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RECENT DEVELOPMENTS IN DIOPHANTINE APPROXIMATION

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The last decades have seen exciting new advances in diophantine approximation. On the other hand, a number of long standing questions have not been resolved. I will give a rather subjective overview of the current state of the area.

As is well known, Dirichlet's box principle can be used to show that given real ξ and $X \geq 1$, there are integers q, p with

$$1 \leq q \leq X, \qquad |q\xi - p| < X^{-1}; \tag{1}$$

and this implies that for irrational ξ there are infinitely many rational approximations p/q with

$$\left|\xi - \frac{p}{q}\right| < \frac{1}{q^2}. \tag{2}$$

One may consider (1) to be a *localized* result, since the range for q is prescribed by X, whereas (2) is non-localized.

Considerable difficulties arise when one tries to approximate ξ by rationals p/q^2, i.e., rationals whose denominator is a square. Write γ_{loc} for the supremum of the numbers γ such that the inequalities

$$1 \leq q \leq X, \qquad |q^2\xi - p| < c(\gamma, \xi)X^{-\gamma}$$

have a solution for every ξ and $X \geq 1$, where $c(\gamma, \xi)$ is a suitable constant. Let γ_{nonl} be the supremum of the numbers γ such that

$$\left|\xi - \frac{p}{q^2}\right| < q^{-\gamma-2} \tag{3}$$

has infinitely many solutions $p, q > 0$ for every irrational ξ. Clearly $\gamma_{loc} \leq \gamma_{nonl}$, and it is easily seen that $\gamma_{nonl} \leq 1$. For a long time the record was held by [H. Heilbronn, 1948] who showed that $\gamma_{nonl} \geq \gamma_{loc} \geq 1/2$. A few years ago, [A. Zaharescu, 1995] gave an ingenious proof that $\gamma_{loc} \geq 4/7$, $\gamma_{nonl} \geq 2/3$.

- *Is it true that* $\gamma_{nonl} = 1$, *or even* $\gamma_{loc} = 1$?

The only reason we have for conjecturing $\gamma_{nonl} = 1$ is that (3) has infinitely many solutions for any $\gamma < 1$, and almost every ξ in the sense of Lebesgue measure.

It is even harder to approximate by rationals p/q^n where $n > 2$. For this and a great many related questions see [R. C. Baker, 1986]. Such questions are usually dealt with by analytic methods. Quite generally, diophantine approximation is not part of algebra or analysis, but straddles both areas.

Again by Dirichlet's box principle, given reals $\xi_1, \ldots, \xi_n$, and given $X \geq 1$, there are integers $q, p_1, \ldots, p_n$ with

$$1 \leq q \leq X, \qquad |q\xi_i - p_i| < X^{-1/n} \quad (i = 1, \ldots, n), \tag{4}$$

and dividing by q we see that $\xi_1, \ldots, \xi_n$ have infinitely many simultaneous approximations $p_1/q, \ldots, p_n/q$ with

$$\left|\xi_i - \frac{p_i}{q}\right| < \frac{1}{q^{1+1/n}} \quad (i = 1, \ldots, n), \tag{5}$$

provided at least one of the ξ_i's is irrational. Here the 1 in the numerator of the right hand side may not be replaced by an arbitrarily small constant. Now if $n = 2$, and we multiply the inequalities (5) together, we obtain

$$\left|\xi_1 - \frac{p_1}{q}\right| \left|\xi_2 - \frac{p_2}{q}\right| < \frac{1}{q^3}. \tag{6}$$

J. E. Littlewood posed the following difficult question:

- *May the* 1 *on the right hand side of* (6) *be replaced by an arbitrarily small constant?*

In other words, given $\epsilon > 0$ and arbitrary ξ_1, ξ_2, are there pairs $p_1/q, p_2/q$ with

$$\left|\xi_1 - \frac{p_1}{q}\right| \left|\xi_2 - \frac{p_2}{q}\right| < \frac{\epsilon}{q^3}? \tag{7}$$

In fact this question is open for many given numbers ξ_1, ξ_2. [J. W. S. Cassels and H. P. F. Swinnerton–Dyer, 1955] could show that (7) may be achieved when $1, \xi_1, \xi_2$ is a basis of a real cubic number field, and a refinement of this result is due to [J. Peck, 1961]. Also, (7) may be achieved for almost every $(\xi_1, \xi_2) \in \mathbb{R}^2$, in the sense of Lebesgue measure. A much stronger result of this type was recently given by [A. Pollington and S. Velani, 2000].

Suppose ρ, σ is a pair of nonnegative reals with $\rho + \sigma = 3$. Let us say (ξ_1, ξ_2) is in class $C(\rho, \sigma)$ if

$$\left|\xi_1 - \frac{p_1}{q}\right| < \epsilon q^{-\rho}, \quad \left|\xi_2 - \frac{p_2}{q}\right| < \epsilon q^{-\sigma}$$

has a solution $p_1/q, p_2/q$ for every $\epsilon > 0$. Littlewood's question would have a positive answer if we had $C(\rho, \sigma) = \mathbb{R}^2$ for some ρ, σ. However, by the method of [W. M. Schmidt, 1969], the complement of $C(\rho, \sigma)$ has the cardinality of the continuum for every ρ, σ. Littlewood's question still has a positive answer if $C(\rho, \sigma) \cup C(\rho', \sigma') = \mathbb{R}^2$ for some pairs ρ, σ and ρ', σ'. But I

- *conjecture that always* $C(\rho, \sigma) \cup C(\rho', \sigma') \neq \mathbb{R}^2$.

It is not even known whether $C(1/3, 2/3) \cup C(2/3, 1/3) = \mathbb{R}^2$.

It is a trivial consequence of Dirichlet's result on (1) that when $L(\mathbf{x})$ is a linear form in $n > 1$ variables with real coefficients, there are for any $\epsilon > 0$ integer points $\mathbf{x} \neq 0$ with $|L(\mathbf{x})| < \epsilon$. A common generalization of this, and of a theorem of [B. J. Birch, 1957] on diophantine equations, says that when $F_1, \ldots, F_R$ are forms of odd degree d with real coefficients in $n > c(d, R)$ variables, then there is for any $\epsilon > 0$ a point $\mathbf{x} \in \mathbb{Z}^n \backslash \{\mathbf{0}\}$ with

$$|F_i(\mathbf{x})| < \epsilon \qquad (i = 1, \ldots, R). \tag{8}$$

The values obtainable for $c(d,R)$ by the present method [W. M. Schmidt, 1980] would be absurdly large. A difficult problem is to

- *find reasonable bounds for* $c(d,R)$.

Even an estimate like $c(d,R) \leq \exp_d(R)$ would be great progress, where $\exp_0(x) = x$, $\exp_d(x) = \exp_{d-1}(e^x)$ for $d > 0$. For recent results on (8) when $d = 3$, see [D. E. Freeman, to appear], who also deals with related questions in his other works. See also the treatise by R. C. Baker quoted above.

We will now turn to more algebraic topics. The exponent 2 in Dirichlet's (2) is best possible. By the Theorem of Thue–Siegel–Roth [K. F. Roth, 1955], the exponent 2 is best possible for approximation to algebraic numbers. Thus when α is algebraic,

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{q^{2+\delta}} \tag{9}$$

where $\delta > 0$, has only finitely many solutions p/q. Here is another challenge:

- *Replace* q^δ *in* (9) *by a function growing more slowly than any positive power of* q.

For instance, one might conjecture that

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{q^2(\log q)^2}$$

has only finitely many solutions. On the other hand, it is widely believed that

- $\left|\alpha - \frac{p}{q}\right| < \epsilon/q^2$ *has infinitely many solutions for every* $\epsilon > 0$ *if* α *is algebraic of degree at least* 3.

This is equivalent to the conjecture that such α has unbounded partial quotients in its continued fraction expansion.

As is well known, Roth's Theorem is not effective: its method of proof allows to bound the *number* of solutions to (9) in terms of α and δ (see, e.g., [E. Bombieri and A. J. Van derPoorten, 1988]), but not the *size* $\max(|p|,|q|)$, hence does not allow to find all the solutions. It therefore would be important to

- *make Roth's Theorem effective.*

The well known abc-conjecture implies Roth's Theorem (see, e.g., [A. Granville and T. J. Tucker, 2002], and an effective version of the conjecture implies an effective Roth's Theorem. The abc-conjecture has many applications to diophantine approximation.

A Thue equation is an equation

$$F(x,y) = m \tag{10}$$

where $m \in \mathbb{N}$ and F is a homogeneous form of degree $n \geq 3$ with integer coefficients and distinct linear factors. We can factor

$$F(x,y) = a(x - \alpha_1 y) \cdots (x - \alpha_n y)$$

with algebraic and distinct α_i's, and any solution of (10) will have some $|x - \alpha_i y|$ small, hence $\left|\alpha_i - \frac{x}{y}\right|$ small, and it easily follows from Roth's Theorem that (10) has only finitely many solutions in integers x, y. This approach is ineffective, i.e., does not allow to find the solutions. To get an effective method, one does not need as much as an effective Roth's Theorem, but only the effective solubility of

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{q^{n-\theta}}$$

with $n = \deg \alpha$ and effective $\theta = \theta(\alpha) > 0$. In fact such a result was proved by [N. I. Feldman, 1971], using A. Baker's theory of linear forms in logarithms. Alan Baker in seminal work of the 1960's gave explicit lower bounds for expressions

$$|\beta_1 \log \alpha_1 + \cdots + \beta_m \log \alpha_m|$$

with algebraic α_i's and β_i's. Many authors, including Baker himself, Wüstholz, Waldschmidt, [E. M. Matveev, 2000], have refined these bounds, and p-adic versions are due to Y. Kunrui. Also, S. David and N. Hirata–Kohno recently established corresponding estimates for elliptic logarithms.

Many mathematicians have contributed to the effective solution of Thue equations, including A. Baker, M. Bennett, E. Bombieri and J. Vaaler, Heuberger, Lettl, Okazaki, Pethö, Thomas, Tichy, Tzanakis, Voutier, Wakabayashi. There are three basic methods: Padé approximation, linear forms in logarithms, and an approach based on Thue and a refined Dyson's Lemma. Others than the author of this survey would be better qualified to report on these developments. Quite generally, solutions of (10) have $\max(|x|,|y|) < \exp(c_1(n)H^{c_2(n)})$ where H is the maximum modulus of m and the coefficients of F.

Let us turn to the number of solutions. [E. Bombieri and W. M. Schmidt, 1987] showed that this number is

$$\leq cn^{1+\omega} \tag{11}$$

where c is an absolute constant and $\omega = \omega(m)$ is the number of distinct prime factors of m. Observe that this bound is independent of the coefficients of F. [C. L. Siegel, 1929] alluded to a conjecture that when a polynomial equation $f(x,y) = 0$ defines an irreducible curve of positive genus, then the number of integer solutions can be bounded in terms of the number of monomials which occur in f with nonzero coefficients. This is not quite true, but according to [J. Mueller and W. M. Schmidt, 1988], for Thue equations the number of solutions may be bounded in terms of m, and the number of monomials of the equation. It would be of interest to see

- *what modified form of Siegel's conjecture is true more generally?*

Often it is just as easy to deal with the Thue inequality

$$|F(x,y)| \leq m$$

as it is to deal with the equation. [J. L. Thunder, 1995] used clever arguments to show that under a natural condition, the number of solutions is $\leq c_0 n m^{2/n}$ with an absolute constant c_0.

A generalization of many of the results mentioned so far from $\mathbb{Q}$ to an algebraic number field K is fairly easy. [E. Wirsing, 1961] introduced a more interesting question: given $\xi \in \mathbb{R}$ and $d \in \mathbb{N}$, how well can ξ be approximated by algebraic numbers of degree $\leq d$? Wirsing himself showed that unless ξ is itself algebraic of degree $\leq d$, there are infinitely many algebraic numbers α of degree $\leq d$ with

$$|\xi - \alpha| < c(\xi) H(\alpha)^{-(d+3)/2}, \tag{12}$$

where $H(\alpha)$ is the naive Height, namely the maximum modulus of the coefficients of the defining polynomial of α. According to [Y. Bugeaud and O. Teulie, 2000] one may even insist that α is of exact degree d. Once it was thought that the exponent in (12) should be $-(d+1)+\epsilon$, or even $-(d+1)$. This is in fact true when $d = 1$ by Dirichlet's Theorem, and was established for $d = 2$ by [H. Davenport and W. M. Schmidt, 1967]. In general, the exponent in (12) was somewhat improved by [K. I. Tishchenko, (to appear)]. For $d > 2$ an exponent such as $-(d+1)$ is now in doubt by a result of D. Roy quoted below. I now make the following, perhaps reckless

- *conjecture: the best exponent in* (12) *is* $-\gamma(d)$ *with* $\gamma(d) \sim d/2$ *as* $d \to \infty$.

There is a variation on the question, where α is restricted to be an algebraic *integer* of degree $\leq d$. It had been thought that in this case the correct exponent should be $-d+\epsilon$, or even $-d$. But [D. Roy, (to appear)] very recently showed the correct exponent for $d = 3$ to be $-\frac{1}{2}(3+\sqrt{5}) > -3$. He derived this from the following. By Dirichlet's result on (4), for any ξ and any $X \geq 1$, there are integers q, p_1, p_2 with

$$1 \leq q \leq X, \quad |q\xi - p_1| < X^{-1/2}, \quad |q\xi^2 - p_2| < X^{-1/2}.$$

However, according to Roy, there are denumerably many numbers ξ for which

$$1 \leq q \leq X, \quad |q\xi - p_1| < c(\xi) X^{-\theta}, \quad |q\xi^2 - p_2| < c(\xi) X^{-\theta}$$

has solutions for every $X \geq 1$, where $\theta = \frac{1}{2}(\sqrt{5}-1) \sim 0.618 > 1/2$. Here θ is best possible. Observe that this is a localized result.

The exponent $1 + 1/n$ in Dirichlet's theorem (5) on simultaneous approximation is best possible. In fact when $\alpha_1, \ldots, \alpha_n$ are algebraic, and $1, \alpha_1, \ldots, \alpha_n$ linearly independent over $\mathbb{Q}$, then

$$\left|\alpha_i - \frac{p_i}{q}\right| < 1/q^{1+\frac{1}{n}+\delta} \quad (i = 1, \ldots, n)$$

where $\delta > 0$, has only finitely many solutions $p_1/q, \ldots, p_n/q$. This is a consequence of the Subspace Theorem, which in its simplest version says that if $L_1, \ldots, L_n$ are linearly independent linear forms in n variables with algebraic coefficients, then the points $\mathbf{x} \in \mathbb{Z}^n \backslash \{\mathbf{0}\}$ with

$$\prod_{i=1}^{n} |L_i(\mathbf{x})| < |\mathbf{x}|^{-\delta}$$

lie in finitely many proper subspaces of $\mathbb{Q}^n$. Here $|\mathbf{x}|$ denotes the Euclidean norm of $\mathbf{x}$.

In a reformulation allowing rational (rather than integral) points, the solutions $\mathbf{x} \in \mathbb{Q}^n \backslash \{\mathbf{0}\}$ of

$$\prod_{i=1}^{n} (|L_i(\mathbf{x})|/|\mathbf{x}|) < H(\mathbf{x})^{-n-\delta},$$

where $H(\mathbf{x})$ is a suitable "Height" of $\mathbf{x}$, lie in finitely many proper subspaces.

A generalization allowing points $\mathbf{x} \in K^n$ where K is a number field is due to Schlickewei. Let $|\cdot|_v$ ($v \in \mathcal{M} = \mathcal{M}(K)$) be suitably normalized absolute values of K such that the product formula holds. Suppose $S \subset \mathcal{M}$ is a finite set containing all the Archimedean absolute values, and for each $v \in S$, let $L_1^v, \ldots, L_n^v$ be linearly independent forms in n variables with coefficients in K. Then the solutions $\mathbf{x} \in K^n \backslash \{\mathbf{x}\}$ of

$$\prod_{v \in S} \prod_{i=1}^{n} (|L_i^v(\mathbf{x})|_v / |\mathbf{x}|_v) < H(\mathbf{x})^{-n-\delta} \tag{13}$$

lie in finitely many proper subspaces of K^n.

In fact, [J. H. Evertse and H. P. Schlickewei, 2002] proved an even more general version, where $\mathbf{x}$ is not confined to K^n, but may be any nonzero point in $\overline{\mathbb{Q}}^n$, where $\overline{\mathbb{Q}}$

is an algebraic closure of K. Furthermore, the solutions of the inequality fall into two classes, the "small solutions" with

$$H(\mathbf{x}) < \max(n^{4n/\delta}, H(L_i^v)(v \in S, 1 \leq i \leq n)),$$

and the others, the "large solutions", lying in the union of at most

$$t = t(n, \delta, \#S)$$

subspaces of $\overline{\mathbb{Q}}^n$. It is important for applications that t does not depend on K or the coefficients of the linear forms L_i^v.

This breakthrough was possible by important work by Roy and Thunder. Siegel's Lemma says that a system of linear equations

$$L_i(\mathbf{x}) = 0 \qquad (i = 1, \ldots, m)$$

in $n > m$ variables defined over $\mathbb{Q}$ has a nontrivial solution $\mathbf{x} \in \mathbb{Q}^n$ with

$$H(\mathbf{x}) \leq c_{n,m}(\max_i H(L_i))^{m/(n-m)},$$

and this has been generalized to a number field K by [R. B. Macfeat, 1971] and independently by [E. Bombieri and J. Vaaler, 1983]. But now $c_{n,m} = c_{n,m}(K)$ depends on K, and in particular on its discriminant. [D. Roy and J. L. Thunder, 1996] showed that if we allow solutions $\mathbf{x} \in \overline{\mathbb{Q}}^n$ (not just K^n), the dependency on K can be eliminated, so that again $c_{n,m}$ depends on n, m only. The proof does not give information on the field $K(\mathbf{x})$ generated over K by their solutions $\mathbf{x}$. It would be of interest to

- *give a bound for the degree $K(\mathbf{x}) : K]$ in the Roy–Thunder result.*

The Subspace Theorem may be applied to Wirsing's question: when ξ is algebraic, and $\delta > 0$, there are only finitely many algebraic numbers α of degree $\leq d$ with

$$|\xi - \alpha| < H(\alpha)^{-d-1-\delta}. \tag{14}$$

[J. H. Evertse and N. Hirata–Kohno, 2002] studied more general "Wirsing systems"

$$|\xi_i - \alpha^{(i)}| < H(\alpha)^{-\phi_i} \qquad (i = 1, \ldots, n) \tag{15}$$

where the $\alpha^{(i)}$ are conjugates of an algebraic number α of fixed degree $d \geq n$. If the ξ_i are algebraic and $\Sigma_i \phi_i > 2d$, there are only finitely many solutions α. In many cases this condition can be relaxed to $\Sigma_i \phi_i > d + 1$, which is more in line with (14).

• *Under what conditions exactly does* $\Sigma_i \phi_i > d + 1$ *suffice for the finiteness of the number of* α*'s with* (15)?

[P. Vojta, 1989] refined the Subspace Theorem by showing that there is a finite union U of proper subspaces of dimension > 1 and depending only on the L_i^v (in particular independent of $\delta > 0$) such that all but finitely many $\mathbf{x} \in K^n \backslash \{\mathbf{0}\}$ with (13) lie in U. See also [W. M. Schmidt, 1993].

• *U can be taken as the union of at most how many subspaces?*

When $n = 3$, $K = \mathbb{Q}$, and $\#S = 1$, then U may be taken as the union of at most 3 proper subspaces; and this is best possible.

Just as Roth's Theorem leads to Thue equations, the Subspace Theorem leads to equations

$$F(\mathbf{x}) = m \tag{16}$$

where $F(\mathbf{x}) = F(x_1, \ldots, x_n)$ with integer coefficients is *decomposable*, i.e., is the product of linear forms. Under quite general circumstances, e.g., when F is a "nondegenerate norm form", there are only finitely many integer solutions. In this case,

• *is there an estimate for the number of solutions analogous to* (11), *in particular depending only on* $n, d = \deg F, \omega(m)$? (*The letter* n *had a different meaning in* (11)).

[J. Thunder, 2001] obtained rather satisfying results on decomposable form inequalities

$$|F(\mathbf{x})| \leq m. \tag{17}$$

The number of solutions is finite for every m precisely if F is of *finite type*, i.e., if the volume of the set of real solutions of (17) is finite, and if the same holds for the real solutions in any n'-dimensional subspace defined over $\mathbb{Q}$. In this case, the number of integer solutions is $\leq c_0 m^{n/d}$ with an effective constant $c_0 = c_0(n,d)$.

Some deep applications of the Subspace Theorem were recently given by Corvaja and Zannier. They gave [P. Corvaja and U. Zannier, 2002b] a new proof of Siegel's theorem on integral points on curves, avoiding the embedding into Jacobians. Part of this theorem says that if an irreducible curve $f(x,y) = 0$ has at least 3 points at infinity, then it contains only finitely many integer points. Corvaja and Zannier prove this result in $1\frac{1}{2}$ pages: there is no loss of generality in assuming that the curve C is nonsingular. If $Q_1, \ldots, Q_r$ $(r \geq 3)$ are the points at infinity, let $\phi_1, \ldots, \phi_d$ be a basis of the space V_N of elements ϕ in the function field of C with

$$\operatorname{div} \phi \geq -N(Q_1 + \cdots + Q_r).$$

They construct linear forms in $\phi_1, \ldots, \phi_d$ to which the Subspace Theorem may be applied if N is chosen sufficiently large. Siegel's Theorem in general follows because when the curve has positive genus, there is an unramified cover containing ≥ 3 points at infinity.

What about the *number* of integral points on a curve with at least 3 points at infinity? Or more generally the number of "S-integral" points, which allow denominators involving a finite set S of "primes" (more precisely, points with coordinates x_i in a number field K, having $|x_i|_v \leq 1$ for all places $v \notin S$). As pointed out, e.g., in [M. Hindry and J. H. Silverman, 2000], there are "small" points $\mathbf{x}$ on the curve with Height

$$H(\mathbf{x}) \leq H^c$$

where H is the maximum Height of the defining equations of the curve, whereas the

number of the remaining points, i.e., the

$$\text{number of "large" points on the curve is} \leq c^{\#S}.$$

The new approach [P. Corvaja and U. Zannier, manuscript a] yields an effective value for c depending only on $\deg C$ and m when the curve $C \subset \mathbb{P}_m$.

But there is more! According to [P. C. and U. Z., manuscript b], suppose X is an irreducible, nonsingular surface with $r \geq 4$ divisors $D_1, \ldots, D_r$ at infinity, such that no three have a point in common, and with intersection matrix $(D_i.D_j)$ of rank 1 and with positive entries. Then the integer points on this surface lie on a curve. A natural question would be whether

• *there is an analogue for irreducible varieties of arbitrary dimension d? Are there suitable conditions on the divisors at infinity for this to happen?*

The same authors [P. C. and U. Z., manuscript c] have results on a wide generalization of decomposable form equations (16), as well as a generalization of the Subspace Theorem. Other generalizations have been given by [G. Faltings and G. Wüstholz, 1994] and [J. H. Evertse and R. Ferretti, to appear].

Consider an exponential equation

$$\sum_{i=1}^{n} a_i \alpha_{i1}^{y_1} \cdots \alpha_{ir}^{y_r} = 0 \tag{18}$$

with given nonzero complex numbers a_i, α_{ij}, to be solved in integers $y_1, \ldots, y_r$. Such an equation arose, e.g., in the contribution by M. Higasikawa at the present conference. The equation may be rewritten as

$$\sum_{i=1}^{n} a_i x_i = 0 \tag{19}$$

where $\mathbf{x} = (x_1, \ldots, x_n)$ runs through the multiplicative group $\Gamma \subset (\mathbb{C}^\times)^n$ of rank $\leq r$ generated by $(\alpha_{1j}, \ldots, \alpha_{nj})$ $(j = 1, \ldots, r)$. In the algebraic case, i.e., when the

a_i, α_{ij} are in a number field K, then each x_i is an S-unit, i.e., has "numerator" and "denominator" in the finite set of numerators and denominators of the α_{ij}, and it turns out that the Subspace Theorem may be applied. Today we know (see [J. H. Evertse, H. P. Schlickewei and W. M. Schmidt, 2002]) that up to proportionality, the number of nondegenerate (i.e., with no vanishing subsum) solutions $\mathbf{x}$ of (19), lying in a group Γ of rank r, is $\leq c(n,r)$. Again there is no dependency on the coefficients, which may be arbitrary complex numbers. This result depends on the Evertse–Schlickewei version of the Subspace Theorem, which in turn depends on the work of Roy–Thunder mentioned above.

The situation is more complicated when the a_i in (18) are polynomials in $\mathbf{x}$. There is a general theorem of [M. Laurent, 1989] which says in particular that if $\alpha_{11}^{y_1} \cdots \alpha_{1r}^{y_r} = \cdots = \alpha_{n1}^{y_1} \cdots \alpha_{nr}^{y_r}$ with $\mathbf{y} = (y_1, \ldots, y_r) \in \mathbb{Z}^r$ implies $\mathbf{y} = \mathbf{0}$, then there are only finitely many nondegenerate solutions to (18). It would be desirable to

• *find a bound for the number of solutions in Laurent's Theorem which depends only on n, r and the degrees of the a_i.*

In the case $r = 1$, i.e., the one variable case, this has been done by [W. M. Schmidt, 1999], and has consequences for linear recurrence sequences. These are sequences $\{u_n\}_{n\in\mathbb{Z}}$ of complex numbers satisfying a recurrence relation

$$u_n = c_1 u_{n-1} + \cdots + c_t u_{n-t} \qquad (n \in \mathbb{Z})$$

with fixed coefficients $c_1, \ldots, c_t$. If the sequence is "non-degenerate" in some sense, then the zero-multiplicity, i.e., the number of n with $u_n = 0$, is $\leq c(t)$.

Of the deep works of Corvaja and Zannier on linear recurrences, let me just mention a result in [P. C. and U. Z., 2002a], that if u_n, v_n are linear recurrence sequences satisfying some natural conditions, and if u_n/v_n is in $\mathbb{Z}$ for infinitely many n, then $\{u_n/v_n\}_{n\in\mathbb{Z}}$ is also a linear recurrence sequence.

Let me finally turn to the analogue of the theory where $\mathbb{Q}$ is replaced by a function field $k(T)$ in one variable. When the characteristic is positive, many issues become more complicated than in the classical case. [C. F. Osgood, 1985] and [P. Vojta. 1987] found a connection with Nevanlinna theory, and work has been done by L.E. Baum, W. M. Bucks, A. Lasjaunias, B. de Mathan, W. H. Mills, D. P. Robbins, M. Ru, J. J. Ruch, M. M. Sweet, D. Thakur, J. F. Voloch, J. T. Y. Wang, P. M. Wong and others. Whereas in the classical case it is widely believed that algebraic α of degree > 2 has unbounded partial quotients in its continued fraction, [L. E. Baum and M. M. Sweet. 1976] exhibited functions of degree 3 over $\mathbb{F}_2(T)$ with bounded partial quotients (i.e., these quotients are polynomials of bounded degree). Many more such instances have since been found; see, e.g., [A. Lasjaunias and J. J. Ruch, 2002]. It is a challenge to

- *find a general criterion on when an algebraic function in positive characteristic has bounded partial quotients.*

It was already known to Mahler that Roth's Theorem is not true in positive characteristic. Given α in $k((T^{-1}))$ (this being the analogue of $\mathbb{R}$), and a suitable absolute value on this field, let $\nu(\alpha)$ be the supremum of the exponents e such that $|\alpha - p/q| < 1/|q|^e$ has infinitely many solutions p/q in $k(T)$. It was shown independently by [W. M. Schmidt, 2000] and [D. Thakur, 1999] that for every rational $\nu \geq 2$, there are algebraic functions α with $\nu(\alpha) = \nu$. It is not known whether

- $\nu(\alpha)$ *for algebraic* α *is necessarily rational*?

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