

WELL-POSEDNESS FOR QUADRATIC NONLINEAR SCHRÖDINGER EQUATIONS

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1. INTRODUCTION

This is a short review of the result obtained in the paper [6], which is joint work with Nobu Kishimoto.

We consider the Cauchy problem of quadratic nonlinear Schrödinger equations as follows;

$$(1.1) \quad \begin{cases} (i\partial_t - \partial_x^2)u = N(u), & (t, x) \in [0, T] \times \mathbb{R}, \\ u(0, x) = u_0(x), & x \in \mathbb{R}. \end{cases}$$

where unknown function u is complex valued and $N(u) = u^2$, $\bar{u}^2$ or $u\bar{u}$. Our aim is to prove the time local well-posedness of (1.1) with low regularity initial data.

We first assume that $u_0 \in H^s$ and recall the known results. Bourgain [2] introduced the Fourier restriction norm $X^{s,b}$ defined below to study the KdV equation and the nonlinear Schrödinger equation;

$$\|u\|_{X^{s,b}} = \|\langle \xi \rangle^s \langle \tau - \xi^2 \rangle^b \tilde{u}\|_{L^2_{\tau,\xi}},$$

where $\langle \cdot \rangle = 1 + |\cdot|$ and $\tilde{u}$ is the Fourier transform of u with respect to t and x . Kenig, Ponce and Vega [4] developed this method and obtained the time local well-posedness of (1.1) with $N(u) = u^2$, $\bar{u}^2$ and $u\bar{u}$ for $s > -3/4$, $s > -3/4$ and $s > -1/4$, respectively. In the proof, the following bilinear estimate plays an important role;

$$\|N(u)\|_{X^{s,b-1}} \leq C\|u\|_{X^{s,b}}^2.$$

Nakanishi, Takaoka and Tsutsumi [7] proved the counter examples of this estimate with $N(u) = u^2$, $\bar{u}^2$ and $u\bar{u}$ for $s \leq -3/4$, $s \leq -3/4$ and $s \leq -1/4$, respectively. This means that we can not improve Kenig, Ponce and Vega's result with the standard Fourier restriction norm method. To overcome this difficulty, Bejenaru and Tao [1] introduced a modified Fourier restriction norm and used a support property of solutions of (1.1), namely, the support of $\tilde{u}$ is in $\{(\tau, \xi) \in \mathbb{R}^2 | \tau \geq 0\}$ when $N(u) = u^2$ and u satisfies (1.1), to obtain the time local well-posedness of (1.1) with $N(u) = u^2$ for $s \geq -1$. When $N(u) = \bar{u}^2$, the problem is more complicated because this property does not hold. Nevertheless, Kishimoto [5], proved the time local well-posedness of (1.1) with $N(u) = \bar{u}^2$ for $s \geq -1$ by using a modified Fourier restriction norm with complicated weight functions. The case $N(u) = u\bar{u}$ is totally different from the cases $N(u) = u^2$ or $\bar{u}^2$. For instance, the data-to-solution map $: u_0 \in H^s \rightarrow C([0, T] : H^s)$ fails to be C^2 when $s < -1/4$ and $N(u) = u\bar{u}$. This is caused by the Energy flow from high frequency parts to low frequency parts. To overcome this difficulty, we introduce the following function space and we assume $u_0 \in H^{s,a}$.

Put

$$H^{s,a} = \{f \in \mathcal{Z}'(\mathbb{R}) \mid \|f\|_{H^{s,a}} < \infty\},$$

$$\|f\|_{H^{s,a}} = \|\langle \xi \rangle^{s-a} |\xi|^a \widehat{f}\|_{L^2},$$

where $\mathcal{Z}'(\mathbb{R}^n)$ denotes the dual space of

$$\mathcal{Z}(\mathbb{R}^n) := \{f \in \mathcal{S}(\mathbb{R}^n) \mid D^\alpha \mathcal{F}f(0) = 0 \text{ for every multi-index } \alpha\}.$$

If we apply the standard Fourier restriction norm method to time local well-posedness of (1.1) with $N(u) = u\bar{u}$ in $H^{s,a}$, we need the following bilinear estimate with $b \geq 1/2$;

$$(1.2) \quad \|u\bar{u}\|_{X^{s,a,b-1}} \leq C \|u\|_{X^{s,a,b}}^2$$

where

$$(1.3) \quad \|u\|_{X^{s,a,b}} = \|\langle \xi \rangle^{s-a} |\xi|^a \langle \tau - \xi^2 \rangle^b \widetilde{u}\|_{L^2_{\tau,\xi}}.$$

Put

$$\widetilde{u}_N(\tau, \xi) = \begin{cases} 1, & |\xi - N| < 1 \text{ and } |\tau - \xi^2| < 1, \\ 0, & \text{otherwise,} \end{cases}$$

and let $N \in \mathbb{N}$ be sufficiently large. Then, we have

$$(1.4) \quad \widetilde{u_N \bar{u}_N}(\tau, \xi) = \widetilde{u}_N * \widetilde{u}_N(\tau, \xi) \sim \psi_{R_0}(\tau, \xi)$$

where ψ_A denotes the characteristic function of the set A and R_0 is the rectangle of dimensions $N \times N^{-1}$ centered at the origin with longest side pointing in the $(1, 2N)$ direction. It follows that

$$R.H.S. \text{ of (1.2)} \leq CN^{2s},$$

$$L.H.S. \text{ of (1.2)} \geq \left(\int_{1/2 < |\xi| < 1} \int \langle \tau - \xi^2 \rangle^{2(b-1)} \psi_{R_0}(\tau, \xi) d\tau d\xi \right)^{1/2} \geq cN^{b-1}.$$

Therefore, (1.2) fails for any $a \in \mathbb{R}$, $s < -1/4$ and $b \geq 1/2$.

To overcome this difficulty, we use the weight function defined in (2.1) instead of $\langle \xi \rangle^{s-a} |\xi|^a \langle \tau - \xi^2 \rangle^b$ in (1.3) and introduce modified Fourier restriction norms $Z^{s,a}$ and $Y^{s,a}$ (see, Section 2) and prove new bilinear estimates (Proposition 3.1) to obtain the following time local well-posedness result.

Theorem 1.1. *Let $s \geq -(2a+1)/4$ and $1/2 > a > -1/2$. Then, (1.1) with $N(u) = u\bar{u}$ is time locally well-posed in $H^{s,a}$.*

Remark 1.2. Since $H^s \subset H^{s,a}$ when $a \geq 0$, we have the existence of the solution for $u_0 \in H^s$ with $s > -1/2$ by Theorem 1.1. However, the solution $u(t)$ is not in H^s for any $t > 0$ when $-1/4 > s > -1/2$.

In Section 2, we give some notations and preliminary lemmas. In Section 3, we prove the main estimates. The proof of Theorem 1.1 follows from a standard argument and these estimates (see, e.g. [5]). So, we omit the proof.

2. NOTATIONS AND PRELIMINARY LEMMAS

Throughout this paper $C > 0$ denotes various constants. The notation $P \lesssim Q$ denote the estimate $P \leq CQ$. We use $P \sim Q$ to denote $P \lesssim Q \lesssim P$.

Put

$$\begin{aligned} P_1 &= \{(\tau, \xi) \in \mathbb{R}^2 \mid |\tau - \xi^2| \leq |\xi|/4 \text{ and } |\xi| \geq 1\}, \\ P_2 &= \{(\tau, \xi) \in \mathbb{R}^2 \mid |\tau - \xi^2| \geq |\xi|/4 \text{ or } |\xi| < 1\}, \end{aligned}$$

and

$$(2.1) \quad w_{s,a}(\tau, \xi) = \begin{cases} \langle \xi \rangle^s \langle \tau - \xi^2 \rangle, & (\tau, \xi) \in P_1, \\ \langle \xi \rangle^{1/2-a} |\xi|^a \langle \tau - \xi^2 \rangle^{1/2+s}, & (\tau, \xi) \in P_2. \end{cases}$$

Note that

$$w_{s,a}(\tau, \xi) \sim \min\{\langle \xi \rangle^{s-a} |\xi|^a \langle \tau - \xi^2 \rangle, \langle \xi \rangle^{1/2-a} |\xi|^a \langle \tau - \xi^2 \rangle^{1/2+s}\}.$$

We define function spaces $Z^{s,a}$ and $Y^{s,a}$ as follows;

$$\begin{aligned} Z^{s,a} &= \{u \in \mathcal{Z}'(\mathbb{R}^2) \mid \|u\|_{Z^{s,a}} < \infty\}, \\ Y^{s,a} &= \{u \in \mathcal{Z}'(\mathbb{R}^2) \mid \|u\|_{Y^{s,a}} < \infty\}, \end{aligned}$$

where

$$\|u\|_{Z^{s,a}} = \|w_{s,a} \tilde{u}\|_{L^2_{\tau,\xi}}, \quad \|u\|_{Y^{s,a}} = \left\| \int \langle \xi \rangle^{s-a} |\xi|^a \tilde{u} d\tau \right\|_{L^2_\xi}.$$

Put

$$\begin{aligned} Q_1 &= \{(\tau, \xi) \in \mathbb{R}^2 \mid |\tau + \xi^2| \leq |\xi|/4 \text{ and } |\xi| \geq 1\}, \\ Q_2 &= \{(\tau, \xi) \in \mathbb{R}^2 \mid |\tau + \xi^2| \geq |\xi|/4 \text{ or } |\xi| < 1\}, \\ w'_{s,a}(\tau, \xi) &= \begin{cases} \langle \xi \rangle^s \langle \tau + \xi^2 \rangle, & (\tau, \xi) \in Q_1, \\ \langle \xi \rangle^{1/2-a} |\xi|^a \langle \tau + \xi^2 \rangle^{1/2+s}, & (\tau, \xi) \in Q_2, \end{cases} \end{aligned}$$

and

$$\|u\|_{\bar{Z}^{s,a}} = \|w'_{s,a} \tilde{u}\|_{L^2_{\tau,\xi}}.$$

Note that $P_j(\tau, \xi) = Q_j(-\tau, -\xi)$ and $\|\tilde{u}\|_{Z^{s,a}} = \|u\|_{\bar{Z}^{s,a}}$.

The following lemmas are basic tools of the Fourier restriction norm method.

Lemma 2.1. *Let $0 \leq p \leq q$ and $p + q > 1$. Then the following estimate holds for all $a, b \in \mathbb{R}$;*

$$\int \langle \tau - a \rangle^{-p} \langle \tau - b \rangle^{-q} d\tau \lesssim \langle a - b \rangle^{-r}$$

where $r = p - [1 - q]_+$. (We recall that $[\lambda]_+ = \lambda$ if $\lambda > 0$, $= \varepsilon > 0$ if $\lambda = 0$ and $= 0$ if $\lambda < 0$).

For the proof of this lemma, see Lemma 4.2 in [3].

For a subset $\Omega \subset \mathbb{R}^4$, we define the characteristic function χ_Ω as follows;

$$\chi_\Omega(\tau, \xi, \tau_1, \xi_1) = \begin{cases} 1, & \text{for } (\tau, \xi, \tau_1, \xi_1) \in \Omega \\ 0, & \text{for } (\tau, \xi, \tau_1, \xi_1) \notin \Omega \end{cases}$$

and put

$$\widetilde{B_\Omega(u, v)} := \int_{\mathbb{R}^2} \chi_\Omega \widetilde{u}(\tau - \tau_1, \xi - \xi_1) \widetilde{v}(\tau_1, \xi_1) d\tau_1 d\xi_1.$$

Lemma 2.2. *If*

$$\sup_{\tau, \xi} \int_{\mathbb{R}^2} \chi_\Omega w_1^{-2}(\tau, \xi) w_2^{-2}(\tau - \tau_1, \xi - \xi_1) w_3^{-2}(\tau_1, \xi_1) d\tau_1 d\xi_1 \lesssim 1$$

or

$$\sup_{\tau_1, \xi_1} \int_{\mathbb{R}^2} \chi_\Omega w_1^{-2}(\tau, \xi) w_2^{-2}(\tau - \tau_1, \xi - \xi_1) w_3^{-2}(\tau_1, \xi_1) d\tau d\xi \lesssim 1$$

hold for measurable functions w_1, w_2 and w_3 on $\mathbb{R}^2$, then we have

$$\|w_1^{-1} \widetilde{B_\Omega(u, v)}\|_{L_{\tau, \xi}^2} \lesssim \|w_2 \widetilde{u}\|_{L_{\tau, \xi}^2} \|w_3 \widetilde{v}\|_{L_{\tau, \xi}^2}.$$

Lemma 2.3. *If*

$$\sup_{\xi} \int_{\mathbb{R}^2} \chi_\Omega w_1^{-2}(\tau, \xi) w_2^{-2}(\tau - \tau_1, \xi - \xi_1) w_3^{-2}(\tau_1, \xi_1) d\tau_1 d\xi_1 d\tau \lesssim 1$$

or

$$\sup_{\xi_1} \int_{\mathbb{R}^2} \chi_\Omega w_1^{-2}(\tau, \xi) w_2^{-2}(\tau - \tau_1, \xi - \xi_1) w_3^{-2}(\tau_1, \xi_1) d\tau d\xi d\tau_1 \lesssim 1$$

hold for measurable functions w_1, w_2 and w_3 on $\mathbb{R}^2$, then we have

$$\|\int w_1^{-1} \widetilde{B_\Omega(u, v)} d\tau\|_{L_\xi^2} \lesssim \|w_2 \widetilde{u}\|_{L_{\tau, \xi}^2} \|w_3 \widetilde{v}\|_{L_{\tau, \xi}^2}.$$

For the proof of Lemmas 2.2, 2.3, see Section 3 in [3].

Let $\widehat{P_l f} = \widehat{f}|_{|\xi| < 1}$ and $\langle \cdot, \cdot \rangle_{L^2}$ be the inner product in L^2 . The following lemma is a variant of the Sobolev inequality.

Lemma 2.4. (i) *Let $b_1 + b_2 + b_3 > 1/2, b_1 \geq 0, b_2 \geq 0, b_3 \geq 0$ and $\alpha, \beta, \gamma \in \mathbb{R}$. Then, we have*

$$(2.2) \quad \langle fg, h \rangle_{L_x^2} \lesssim \|\langle \xi - \alpha \rangle^{b_1} \widehat{f}\|_{L_\xi^2} \|\langle \xi - \beta \rangle^{b_2} \widehat{g}\|_{L_\xi^2} \|\langle \xi - \gamma \rangle^{b_3} \widehat{h}\|_{L_\xi^2}$$

where implicit constant depends only on b_1, b_2 and b_3 .

(ii) *Let $s_1 + s_2 + s_3 > 1/2, s_1 + s_2 \geq 0, s_2 + s_3 \geq 0$ and $s_3 + s_1 \geq 0$. Then, we have*

$$(2.3) \quad \langle fg, h \rangle_{L_x^2} \lesssim \|\langle \xi \rangle^{s_1} \widehat{f}\|_{L_\xi^2} \|\langle \xi \rangle^{s_2} \widehat{g}\|_{L_\xi^2} \|\langle \xi \rangle^{s_3} \widehat{h}\|_{L_\xi^2}$$

where implicit constant depends only on s_1, s_2 and s_3 .

(iii) *Let $-1/2 < a < 1/2$. Then, we have*

$$(2.4) \quad \langle (P_l f)g, h \rangle_{L_x^2} \lesssim \| |\xi|^a \widehat{f} \|_{L_\xi^2} \| \widehat{g} \|_{L_\xi^2} \| \widehat{h} \|_{L_\xi^2},$$

$$(2.5) \quad \langle (P_l f)(P_l g), h \rangle_{L_x^2} \lesssim \| |\xi|^a \widehat{f} \|_{L_\xi^2} \| |\xi|^{-a} \widehat{g} \|_{L_\xi^2} \| \widehat{h} \|_{L_\xi^2},$$

$$(2.6) \quad \langle (P_l f)(P_l g), P_l h \rangle_{L_x^2} \lesssim \| |\xi|^a \widehat{f} \|_{L_\xi^2} \| |\xi|^a \widehat{g} \|_{L_\xi^2} \| |\xi|^{-a} \widehat{h} \|_{L_\xi^2},$$

where the implicit constants depend only on a .

Proof. By the Plancherel theorem, the Hölder inequality and the Young inequality, we have

$$\begin{aligned} \langle fg, h \rangle_{L_x^2} &\sim \langle \widehat{f} * \widehat{g}, \widehat{h} \rangle_{L_\xi^2} \lesssim \|\widehat{f}\|_{L_\xi^{p_1}} \|\widehat{g}\|_{L_\xi^{p_2}} \|\widehat{h}\|_{L_\xi^{p_3}} \\ &\lesssim \|\langle \xi - \alpha \rangle^{-b_1}\|_{L_\xi^{q_1}} \|\langle \xi - \beta \rangle^{-b_2}\|_{L_\xi^{q_2}} \|\langle \xi - \gamma \rangle^{-b_3}\|_{L_\xi^{q_3}} \\ &\quad \times \|\langle \xi - \alpha \rangle^{b_1} \widehat{f}\|_{L_\xi^2} \|\langle \xi - \beta \rangle^{b_2} \widehat{g}\|_{L_\xi^2} \|\langle \xi - \gamma \rangle^{b_3} \widehat{h}\|_{L_\xi^2}, \end{aligned}$$

for any $1 \leq p_j \leq 2$ and $2 \leq q_j \leq \infty$ satisfying $1/p_1 + 1/p_2 + 1/p_3 = 2$ and $1/q_j + 1/2 = 1/p_j$. Since $b_1 + b_2 + b_3 > 1/2$ and $1/q_1 + 1/q_2 + 1/q_3 = 1/2$, we can take q_j such that $q_j > 1/b_j$ for $b_j > 0$ and $q_j = \infty$ for $b_j = 0$. Thus, we obtain (2.2).

For the proof of (2.3), we can assume $s_1 \geq s_2 \geq s_3$ without loss of generality. Since the case $s_3 \geq 0$ follows from (2.2), we only need to show the case $s_2 \geq 0 > s_3$. By using the triangle inequality $\langle \xi \rangle \leq \langle \xi_1 \rangle + \langle \xi - \xi_1 \rangle$ and the Plancherel theorem, we have

$$\begin{aligned} \langle fg, h \rangle_{L_x^2} &\sim \left\langle \int \widehat{f}(\xi - \xi_1) \widehat{g}(\xi_1) d\xi_1, \widehat{h}(\xi) \right\rangle_{L_\xi^2} \\ &\lesssim \left\langle \int \widehat{f}(\xi - \xi_1) \langle \xi_1 \rangle^{-s_3} \widehat{g}(\xi_1) d\xi_1, \langle \xi \rangle^{s_3} \widehat{h}(\xi) \right\rangle_{L_\xi^2} \\ &\quad + \left\langle \int \langle \xi - \xi_1 \rangle^{-s_3} \widehat{f}(\xi - \xi_1) \widehat{g}(\xi_1) d\xi_1, \langle \xi \rangle^{s_3} \widehat{h}(\xi) \right\rangle_{L_\xi^2}. \end{aligned}$$

Therefore, this case also follows from (2.2).

By the Plancherel theorem, the Hölder inequality and the Young inequality, we have

$$\langle (P_l f)g, h \rangle_{L_x^2} \sim \langle \widehat{P_l f} * \widehat{g}, \widehat{h} \rangle_{L_\xi^2} \lesssim \|\widehat{P_l f}\|_{L_\xi^1} \|\widehat{g}\|_{L_\xi^2} \|\widehat{h}\|_{L_\xi^2}.$$

Since $\|\widehat{P_l f}\|_{L_\xi^1} \leq \| |\xi|^{-a} \|_{L_\xi^2(-1,1)} \| |\xi|^a \widehat{f} \|_{L_\xi^2} \lesssim \| |\xi|^a \widehat{f} \|_{L_\xi^2}$, we obtain (2.4).

For the proof of (2.5), we can assume $a \geq 0$ without loss of generality. From (2.4), we have

$$\langle (P_l f)(P_l g), h \rangle_{L_x^2} \lesssim \| |\xi|^a \widehat{f} \|_{L_\xi^2} \|\widehat{P_l g}\|_{L_\xi^2} \|\widehat{h}\|_{L_\xi^2}.$$

Since $\|\widehat{P_l g}\|_{L_\xi^2} \leq \| |\xi|^{-a} \widehat{g} \|_{L_\xi^2}$, we obtain (2.5).

For the proof of (2.6), we can assume $a \geq 0$ without loss of generality. From the Plancherel theorem, we have

$$\langle (P_l f)(P_l g), P_l h \rangle_{L_x^2} \sim \left\langle \int \widehat{P_l f}(\xi - \xi_1) \widehat{P_l g}(\xi_1) d\xi_1, \widehat{P_l h}(\xi) \right\rangle_{L_\xi^2}.$$

Since $\max\{|\xi - \xi_1|^a, |\xi_1|^a\} \gtrsim |\xi|^a$, (2.6) follows from (2.4). $\square$

From this lemma, we obtain the following space time estimates.

Proposition 2.5. *Let $b_1 + b_2 + b_3 > 1/2$, $b_1 \geq 0$, $b_2 \geq 0$, $b_3 \geq 0$ and $i, j, k = 1$ or -1 .*

(i) Moreover, we assume that $s_1 + s_2 + s_3 > 1/2$, $s_1 + s_2 \geq 0$, $s_2 + s_3 \geq 0$ and $s_3 + s_1 \geq 0$. Then, we have

$$\begin{aligned} (2.7) \quad \langle fg, h \rangle_{L_{t,x}^2} &\lesssim \|\langle \xi \rangle^{s_1} \langle \tau - i\xi^2 \rangle^{b_1} \widetilde{f}\|_{L_{\tau,\xi}^2} \|\langle \xi \rangle^{s_2} \langle \tau - j\xi^2 \rangle^{b_2} \widetilde{g}\|_{L_{\tau,\xi}^2} \|\langle \xi \rangle^{s_3} \langle \tau - k\xi^2 \rangle^{b_3} \widetilde{h}\|_{L_{\tau,\xi}^2}. \end{aligned}$$

(ii) Moreover, we assume $-1/2 < a < 1/2$. Then, we have

$$(2.8) \quad \begin{aligned} & \langle (P_l f)g, h \rangle_{L_{t,x}^2} \\ & \lesssim \| |\xi|^a \langle \tau - i\xi^2 \rangle^{b_1} \tilde{f} \|_{L_{\tau,\xi}^2} \| \langle \tau - j\xi^2 \rangle^{b_2} \tilde{g} \|_{L_{\tau,\xi}^2} \| \langle \tau - k\xi^2 \rangle^{b_3} \tilde{h} \|_{L_{\tau,\xi}^2}, \end{aligned}$$

$$(2.9) \quad \begin{aligned} & \langle (P_l f)(P_l g), h \rangle_{L_{t,x}^2} \\ & \lesssim \| |\xi|^a \langle \tau - i\xi^2 \rangle^{b_1} \tilde{f} \|_{L_{\tau,\xi}^2} \| |\xi|^{-a} \langle \tau - j\xi^2 \rangle^{b_2} \tilde{g} \|_{L_{\tau,\xi}^2} \| \langle \tau - k\xi^2 \rangle^{b_3} \tilde{h} \|_{L_{\tau,\xi}^2}, \end{aligned}$$

$$(2.10) \quad \begin{aligned} & \langle (P_l f)(P_l g), P_l h \rangle_{L_{t,x}^2} \\ & \lesssim \| |\xi|^a \langle \tau - i\xi^2 \rangle^{b_1} \tilde{f} \|_{L_{\tau,\xi}^2} \| |\xi|^a \langle \tau - j\xi^2 \rangle^{b_2} \tilde{g} \|_{L_{\tau,\xi}^2} \| |\xi|^{-a} \langle \tau - k\xi^2 \rangle^{b_3} \tilde{h} \|_{L_{\tau,\xi}^2}. \end{aligned}$$

Proof. Fix $\xi, \xi_1 \in \mathbb{R}$. Then, from (2.2), we have

$$\begin{aligned} & \int \tilde{f}(\tau_1, \xi_1) \tilde{g}(\tau - \tau_1, \xi - \xi_1) \tilde{h}(\tau, \xi) d\tau_1 d\tau \\ & \lesssim \| \langle \cdot - i\xi_1^2 \rangle^{b_1} \tilde{f}(\cdot, \xi_1) \|_{L^2} \| \langle \cdot - j(\xi - \xi_1)^2 \rangle^{b_2} \tilde{g}(\cdot, \xi - \xi_1) \|_{L^2} \| \langle \cdot - k\xi^2 \rangle^{b_3} \tilde{h}(\cdot, \xi) \|_{L^2} \end{aligned}$$

where implicit constant does not depend on ξ, ξ_1 . Therefore, the left-hand side of (2.7) is bounded by

$$\int \| \langle \cdot - i\xi_1^2 \rangle^{b_1} \tilde{f}(\cdot, \xi_1) \|_{L^2} \| \langle \cdot - j(\xi - \xi_1)^2 \rangle^{b_2} \tilde{g}(\cdot, \xi - \xi_1) \|_{L^2} \| \langle \cdot - k\xi^2 \rangle^{b_3} \tilde{h}(\cdot, \xi) \|_{L^2} d\xi_1 d\xi,$$

which is bounded by the right-hand side of (2.7) by (2.3). In the same manner, (2.8)–(2.10) follow from (2.2), (2.4)–(2.6). $\square$

3. BILINEAR ESTIMATES

Proposition 3.1. *Let $0 > s \geq -(2a + 1)/4$ and $1/2 > a > -1/2$. Then the following estimates hold;*

$$(3.1) \quad \| \mathcal{F}^{-1} \langle \tau - \xi^2 \rangle^{-1} \widetilde{u\bar{v}} \|_{Z^{s,a}} \lesssim \| u \|_{Z^{s,a}} \| v \|_{Z^{s,a}},$$

$$(3.2) \quad \| \mathcal{F}^{-1} \langle \tau - \xi^2 \rangle^{-1} \widetilde{uv} \|_{Y^{s,a}} \lesssim \| u \|_{Z^{s,a}} \| v \|_{Z^{s,a}}.$$

Moreover, the same estimates hold with $u\bar{v}$ replaced by uv or $\bar{u}\bar{v}$.

We prove only the case $u\bar{u}$ because the case uv and $\bar{u}\bar{v}$ are easier.

Proof. We first consider (3.1), which is equivalent to

$$\| \mathcal{F}^{-1} \langle \tau - \xi^2 \rangle^{-1} \widetilde{u\bar{v}} \|_{Z^{s,a}} \lesssim \| u \|_{Z^{s,a}} \| v \|_{\bar{Z}^{s,a}}.$$

Put

$$\Omega_{i,j,k} = \{ (\tau, \xi, \tau_1, \xi_1) | (\tau, \xi) \in P_i, (\tau - \tau_1, \xi - \xi_1) \in P_j, (\tau_1, \xi_1) \in Q_k \}$$

for $i, j, k = 1$ or 2 . Then, we have

$$B_{\mathbb{R}^4}(u, v) = \sum_{i,j,k} B_{\Omega_{i,j,k}}(u, v).$$

Therefore, we only need to show

$$(3.3) \quad \| \mathcal{F}^{-1} \langle \tau - \xi^2 \rangle^{-1} \widetilde{B_{\Omega}(u, v)} \|_{Z^{s,a}} \lesssim \| u \|_{Z^{s,a}} \| v \|_{\bar{Z}^{s,a}}$$

with $\Omega = \Omega_{i,j,k}$ for $i, j, k = 1$ or 2 . Put $M_1 = \max\{|\tau - \xi^2|, |\tau - \tau_1 - (\xi - \xi_1)^2|, |\tau_1 + \xi_1^2|\}$. Then, we have the following algebraic property;

$$M_1 \geq (|\tau - \xi^2| + |\tau - \tau_1 - (\xi - \xi_1)^2| + |\tau_1 + \xi_1^2|)/3 \geq 2|\xi\xi_1|/3,$$

which plays an important role in our proof.

(a-1) We prove that $\Omega_{1,1,1}$ is empty. If $M_1 = |\tau - \xi^2|$ and $(\tau, \xi) \in P_1$, then $2|\xi\xi_1|/3 \leq M_1 \leq |\xi|/4$. Therefore, we have $|\xi_1| \leq 3/8$, which contradicts $(\tau_1, \xi_1) \in Q_1$. If $M_1 = |\tau_1 + \xi_1^2|$ and $(\tau_1, \xi_1) \in Q_1$, then $2|\xi\xi_1|/3 \leq M_1 \leq |\xi_1|/4$. Therefore, we have $|\xi| \leq 3/8$, which contradicts $(\tau, \xi) \in P_1$. If $M_1 = |\tau - \tau_1 - (\xi - \xi_1)^2|$ and $(\tau - \tau_1, \xi - \xi_1) \in P_1$, then $2|\xi\xi_1|/3 \leq M_1 \leq |\xi - \xi_1|/4 \leq \max\{|\xi|, |\xi_1|\}/2$. Therefore, we have $|\xi| \leq 3/4$ or $|\xi_1| \leq 3/4$, which contradicts $(\tau, \xi) \in P_1$ and $(\tau_1, \xi_1) \in Q_1$. Thus, we obtain (3.3) with $\Omega = \Omega_{1,1,1}$.

(a-2) (3.3) with $\Omega = \Omega_{2,1,1}$ is equivalent to

$$(3.4) \quad \begin{aligned} & \|\langle \xi \rangle^{1/2-a} |\xi|^a \langle \tau - \xi^2 \rangle^{-1/2+s} \widetilde{B_{\Omega_{2,1,1}}}(u, v)\|_{L^2_{\tau, \xi}} \\ & \lesssim \|\langle \xi \rangle^s \langle \tau - \xi^2 \rangle \widetilde{u}\|_{L^2_{\tau, \xi}} \|\langle \xi \rangle^s \langle \tau + \xi^2 \rangle \widetilde{v}\|_{L^2_{\tau, \xi}}. \end{aligned}$$

We devide $\Omega_{2,1,1}$ into two parts;

$$\begin{aligned} A_1 &= \{(\tau, \xi, \tau_1, \xi_1) \in \Omega_{2,1,1} \mid |\xi| < 1\}, \\ A_2 &= \{(\tau, \xi, \tau_1, \xi_1) \in \Omega_{2,1,1} \mid |\xi| \geq 1\}. \end{aligned}$$

From Lemma 2.2, (3.4) with $\Omega_{2,1,1}$ replaced by A_1 can be reduced to

$$\sup_{\tau_1, \xi_1} \int \frac{\chi_{A_1} \langle \xi_1 \rangle^{-2s} |\xi|^{2a} \langle \xi - \xi_1 \rangle^{-2s}}{\langle \tau_1 + \xi_1^2 \rangle^2 \langle \tau - \xi^2 \rangle^{1-2s} \langle \tau - \tau_1 - (\xi - \xi_1)^2 \rangle^2} d\tau d\xi \lesssim 1.$$

Since $\langle M_1 \rangle \sim \langle \xi\xi_1 \rangle$ and $\langle \xi_1 \rangle \sim \langle \xi - \xi_1 \rangle \sim |\xi_1|$, from Lemma 2.1, the left hand side is bounded by

$$\int \frac{|\xi|^{2a} |\xi_1|^{-4s}}{\langle M_1 \rangle^{1-2s}} d\xi \lesssim \int \frac{|\xi\xi_1|^{-4s-1}}{\langle \xi\xi_1 \rangle^{1-2s}} |\xi_1| d\xi \lesssim \int \frac{|p|^{-4s-1}}{\langle p \rangle^{1-2s}} dp \lesssim 1.$$

Here, we put $p = \xi\xi_1$ and used $2a \geq -4s - 1$ and $1 - 2s > -4s$.

From Lemma 2.2, (3.4) with $\Omega_{2,1,1}$ replaced by A_2 can be reduced to

$$\sup_{\tau, \xi} \int \frac{\chi_{A_2} |\xi| \langle \xi_1 \rangle^{-2s} \langle \xi - \xi_1 \rangle^{-2s}}{\langle \tau - \xi^2 \rangle^{1-2s} \langle \tau_1 + \xi_1^2 \rangle^2 \langle \tau - \tau_1 - (\xi - \xi_1)^2 \rangle^2} d\tau_1 d\xi_1 \lesssim 1.$$

In the same manner as (a-1), it follows that $M_1 = \langle \tau - \xi^2 \rangle \sim \langle \xi\xi_1 \rangle$ from $(\tau - \tau_1, \xi - \xi_1) \in P_1, (\tau_1, \xi_1) \in Q_1$ and $|\xi| \geq 1$. Therefore, from Lemma 2.1, the left hand side is bounded by

$$\int \frac{\langle \xi_1 \rangle^{-2s} \langle \xi - \xi_1 \rangle^{-2s}}{\langle \xi\xi_1 \rangle^{1-2s} \langle \tau - \xi^2 + 2\xi\xi_1 \rangle^2} |\xi| d\xi_1 \lesssim \int \frac{\langle p \rangle^{-4s}}{\langle p \rangle^{1-2s} \langle \tau - \xi^2 + 2p \rangle^2} dp \lesssim 1.$$

Here, we put $p = \xi\xi_1$ and used $1 - 2s \geq -4s$.

(a-3) (3.3) with $\Omega = \Omega_{1,2,1}$ is equivalent to

$$(3.5) \quad \|\langle \xi \rangle^s \widetilde{B_{\Omega_{1,2,1}}}(u, v)\|_{L^2_{\tau, \xi}} \lesssim \|\langle \xi \rangle^{1/2-a} |\xi|^a \langle \tau - \xi^2 \rangle^{1/2+s} \widetilde{u}\|_{L^2_{\tau, \xi}} \|\langle \xi \rangle^s \langle \tau + \xi^2 \rangle \widetilde{v}\|_{L^2_{\tau, \xi}}.$$

We devide $\Omega_{1,2,1}$ into two parts;

$$A_1 = \{(\tau, \xi, \tau_1, \xi_1) \in \Omega_{1,2,1} \mid |\xi - \xi_1| < 1\},$$

$$A_2 = \{(\tau, \xi, \tau_1, \xi_1) \in \Omega_{1,2,1} \mid |\xi - \xi_1| \geq 1\}.$$

Since $\langle \xi \rangle \sim \langle \xi_1 \rangle$ and $\langle \xi - \xi_1 \rangle \sim 1$ in A_1 , (3.5) with $\Omega_{1,2,1}$ replaced by A_1 can be reduced to

$$\|(\widetilde{Pl}u)v\|_{L^2_{\tau,\xi}} \lesssim \| |\xi|^a \langle \tau - \xi^2 \rangle^{1/2+s} \widetilde{u} \|_{L^2_{\tau,\xi}} \| \langle \tau + \xi^2 \rangle \widetilde{v} \|_{L^2_{\tau,\xi}},$$

which follows from the duality argument and (2.8) in Proposition 2.5. Since $\langle \tau - \tau_1 - (\xi - \xi_1)^2 \rangle \gtrsim \langle \xi - \xi_1 \rangle$ in A_2 , (3.5) with $\Omega_{1,2,1}$ replaced by A_2 can be reduced to

$$\| \langle \xi \rangle^s \widetilde{u} \widetilde{v} \|_{L^2_{\tau,\xi}} \lesssim \| \langle \xi \rangle^{1+s} \widetilde{u} \|_{L^2_{\tau,\xi}} \| \langle \xi \rangle^s \langle \tau + \xi^2 \rangle \widetilde{v} \|_{L^2_{\tau,\xi}},$$

which follows from the duality argument and (2.7) in Proposition 2.5.

(a-4) (3.3) with $\Omega = \Omega_{2,2,1}$ is equivalent to

$$(3.6) \quad \begin{aligned} & \| \langle \xi \rangle^{1/2-a} |\xi|^a \langle \tau - \xi^2 \rangle^{-1/2+s} \widetilde{B_{\Omega_{2,2,1}}}(u, v) \|_{L^2_{\tau,\xi}} \\ & \lesssim \| \langle \xi \rangle^{1/2-a} |\xi|^a \langle \tau - \xi^2 \rangle^{1/2+s} \widetilde{u} \|_{L^2_{\tau,\xi}} \| \langle \xi \rangle^s \langle \tau + \xi^2 \rangle \widetilde{v} \|_{L^2_{\tau,\xi}}. \end{aligned}$$

We devide $\Omega_{2,2,1}$ into four parts;

$$A_1 = \{(\tau, \xi, \tau_1, \xi_1) \in \Omega_{2,2,1} \mid |\xi| < 1, |\xi - \xi_1| < 1\},$$

$$A_2 = \{(\tau, \xi, \tau_1, \xi_1) \in \Omega_{2,2,1} \mid |\xi| < 1, |\xi - \xi_1| \geq 1\},$$

$$A_3 = \{(\tau, \xi, \tau_1, \xi_1) \in \Omega_{2,2,1} \mid |\xi| \geq 1, |\xi - \xi_1| < 1\},$$

$$A_4 = \{(\tau, \xi, \tau_1, \xi_1) \in \Omega_{2,2,1} \mid |\xi| \geq 1, |\xi - \xi_1| \geq 1\}.$$

Since $\langle \xi \rangle \sim \langle \xi_1 \rangle \sim \langle \xi - \xi_1 \rangle \sim 1$ in A_1 , (3.6) with $\Omega_{2,2,1}$ replaced by A_1 can be reduced to

$$\| |\xi|^a \langle \tau - \xi^2 \rangle^{-1/2+s} \widetilde{Pl\{(\widetilde{Pl}u)v\}} \|_{L^2_{\tau,\xi}} \lesssim \| |\xi|^a \langle \tau - \xi^2 \rangle^{1/2+s} \widetilde{u} \|_{L^2_{\tau,\xi}} \| \langle \tau + \xi^2 \rangle \widetilde{v} \|_{L^2_{\tau,\xi}},$$

which follows from the duality argument and (2.9) in Proposition 2.5.

Since $\langle \xi \rangle \sim 1$ and $\langle \xi_1 \rangle \sim \langle \xi - \xi_1 \rangle$ in A_2 , (3.6) with $\Omega_{2,2,1}$ replaced by A_2 can be reduced to

$$\| |\xi|^a \langle \tau - \xi^2 \rangle^{-1/2+s} \widetilde{Pl(uv)} \|_{L^2_{\tau,\xi}} \lesssim \| \langle \xi \rangle^{1/2+s} \langle \tau - \xi^2 \rangle^{1/2+s} \widetilde{u} \|_{L^2_{\tau,\xi}} \| \langle \tau + \xi^2 \rangle \widetilde{v} \|_{L^2_{\tau,\xi}},$$

which follows from (2.8) in Proposition 2.5.

Since $|\tau - \xi^2| \sim |\xi| \sim |\xi_1| \gtrsim 1$, $\langle \xi - \xi_1 \rangle \sim 1$ in A_3 , (3.6) with $\Omega_{2,2,1}$ replaced by A_3 can be reduced to

$$\| \widetilde{uv} \|_{L^2_{\tau,\xi}} \lesssim \| |\xi|^a \langle \tau - \xi^2 \rangle^{1/2+s} \widetilde{u} \|_{L^2_{\tau,\xi}} \| \langle \tau + \xi^2 \rangle \widetilde{v} \|_{L^2_{\tau,\xi}},$$

which follows from (2.8) in Proposition 2.5.

Since $|\xi| \sim |\xi_1| \gtrsim 1$, $\langle \tau - \xi^2 \rangle \gtrsim \langle \xi \rangle$ and $\langle \tau - \tau_1 - (\xi - \xi_1)^2 \rangle \gtrsim \langle \xi - \xi_1 \rangle$ in A_4 , (3.6) with $\Omega_{2,2,1}$ replaced by A_4 can be reduced to

$$\| \langle \xi \rangle^s \widetilde{uv} \|_{L^2_{\tau,\xi}} \lesssim \| \langle \xi \rangle^{1+s} \widetilde{u} \|_{L^2_{\tau,\xi}} \| \langle \xi \rangle^s \langle \tau + \xi^2 \rangle \widetilde{v} \|_{L^2_{\tau,\xi}},$$

which follows from (2.7) in Proposition 2.5.

(a-5) We can prove (3.3) with $\Omega = \Omega_{1,1,2}$ in the same manner as (a-3).

(a-6) We can prove (3.3) with $\Omega = \Omega_{2,1,2}$ in the same manner as (a-4).

(a-7) Since $\langle \xi \rangle^s \leq \langle \tau - \xi^2 \rangle^s$ in $\Omega_{1,2,2}$, (3.3) with $\Omega = \Omega_{1,2,2}$ can be reduced to

$$(3.7) \quad \begin{aligned} & \|\langle \tau - \xi^2 \rangle^s \widetilde{B_{\Omega_{1,2,2}}}(u, v)\|_{L_{\tau, \xi}^2} \\ & \lesssim \|\langle \xi \rangle^{1/2-a} |\xi|^a \langle \tau - \xi^2 \rangle^{1/2+s} \widetilde{u}\|_{L_{\tau, \xi}^2} \|\langle \xi \rangle^{1/2-a} |\xi|^a \langle \tau + \xi^2 \rangle^{1/2+s} \widetilde{v}\|_{L_{\tau, \xi}^2}. \end{aligned}$$

We devide $\Omega_{1,2,2}$ into three parts;

$$\begin{aligned} A_1 &= \{(\tau, \xi, \tau_1, \xi_1) \in \Omega_{1,2,2} \mid |\xi - \xi_1| < 1/2, |\xi_1| \geq 1/2\}, \\ A_2 &= \{(\tau, \xi, \tau_1, \xi_1) \in \Omega_{1,2,2} \mid |\xi - \xi_1| \geq 1/2, |\xi| < 1/2\}, \\ A_3 &= \{(\tau, \xi, \tau_1, \xi_1) \in \Omega_{1,2,2} \mid |\xi - \xi_1| \geq 1/2, |\xi| \geq 1/2\}. \end{aligned}$$

(3.7) with $\Omega_{1,2,2}$ replaced by A_1 or A_2 follow from (2.8) in Proposition 2.5 and (3.7) with $\Omega_{1,2,2}$ replaced by A_3 follows from (2.7) in Proposition 2.5.

(a-8) (3.3) with $\Omega = \Omega_{2,2,2}$ is equivalent to

$$(3.8) \quad \begin{aligned} & \|\langle \xi \rangle^{1/2-a} |\xi|^a \langle \tau - \xi^2 \rangle^{-1/2+s} \widetilde{B_{\Omega_{2,2,2}}}(u, v)\|_{L_{\tau, \xi}^2} \\ & \lesssim \|\langle \xi \rangle^{1/2-a} |\xi|^a \langle \tau - \xi^2 \rangle^{1/2+s} \widetilde{u}\|_{L_{\tau, \xi}^2} \|\langle \xi \rangle^{1/2-a} |\xi|^a \langle \tau + \xi^2 \rangle^{1/2+s} \widetilde{v}\|_{L_{\tau, \xi}^2}. \end{aligned}$$

We devide $\Omega_{2,2,2}$ into seven parts;

$$\begin{aligned} A_1 &= \{(\tau, \xi, \tau_1, \xi_1) \in \Omega_{2,2,2} \mid |\xi| < 1, |\xi - \xi_1| < 1/2, |\xi_1| < 1/2\}, \\ A_2 &= \{(\tau, \xi, \tau_1, \xi_1) \in \Omega_{2,2,2} \mid |\xi| < 1, |\xi - \xi_1| \geq 1/2, |\xi_1| < 1/2\}, \\ A_3 &= \{(\tau, \xi, \tau_1, \xi_1) \in \Omega_{2,2,2} \mid |\xi| < 1, |\xi - \xi_1| < 1/2, |\xi_1| \geq 1/2\}, \\ A_4 &= \{(\tau, \xi, \tau_1, \xi_1) \in \Omega_{2,2,2} \mid |\xi| < 1, |\xi - \xi_1| \geq 1/2, |\xi_1| \geq 1/2\}, \\ A_5 &= \{(\tau, \xi, \tau_1, \xi_1) \in \Omega_{2,2,2} \mid |\xi| \geq 1, |\xi - \xi_1| \geq 1/2, |\xi_1| < 1/2\}, \\ A_6 &= \{(\tau, \xi, \tau_1, \xi_1) \in \Omega_{2,2,2} \mid |\xi| \geq 1, |\xi - \xi_1| < 1/2, |\xi_1| \geq 1/2\}, \\ A_7 &= \{(\tau, \xi, \tau_1, \xi_1) \in \Omega_{2,2,2} \mid |\xi| \geq 1, |\xi - \xi_1| \geq 1/2, |\xi_1| \geq 1/2\}. \end{aligned}$$

(3.8) with $\Omega_{2,2,2}$ replaced by A_1 follows from (2.10) in Proposition 2.5, (3.8) with $\Omega_{2,2,2}$ replaced by A_2 or A_3 follow from (2.9) in Proposition 2.5 and (3.8) with $\Omega_{2,2,2}$ replaced by A_4 or A_5 or A_6 follow from (2.8) in Proposition 2.5. Since $\langle \xi \rangle^{1/2-a} |\xi|^a \langle \tau - \xi^2 \rangle^{-1/2+s} \leq \langle \tau - \xi^2 \rangle^s$ in A_7 , (3.8) with $\Omega_{2,2,2}$ replaced by A_7 can be reduced to

$$\|\langle \tau - \xi^2 \rangle^s \widetilde{u\widetilde{v}}\|_{L_{\tau, \xi}^2} \lesssim \|\langle \xi \rangle^{1/2} \langle \tau - \xi^2 \rangle^{1/2+s} \widetilde{u}\|_{L_{\tau, \xi}^2} \|\langle \xi \rangle^{1/2} \langle \tau + \xi^2 \rangle^{1/2+s} \widetilde{v}\|_{L_{\tau, \xi}^2},$$

which follows from (2.7) in Proposition 2.5.

We next consider (3.2), which is equivalent to

$$\|\mathcal{F}^{-1} \langle \tau - \xi^2 \rangle^{-1} \widetilde{u\widetilde{v}}\|_{Y^{s,a}} \lesssim \|u\|_{Z^{s,a}} \|v\|_{\bar{Z}^{s,a}}$$

Because

$$\|\mathcal{F}^{-1} \langle \tau - \xi^2 \rangle^{-1} \widetilde{B_{\Omega}}(u, v)\|_{Y^{s,a}} \lesssim \|\mathcal{F}^{-1} \langle \tau - \xi^2 \rangle^{-1} \widetilde{B_{\Omega_{1,j,k}}}(u, v)\|_{X^{s,a}}$$

for $\Omega = \Omega_{1,j,k}$ with $j, k = 1$ or 2 , we only need to show

$$(3.9) \quad \|\mathcal{F}^{-1} \langle \tau - \xi^2 \rangle^{-1} \widetilde{B_{\Omega}}(u, v)\|_{Y^{s,a}} \lesssim \|u\|_{Z^{s,a}} \|v\|_{\bar{Z}^{s,a}}$$

for $\Omega = \Omega_{2,j,k}$ with $j, k = 1$ or 2 .

(b-1) We divide $\Omega_{2,1,1}$ into two parts;

$$A_1 = \{(\tau, \xi, \tau_1, \xi_1) \in \Omega_{2,1,1} \mid |\xi| < 1\},$$

$$A_2 = \{(\tau, \xi, \tau_1, \xi_1) \in \Omega_{2,1,1} \mid |\xi| \geq 1\}.$$

Since $\langle \xi_1 \rangle \sim \langle \xi - \xi_1 \rangle \sim |\xi_1| \gtrsim 1$ in A_1 , from Lemma 2.3, (3.9) with $\Omega = A_1$ can be reduced to

$$\sup_{\xi_1} \int_{A_1} \frac{|\xi|^{2a} |\xi_1|^{-4s}}{\langle \tau - \xi^2 \rangle^2 \langle \tau_1 + \xi_1^2 \rangle^2 \langle \tau - \tau_1 - (\xi - \xi_1)^2 \rangle^2} d\tau_1 d\tau d\xi \lesssim 1.$$

Since $\langle M_1 \rangle \sim \langle \xi \xi_1 \rangle$, from Lemma 2.1, the left hand side is bounded by

$$\int \frac{|\xi|^{2a} |\xi_1|^{-4s}}{\langle M_1 \rangle^2} d\xi \lesssim \int \frac{|\xi \xi_1|^{-4s-1}}{\langle \xi \xi_1 \rangle^2} \frac{1}{|\xi_1|} d\xi \lesssim \int \frac{|p|^{-4s-1}}{\langle p \rangle^2} dp \lesssim 1.$$

Here, we put $p = \xi \xi_1$ and used $2a \geq -4s - 1$ and $2 > -4s$.

From Lemma 2.3, (3.9) with $\Omega = A_2$ can be reduced to

$$\sup_{\xi_1} \int_{A_2} \frac{\langle \xi_1 \rangle^{-2s} \langle \xi \rangle^{2s} \langle \xi - \xi_1 \rangle^{-2s}}{\langle \tau - \xi^2 \rangle^2 \langle \tau_1 + \xi_1^2 \rangle^2 \langle \tau - \tau_1 - (\xi - \xi_1)^2 \rangle^2} d\tau_1 d\xi d\tau \lesssim 1.$$

Since $\langle M_1 \rangle \sim \langle \xi \xi_1 \rangle$, from Lemma 2.1, the left hand side is bounded by

$$\int \frac{\langle \xi_1 \rangle^{-2s} \langle \xi \rangle^{2s} (\langle \xi_1 \rangle^{-2s} + \langle \xi \rangle^{-2s})}{\langle \xi \xi_1 \rangle^2} d\xi \lesssim \int \frac{\langle \xi_1 \rangle^{-4s} \langle \xi \rangle^{2s}}{\langle \xi \xi_1 \rangle^2} d\xi + \int \frac{\langle \xi_1 \rangle^{-2s}}{\langle \xi \xi_1 \rangle^2} d\xi \lesssim 1.$$

(b-2) (3.9) with $\Omega = \Omega_{2,2,1}$ is equivalent to

$$\begin{aligned} & \left\| \int \langle \xi \rangle^{s-a} |\xi|^a \langle \tau - \xi^2 \rangle^{-1} \widetilde{B_{\Omega_{2,2,1}}}(u, v) d\tau \right\|_{L_\xi^2} \\ & \lesssim \|\langle \xi \rangle^{1/2-a} |\xi|^a \langle \tau - \xi^2 \rangle^{1/2+s} \widetilde{u}\|_{L_{\tau, \xi}^2} \|\langle \xi \rangle^s \langle \tau + \xi^2 \rangle \widetilde{v}\|_{L_{\tau, \xi}^2}, \end{aligned}$$

which follows from Proposition 2.5 in the same manner as (a-4) because the left-hand side is bounded by

$$\|\langle \xi \rangle^{s-a} |\xi|^a \langle \tau - \xi^2 \rangle^{-1/2+\varepsilon} \widetilde{B_{\Omega_{2,2,1}}}(u, v)\|_{L_{\tau, \xi}^2}$$

for any $\varepsilon > 0$.

(b-3) For $\Omega = \Omega_{2,1,2}$, we can prove the estimate in the same manner as (b-2).

(b-4) For $\Omega = \Omega_{2,2,2}$, we only need to show

$$\begin{aligned} & \left\| \int \langle \xi \rangle^{s-a} |\xi|^a \langle \tau - \xi^2 \rangle^{-1} \widetilde{B_{\Omega_{2,2,2}}}(u, v) d\tau \right\|_{L_\xi^2} \\ & \lesssim \|\langle \xi \rangle^{1/2-a} |\xi|^a \langle \tau - \xi^2 \rangle^{1/2+s} \widetilde{u}\|_{L_\xi^2} \|\langle \xi \rangle^{1/2-a} |\xi|^a \langle \tau + \xi^2 \rangle^{1/2+s} \widetilde{v}\|_{L_\xi^2}, \end{aligned}$$

which follows from Proposition 2.5 in the same manner as (a-8) because the left-hand side is bounded by

$$\|\langle \xi \rangle^{s-a} |\xi|^a \langle \tau - \xi^2 \rangle^{-1/2+\varepsilon} \widetilde{B_{\Omega_{2,2,2}}}(u, v)\|_{L_{\tau, \xi}^2}$$

for any $\varepsilon > 0$.

□

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