisms of a complex K3 surface on $H^*(X, \mathbf{Q}) \cong \mathbf{Q}^{24}$ is a Mathieu representation. From this Mukai deduces that G is isomorphic to a subgroup of M_{23} with at least 5 orbits. In positive characteristic the formula for the number of fixed points is no longer true and the representation of G on the l-adic cohomology $H^*_{et}(X, \mathbf{Q}_l) \cong \mathbf{Q}_l^{24}$ is not Mathieu in general for any $l \neq p$. For $p > 11$, from Theorem 2.2 and 2.5, we have the following :

Theorem 2.6. *Let G be a finite group of symplectic automorphisms of a $K3$ surface in characteristic $p > 11$. Then the natural representation of G on $H^*_{et}(X, \mathbb{Q}_l) \cong \mathbb{Q}_l^{24}$ is Mathieu for $l \neq p$.*

Even in the case of characteristic $p > 11$, one cannot expect the same classification of finite symplectic groups G as in Mukai's list. The problem is the existence of supersingular $K3$ surfaces. For such surfaces, the G-invariant subspace of $H^*_{et}(X, \mathbb{Q}_l) \cong \mathbb{Q}_l^{24}$ has dimension ≥ 3. Nevertheless, we hope to have a complete classification of finite symplectic groups in positive characteristic.

References

[DK] I. Dolgachev, J. Keum, *Wild p-cyclic actions on K3 surfaces*, J. Algebraic Geometry (2001), 101-131.

[Ko1] S. Kondo, *On automorphisms of algebraic $K3$ surfaces which acts trivially on Picard groups*, Proc. Japan Acad. Ser. A Math Sci. **62(9)** (1986), 356-359.

[Ko2] S. Kondo, *Niemeier Lattices, Mathieu groups, and finite groups of symplectic automorphisms of $K3$ surfaces*, Duke Math. J. **92** (1998) 593–598.

[M] S. Mukai, *Finite groups of automorphisms of $K3$ surfaces and the Mathieu group*, Invent. Math. **94** (1988), 183-221.

[N] V. V. Nikulin, *Finite automorphism groups of Kähler K3 surfaces*, Trans. Moscow Math. Soc. **38** (1980) 71–135.

[O] A. Ogus, *Supersingular $K3$ crystals*, Astérisque **64** (1979), 3-86.

[PS] I. Piatetski-Shapiro, I.R. Shafarevich, *A Torelli theorem for algebraic surfaces of type $K3$*, Math. USSR Izv. **5** (1971), 547-587.

[V] S. P. Vorontsov, *Automorphisms of even lattices arising in connection with automorphisms of algebraic $K3$ surfaces*, Vestnik Moscow Univ. Mat. Mekh. (1983), 19-21.

School of Mathematics, Korea Institute for Advanced Study, Seoul 130-722, Korea
E-mail address: jhkeum@kias.re.kr