

ON p -ADIC FAMILIES OF THE D -TH SAITO-KUROKAWA LIFTS

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1. Introduction

Let p be an odd prime, N an odd positive integer not divisible by p and D the discriminant of an imaginary quadratic field satisfying $p \mid D$. We will establish p -adic interpolation of Fourier coefficients of the D -th Saito-Kurokawa lifts of primitive forms of level N varying in a Coleman family. This generalizes known results for Hida families of tame level $N = 1$ to the case of Coleman families of tame level N . The main theorem is Theorem 6.7.

Notation and terminology. Throughout the paper, we fix an odd prime p , a positive integer N satisfying $(N, 2p) = 1$ and a non-negative rational number α . We assume that $Np \geq 4$ to ensure that $\Gamma_1(Np)$ is torsion-free. We denote by $\mathbb{Q}$ and $\bar{\mathbb{Q}}_p$ an algebraic closure of the rational number field $\mathbb{Q}$, and the p -adic number field $\mathbb{Q}_p$, respectively. Let $\mathbb{C}$ be the complex number field and $\mathbb{C}_p$ the p -adic completion of $\bar{\mathbb{Q}}_p$. We fix two embeddings $i_\infty : \bar{\mathbb{Q}} \hookrightarrow \mathbb{C}$ and $i_p : \bar{\mathbb{Q}} \hookrightarrow \mathbb{C}_p$, and an isomorphism $\mathbb{C}_p \xrightarrow{\sim} \mathbb{C}$ which commutes with i_∞ and i_p . Let val_p be the normalized p -adic additive valuation on $\mathbb{C}_p$ so that $\text{val}_p(p) = 1$. For $z \in \mathbb{C}$, we define $\sqrt{z} = z^{1/2}$ so that $\pi/2 < \arg(z^{1/2}) \leq \pi/2$ and put $z^{k/2} := (\sqrt{z})^k$ for each integer k . We put $e(z) := \exp(2\pi\sqrt{-1}z)$ and $e^m(z) := e(mz)$. For a Dirichlet character χ , we denote by c_χ the conductor of χ , χ_0 the primitive character attached to χ and $G(\chi) := \sum_{i=0}^{c_\chi-1} \chi_0(i)e(i/c_\chi)$. For a non-zero integer a , we let χ_a denote the *Kronecker symbol* $\chi_a(b) := (\frac{a}{b})$ defined by [MFM, (3.1.9)]. We call D a *fundamental discriminant* if D is either 1 or the discriminant of a quadratic field. We denote by $\mathbb{1}_M$ the trivial Dirichlet character modulo M , i.e., for any integer n , $\mathbb{1}_M(n) := 1$ if $(n, M) = 1$ and $\mathbb{1}_M(n) := 0$ otherwise. By $d \parallel n$, we mean $d \mid n$ and $(d, n/d) = 1$.

2. Siegel cusp forms

Let g be a positive integer. Note that our concern is only for $g = 1, 2$.

2.1. Definition of Siegel cusp forms

Put $1_g := \text{diag}(1, \dots, 1)$, $0_g := \text{diag}(0, \dots, 0) \in M_g(\mathbb{Z})$ and

$$(2-1-1) \quad J_g := \begin{pmatrix} 0_g & -1_g \\ 1_g & 0_g \end{pmatrix} \in \text{GL}_{2g}(\mathbb{Z})$$

For a commutative ring R and an integer M ,

$$(2-1-2) \quad \text{GSp}_g(R) := \{\gamma \in \text{GL}_{2g}(R) \mid {}^t\gamma J_g \gamma = \nu(\gamma) J_g \text{ for some } \nu(\gamma) \in R^\times\},$$

$$(2-1-3) \quad \text{Sp}_g(R) := \{\gamma \in \text{GL}_{2g}(R) \mid {}^t\gamma J_g \gamma = J_g\} = \text{Ker}(\nu),$$

$$(2-1-4) \quad \Gamma_0^g(M) := \{\gamma \in \text{Sp}_g(\mathbb{Z}) \mid c_\gamma \equiv 0_g \pmod{M}\},$$

where we denote by c_γ the left lower $g \times g$ matrix of γ and from now on we use the notation

$$(2-1-5) \quad \begin{pmatrix} a_\gamma & b_\gamma \\ c_\gamma & d_\gamma \end{pmatrix} := \gamma \in \text{GL}_{2g}(R).$$

For a subring R of the real number field $\mathbb{R}$, we put

$$(2-1-6) \quad \mathrm{GSp}_g^+(R) := \{\gamma \in \mathrm{GL}_{2g}(R) \mid {}^t\gamma J_g \gamma = \nu(\gamma) J_g \text{ for some } \nu(\gamma) > 0\}.$$

We denote by $\mathfrak{H}_g := \{Z \in \mathrm{Sym}_g(\mathbb{C}) \mid \mathrm{Im}(Z) > 0\}$ the Siegel upper-half plane of genus g , where $\mathrm{Sym}_g(\mathbb{C})$ is the set of symmetric $g \times g$ matrices whose entries in $\mathbb{C}$. Let $\gamma \in \mathrm{GSp}_g^+(\mathbb{R})$ act on $Z \in \mathfrak{H}_g$ by

$$(2-1-7) \quad \gamma Z := (a_\gamma Z + b_\gamma)(c_\gamma Z + d_\gamma)^{-1}.$$

Definition 2.1. Let χ be a Dirichlet character modulo M . A *Siegel modular form* F of genus g , weight k , level M and character χ is a holomorphic function on $\mathfrak{H}_g$ satisfying

$$(2-1-8) \quad F|_k \gamma(Z) := \det(c_\gamma Z + d_\gamma)^{-k} F(\gamma Z) = \chi^{-1}(\det(a_\gamma)) F(Z)$$

for any $\gamma \in \Gamma_0^g(M)$ and additional conditions of holomorphy at the cusps when $g = 1$. We denote the space of all such functions F by $M_k^g(M, \chi)$. We call $F \in M_k^g(M, \chi)$ a *Siegel cusp form* of genus g , weight k , level M and character χ if $F|_k \gamma| \Phi = 0$ for any $\gamma \in \mathrm{Sp}_g(\mathbb{Z})$, where Φ is the Siegel operator (see [QFHO, Section 2.3.4]). We denote the space of such all functions F by $S_k^g(M, \chi)$. Note that a Siegel cusp form of genus g , weight k , level M and character χ is a cusp form of weight k and character χ for $\Gamma_0^g(M)$ in terms of [QFHO, p.78] and $S_k^g(M, \chi)$ is written as $\mathfrak{N}_k(\Gamma_0^g(M), \chi)$ in [QFHO, p.82].

For any $F \in S_k^g(M, \chi)$ and $Z \in \mathfrak{H}_g$, we write the Fourier expansion of F as

$$(2-1-9) \quad F(Z) = \sum_{T \in \mathcal{L}_{>0}} a_T(F) e(\mathrm{tr} T Z),$$

where $\mathcal{L}_{>0}$ is the set of positive definite half-integral symmetric $g \times g$ matrices (see [QFHO, Theorem 2.3.12] for the Fourier expansions).

2.2. Hecke algebras

For a group G , a subgroup Γ of G and a commutative ring R , we put

$$(2-2-1) \quad \mathcal{H}_R(G, \Gamma) := R[\Gamma \backslash G / \Gamma].$$

Put $\mathbb{Z}_{(M)} := \bigcap_{\ell \mid M} \mathbb{Z}_{(\ell)}$ (the intersection of the localizations $\mathbb{Z}_{(\ell)}$ of $\mathbb{Z}$ at $\ell \mathbb{Z}$ for all primes $\ell \mid M$) and

$$(2-2-2) \quad \Delta_0^g(M) := \mathrm{GSp}_g^+(\mathbb{Q}) \cap \mathrm{GL}_{2g}(\mathbb{Z}_{(M)}) \cap M_{2g}(\mathbb{Z})$$

For a prime $\ell \nmid M$, we denote the integral Hecke algebra at ℓ over $\mathbb{Z}$ by

$$(2-2-3) \quad \mathcal{H}_\ell^g(M) := \mathcal{H}_\mathbb{Z}(\Delta_0^g(M) \cap \mathrm{GL}_{2g}(\mathbb{Z}[\ell^{-1}]), \Gamma_0^g(M)).$$

Then $\mathcal{H}_\ell^g(M)$ is generated over $\mathbb{Z}$ by the following elements:

$$(2-2-4) \quad T^g(\ell) := \Gamma_0^g(M) \mathrm{diag}(1_g, \ell 1_g) \Gamma_0^g(M),$$

$$(2-2-5) \quad T_i^g(\ell^2) := \Gamma_0^g(M) \mathrm{diag}(1_{g-i}, \ell 1_i, \ell^2 1_{g-i}, \ell 1_i) \Gamma_0^g(M)$$

for $i = 1, 2, \dots, g$ (see [QFHO, Theorem 3.3.23] and note $\underline{L}_p^n(q) = \mathcal{H}_p^n(q) \otimes_{\mathbb{Z}} \mathbb{Q}$). We define

$$(2-2-6) \quad \mathcal{H}^g(M) := \otimes'_\ell \mathcal{H}_\ell^g(M),$$

where $\otimes'_\ell$ is the restricted tensor product running over all primes ℓ , i.e., $\mathcal{H}^g(M)$ is defined to be the $\mathbb{Z}$ -algebra generated by $\mathcal{H}_\ell^g(M)$ for all primes ℓ over $\mathbb{Z}$. Note that $\mathcal{H}^g(M) = \mathcal{H}_\mathbb{Z}(\Delta_0^g(M), \Gamma_0^g(M))$ and

that $\mathcal{H}^g(M)$ is commutative ([QFHO, Theorem 3.3.7 and 3.3.12]). We let $T := \Gamma_0^g(M) \gamma \Gamma_0^g(M) = \bigcup_i \Gamma_0^g(M) \gamma_i \in \mathcal{H}^g(M)$ act on $F \in S_k^g(M, \chi)$ by

$$(2-2-7) \quad T(F)(Z) := \nu(\gamma)^{g(2k-g-1)/2} \sum_i \chi(\det(a_\gamma)) F|_k \gamma_i(Z)$$

Since $\nu(\ell 1_{2g}) = \ell^g$ by the definition (2-1-2), the definition (2-2-7) requires that

$$(2-2-8) \quad T_g^g(\ell) \text{ acts as } \chi(\ell^g) \ell^{g^2(k-g-1)/2} \text{ on } S_k^g(M, \chi) \text{ if } \ell \nmid M.$$

We refer to $F \in S_k^g(M, \chi)$ as a *Hecke eigenform* if F is an eigenvector for any $T \in \mathcal{H}^g(M)$.

2.3. Hecke fields

For a Hecke eigenform $F \in S_k^g(M, \chi)$, we define $\lambda_F \in \text{Hom}_{\mathbb{C}\text{-alg}}(\mathbb{T}^\Sigma(S_k^g(M, \chi)), \mathbb{C})$ by $\lambda_F(T)$ to be the eigenvalue of T at F and the *Hecke field* $\mathbb{Q}(F)$ of F by

$$(2-3-1) \quad \mathbb{Q}(F) := \mathbb{Q}(\{\lambda_F(T) \mid T \in \mathcal{H}^g(M)\}).$$

Moreover, we denote by $\mathbb{Z}(F)$ the ring of integers of $\mathbb{Q}(F)$, $\mathbb{Z}_{(p)}(F)$ the localization of $\mathbb{Z}(F)$ at the prime above $p\mathbb{Z}$, $\mathbb{Z}_p(F)$ the p -adic completion of $\mathbb{Z}(F)$, and $\mathbb{Q}_p(F) := \text{Frac}(\mathbb{Z}_p(F))$ the field of fractions. For a Dirichlet character ψ , we denote by $\mathbb{Q}(F, \psi)$ the field obtained by adjoining the values of ψ to $\mathbb{Q}(F)$, $\mathbb{Z}(F, \psi)$ the ring of integers of $\mathbb{Q}(F, \psi)$, $\mathbb{Z}_{(p)}(F, \psi)$ the localization of $\mathbb{Z}(F, \psi)$ at the prime above $p\mathbb{Z}$, $\mathbb{Z}_p(F, \psi)$ the p -adic completion of $\mathbb{Z}(F, \psi)$, and $\mathbb{Q}_p(F, \psi) := \text{Frac}(\mathbb{Z}_p(F, \psi))$ the field of fractions.

2.4. Petersson inner products

For $F, G \in S_k^g(M, \chi)$, the normalized *Peterson inner product* of F and G is defined by

$$(2-4-1) \quad \langle F, G \rangle := [\text{Sp}_g(\mathbb{Z}) : \Gamma_0^g(M)]^{-1} \int_{\Gamma_0^g(M) \backslash \mathfrak{H}_g} F(Z) \overline{G(Z)} \det(Y)^{k-(g+1)} dZ,$$

where $Z = X + \sqrt{-1}Y = (x_{\alpha\beta}) + \sqrt{-1}(y_{\alpha\beta})$,

$$(2-4-2) \quad dZ := \prod_{1 \leq \alpha \leq \beta \leq g} dx_{\alpha\beta} dy_{\alpha\beta},$$

and $(\det Y)^{-(g+1)} dZ$ is a $\text{Sp}_g(\mathbb{R})$ -invariant measure ([QFHO, Proposition 1.2.9]). By [QFHO, Theorem 2.5.3], $\langle F, G \rangle$ is absolutely convergent, independent of choice of the subgroup $\Gamma_0^g(N)$ with $F, G \in S_k^g(N, \chi)$ and a positive definite Hermitian form.

2.5. Notation and terminology for genus 1

We often omit g from any notation when $g = 1$ for simplicity. We denote by $S_k^{\text{new}}(M, \varepsilon)$ the orthogonal complement of the subspace of old forms of level M in $S_k(M, \varepsilon)$ with respect to the Petersson inner product. We refer to a Hecke eigenform in $S_k^{\text{new}}(M, \varepsilon)$ as a *primitive form* of level M if $T^1(n)f = a_n(f)f$ for all positive integers n , where $a_n(f)$ is the n -th Fourier coefficient of f . We denote by $S_k(M, \varepsilon)_\alpha$ the subspace of $S_k(M, \varepsilon)$ spanned by the generalized eigenspaces for eigenvalues λ of T_p with $\text{val}_p(\lambda) = \alpha$. Let $\mathbb{Z}[\varepsilon]$ be the ring generated by the values of ε over $\mathbb{Z}$. For a $\mathbb{Z}[\varepsilon]$ -algebra R and $q := \exp(2\pi\sqrt{-1}z)$, we put

$$(2-5-1) \quad S_k(M, \varepsilon; R)_\alpha := (S_k(M, \varepsilon)_\alpha \cap \mathbb{Z}[\varepsilon][[q]]) \otimes_{\mathbb{Z}[\varepsilon]} R,$$

$$(2-5-2) \quad S_k^{\text{new}}(M, \varepsilon; R)_\alpha := (S_k^{\text{new}}(M, \varepsilon) \cap S_k(M, \varepsilon; \mathbb{Z}[\varepsilon])_\alpha) \otimes_{\mathbb{Z}[\varepsilon]} R.$$

For $f \in S_k(M, \varepsilon)$ and a Dirichlet character ψ , we denote by $f \otimes \psi \in S_k(L, \varepsilon\psi^2)$ the ψ -twist of f defined by $a_n(f \otimes \psi) := \psi_0(n)a_n(f)$ for all $n \geq 1$, where L is the least common multiple of M , c_ψ^2 , and $c_\psi c_\varepsilon$ ([MFM, Lemma 4.3.10.(2)]). We put $L(s, f) := \sum_{n=1}^{\infty} a_n(f)n^{-s}$.

2.6. Notation for genus 2

Let $F \in S_k^2(M, \chi)$ be a Hecke eigenform such that $T^2(\ell)F = \lambda_F(\ell)F$ for any prime ℓ and $T_1^2(\ell^2)F = \lambda_F(\ell^2)F$ for each prime $\ell \nmid M$, where $T^2(\ell)$ is defined as (2-2-4) even when $\ell \mid M$. Recall that we have always $T_2^2(\ell)F = \chi(\ell^2)\ell^{2(k-3)}F$ by (2-2-8). The spinor L -function $L(s, F, \text{spin})$ attached to F is defined by

$$(2-6-1) \quad L(s, F, \text{spin}) := \prod_{\ell \mid M} (1 - \lambda_F(\ell)\ell^{-s})^{-1} \prod_{\ell \nmid M} Q_\ell(\ell^{-s})^{-1}$$

with

$$(2-6-2) \quad \begin{aligned} Q_\ell(X) := & 1 - \lambda_F(\ell)X + \left(\ell\lambda_F(\ell^2) + \chi(\ell^2)(\ell^2 + 1)\ell^{2k-5} \right) X^2 \\ & - \chi(\ell^2)\lambda_F(\ell)\ell^{2k-3}X^3 + \chi(\ell^4)\ell^{4k-6}X^4. \end{aligned}$$

3. D -th Saito-Kurokawa lifts

Let $k \geq 2$ be an integer, M an odd positive integer and χ a Dirichlet character modulo M .

3.1. Kohnen plus spaces

Put $\tilde{\chi} := \chi_\epsilon \chi$ with $\epsilon := \chi(-1)$. We denote the *Kohnen plus space* by

$$(3-1-1) \quad S_{k-1/2}^+(M, \chi) := \left\{ g \in S_{2k+1}^{\text{Sh}}(4M, \tilde{\chi}) \mid a_n(g) = 0 \text{ if } \chi(-1)(-1)^{k-1}n \equiv 2, 3 \pmod{4} \right\},$$

where $S_{2k+1}^{\text{Sh}}(4M, \tilde{\chi})$ is the space of cusp forms of half-integral weight $k - 1/2$ with level $4M$ and a character $\tilde{\chi}$ modulo $4M$ in the sense of Shimura [Shi73, p. 447]. For $g \in S_{k-1/2}^+(M, \chi)$ and each prime ℓ , the Hecke operator $T^+(\ell)$ is defined by

$$(3-1-2) \quad a_n(T^+(\ell)g) = a_{\ell^2 n}(g) + \tilde{\chi}\chi_{(-1)^{k-1}n}(\ell)\ell^{k-2}a_n(g) + \chi(\ell^2)\ell^{2k-3}a_{n/\ell^2}(g)$$

for any positive integer n with $\chi(-1)(-1)^{k-1}n \equiv 0, 1 \pmod{4}$. For $g, h \in S_{k-1/2}^+(M, \chi)$, we define the Petersson inner product by

$$(3-1-3) \quad \langle g, h \rangle_{4M} := \int_{\Gamma_0(4M) \backslash \mathfrak{H}} g(z) \overline{h(z)} y^{k-5/2} dx dy \quad (z = x + \sqrt{-1}y),$$

$$(3-1-4) \quad \langle g, h \rangle := [\text{SL}_2(\mathbb{Z}) : \Gamma_0(4M)]^{-1} \langle g, h \rangle_{4M}.$$

3.2. D -th Shintani lifts θ_D

Let D be a fundamental discriminant with $\chi(-1)(-1)^{k-1}D > 0$ and $(D, c_\chi) = 1$. We define the D -th *Shimura lift* Sh_D by

$$(3-2-1) \quad \text{Sh}_D(g) := \sum_{n \geq 1} \left(\sum_{d \mid n} \chi_D \chi(d) d^{k-2} a_{n^2|D|/d^2}(g) \right) q^n$$

(see [KT04, (3-1)]).

Remark 3.1. Although [KT04] assumes $(D, M) = 1$, we see that the results below do not require this assumption except for Theorem 3.3. Note that $\text{Sh}_D = 0$ if $(D, c_\chi) \neq 1$ by [KT04, (3-2)].

Assume that

$$(3-2-2) \quad \text{either } k \geq 3, M \text{ is square-free, or } M \text{ is cubic-free and } \chi = \mathbb{1}.$$

Then the image of the D -th Shimura lift Sh_D is contained in the space of cusp forms ([Koh85, p.241, 1.4-9]). Now we define the D -th Shintani lift $\theta_D : S_{2k-2}(M, \chi^2) \rightarrow S_{k-1/2}^+(M, \chi)$ as the adjoint map of Sh_D with respect to the Petersson inner products, i.e., the map satisfying

$$(3-2-3) \quad \langle g, \theta_D(f) \rangle = \langle \text{Sh}_D(g), f \rangle$$

for every $g \in S_{k-1/2}^+(M, \chi)$ and $f \in S_{2k-2}(M, \chi^2)$. Since the D -th Shimura lift Sh_D is Hecke equivariant in the sense that $T^1(\ell) \circ \text{Sh}_D = \text{Sh}_D \circ T^+(\ell)$ for all primes ℓ by [KT04, Theorem 3.1] and the Hecke operators are Hermitian operators with respect to the Petersson inner products, we have the following:

Theorem 3.2. *Let $k \geq 2$ be an integer, M an odd positive integer, χ a Dirichlet character modulo M and D a fundamental discriminant with $\chi(-1)(-1)^{k-1}D > 0$ and $(D, c_\chi) = 1$. Assume (3-2-2). Then the D -th Shintani lift θ_D is a $\mathbb{C}$ -homomorphism from $S_{2k-2}(M, \chi^2)$ into $S_{k-1/2}^+(M, \chi)$ and Hecke equivariant in the sense that $T^+(\ell) \circ \theta_D = \theta_D \circ T^1(\ell)$ for all primes ℓ .*

Suppose that $c_\chi \parallel M$. Let ℓ be a prime factor of M/c_χ . We put $v_\ell := \text{val}_\ell(M/c_\chi) = \text{val}_\ell(M)$. Let γ_ℓ be an element in $\text{SL}_2(\mathbb{Z})$ such that

$$(3-2-4) \quad \gamma_\ell \equiv \begin{cases} J_1 & (\text{mod } \ell^{2v_\ell}), \\ 1_2 & (\text{mod } (M/\ell^{v_\ell})^2). \end{cases}$$

We put $\eta_\ell := \gamma_\ell \cdot \text{diag}(\ell^{v_\ell}, 1)$ (see [MFM, (4.6.21)]). We define the eigenvalue of f for the Atkin-Lehner involution η_ℓ by

$$(3-2-5) \quad w_\ell(f) := \chi_0^2(\ell^{v_\ell}) a_1(f|_{2k-2}\eta_\ell).$$

If $v_\ell = 1$, then we have $a_1(f|_{2k-2}\eta_\ell) = -\chi_0^{-2}(\ell)\ell^{-k+2}a_\ell(f)$ by [MFM, Corollary 4,6,18.(2)] and hence

$$(3-2-6) \quad w_\ell(f) = -\ell^{-k+2}a_\ell(f) \in \{\pm 1\}$$

by [MFM, Theorem 4.6.17.(2)].

Theorem 3.3 ([KT04, (4-19, 20, 21, and 22)]). *Let $f \in S_{2k-2}^{\text{new}}(M, \chi^2)$ be a primitive form. Suppose that $c_\chi \parallel M$ and $(D, M) = 1$. We put*

$$(3-2-7) \quad R_D(f) := \prod_{\ell} \left(1 + \chi_D \chi_0(\ell^{v_\ell}) w_\ell(f) \left(\frac{1 - \chi_D \chi_0^{-1}(\ell) \ell^{-k+1} a_\ell(f)}{1 - \chi_D \chi_0(\ell) \ell^{-k+1} a_\ell(f)^c} \right) \right),$$

where $\prod_{\ell}$ is taken over all prime factors ℓ of M/c_χ and $a_\ell(f)^c$ is the complex conjugate of $a_\ell(f)$. Then

$$(3-2-8) \quad a_{|D|}(\theta_D(f)) = R_D(f) |D|^{k-3/2} c_\chi^{2k-3} \pi^{-(k-1)} \Gamma(k-1) L(k-1, f \otimes \chi_D \chi^{-1}).$$

Remark 3.4. Let the notation and the assumption be the same as the theorem above. If $\chi^2 = \mathbb{1}$ and M/c_χ is square-free, then $R_D(f) \in \{0, 2^{\nu(M/c_\chi)}\}$ by [Shi72, Proposition 1.3] and (3-2-6), where $\nu(M/c_\chi)$ is the number of distinct prime factors of M/c_χ . In particular, if $\chi = \mathbb{1}$, then the following conditions are equivalent:

- (1) $R_D(f) \neq 0$.
- (2) $R_D(f) = 2^{\nu(M)}$.
- (3) $\chi_D(\ell) = w_\ell(f)$ for any prime divisor ℓ of M .

In this case, the formula (3-2-8) is nothing but the result of Kohnen in [Koh85] and the sign of the functional equation of $L(s, f \otimes \chi_D)$ is $(-1)^{k-1} \chi_D(-1)$, i.e., if $(-1)^{k-1} \chi_D(-1) = -1$, then $L(k-1, f \otimes \chi_D) = 0$.

3.3. D -th Saito-Kurokawa lifts SK_D

Let $k \geq 2$ be an even integer, $M \geq 1$ an odd integer, χ a Dirichlet character modulo M with $\chi(-1) = 1$ and D a fundamental discriminant with $D < 0$ and $(D, c_\chi) = 1$. Assume (3-2-2). We then define the D -th *Saito-Kurokawa lift* SK_D by composing θ_D with the Eichler-Zagier map EZ and the Maass lift L (see [Mak, Section 4 and 5] for definition of EZ and L , respectively):

$$(3-3-1) \quad \text{SK}_D : S_{2k-2}(M, \chi^2) \xrightarrow{\theta_D} S_{k-1/2}^+(M, \chi) \xrightarrow{\text{EZ}} J_{k,1}^{\text{cusp}}(M, \chi) \xrightarrow{L} S_k^2(M, \chi).$$

Let $f \in S_{2k-2}(M, \chi^2)$ be any element. Since EZ is an isomorphism ([Mak, Theorem 4.2]) and L is an injective homomorphism ([Mak, Theorem 5.1]), we see that

$$(3-3-2) \quad \text{SK}_D(f) \neq 0 \text{ if and only if } \theta_D(f) \neq 0$$

(see [Mak, Theorem 3.6] for the non-vanishing criterion for $\theta_D(f)$). For $T = [a, b, c] \in \mathcal{L}_{>0}$, the T -th Fourier coefficient of $\text{SK}_D(f)$ is given by

$$(3-3-3) \quad a_T(\text{SK}_D(f)) = \sum_{\substack{0 < d \mid (a,b,c) \\ (d,M)=1}} \chi(d) d^{k-1} a_{\det(T)/d^2}(\theta_D(f)).$$

For any prime ℓ ,

$$(3-3-4) \quad T^2(\ell) \circ \text{SK}_D = \text{SK}_D \circ (T^1(\ell) + \chi(\ell)(\ell^{k-2} + \ell^{k-1})),$$

$$(3-3-5) \quad (\ell T_1^2(\ell^2) + \chi(\ell)^2(\ell^2 + 1)\ell^{2k-5}) \circ \text{SK}_D = \text{SK}_D \circ (\chi(\ell)(\ell^{k-1} + \ell^{k-2})T^1(\ell) + 2\chi(\ell)^2\ell^{2k-3}).$$

Namely, we have the following:

Theorem 3.5. *Let $k \geq 2$ be an even integer, $M \geq 1$ an odd integer, χ a Dirichlet character modulo M with $\chi(-1) = 1$ and D a fundamental discriminant with $D < 0$ and $(D, c_\chi) = 1$. Assume (3-2-2). Let $f \in S_{2k-2}^{\text{new}}(M, \chi^2)$ be a primitive form. If $\theta_D(f) \neq 0$, then $\text{SK}_D(f) \in S_k^2(M, \chi)$ is a Hecke eigenform satisfying*

$$(3-3-6) \quad L(s, \text{SK}_D(f), \text{spin}) = L(s-k+1, \chi)L(s-k+2, \chi)L(s, f).$$

Remark 3.6. It is known that the image of SK_D is characterized by the generalized Maass relation (see [Hei17]).

4. p -adic families

Let K be a complete discretely valued subfield of $\mathbb{C}_p$. The *weight space* $\mathcal{W}$ attached to $\mathcal{O}_K[[\mathbb{Z}_p^\times]]$ is the rigid analytic variety whose $\mathbb{C}_p$ -valued points are given by

$$(4-0-1) \quad \text{Hom}^{\text{cont}}(\mathbb{Z}_p^\times, \mathbb{C}_p^\times) \cong \text{Hom}_{\mathcal{O}_K\text{-alg}}^{\text{cont}}(\mathcal{O}_K[[\mathbb{Z}_p^\times]], \mathbb{C}_p).$$

For a K -Banach algebra R and an R -valued point $k \in \mathcal{W}(R)$, we will use a notation t^k instead of $k(t)$ for $t \in \mathbb{Z}_p^\times$. For a K -rigid analytic variety X , we denote by $A(X)$ the ring of rigid analytic functions on X and $A^\circ(X)$ the subring consisting of elements that are power bounded with respect to the supremum semi-norm $||$ (see [BGR, Definition 6.2.1/2]). By [BGR, Proposition 6.2.3/1], we have $A^\circ(X) = \{f \in A(X) \mid |f| \leq 1\}$. We denote by $B_K[a, r]$ the affinoid closed disk over K of radius $r \in |K|$ about $a \in \mathcal{O}_K$ whose $\mathbb{C}_p$ -valued points are given by

$$(4-0-2) \quad B_K[a, r](\mathbb{C}_p) = \{x \in \mathcal{O}_K \mid |x - a|_p < r\}.$$

4.1. Coleman families

Let $f \in S_w^{\text{new}}(N, \varepsilon)_\alpha$ be a primitive form. Assume that the characteristic polynomial

$$(4-1-1) \quad X^2 - a_p(f)X + \varepsilon(p)p^{w-1} \in \mathbb{Z}(f)[X]$$

of $T^1(p)$ on the subspace spanned by f and $f|V_p$ has no double roots, i.e., $a_p(f)^2 \neq \varepsilon(p)p^{w-1}$. Let $\alpha_p(f)$ be the root of this polynomial satisfying $\text{val}_p(\alpha_p(f)) = \alpha$. We refer to

$$(4-1-2) \quad f^*(z) := f(z) - \varepsilon(p)p^{w-1}\alpha_p(f)^{-1}f(pz)$$

as the p -stabilization of f . The p -stabilization f^* is the Hecke eigenform of level Np with the same eigenvalues as f outside p and $T^1(p)$ -eigenvalue $a_p(f^*) = \alpha_p(f)$.

Theorem 4.1 ([Col97]). *Let $f \in S_{w_0}^{\text{new}}(N, \varepsilon)_\alpha$ be a primitive form with $w_0 > \alpha + 1$ and K a complete discretely valued subfield of $\mathbb{C}_p$ containing the p -adic completion of the Hecke field $\mathbb{Q}(f^*)$. Assume $a_p(f)^2 \neq \varepsilon(p)p^{w_0-1}$. Then there exists a positive integer M and a formal power series*

$$(4-1-3) \quad \mathbf{f} = \sum_{n=1}^{\infty} a_n(\mathbf{f})q^n \in A^\circ(B_K[w_0, p^{-M}])([q])$$

such that for any w in

$$(4-1-4) \quad W(M) := \{w \in \mathbb{Z} \mid w \equiv w_0 \pmod{(p-1)p^M}, w > \alpha + 1\}$$

except for at most one, the specialization $\mathbf{f}(w)$ at w given by

$$(4-1-5) \quad \mathbf{f}(w) := \sum_{n=1}^{\infty} a_n(\mathbf{f})(w)q^n$$

is the p -stabilization of some primitive form in $S_w^{\text{new}}(N, \varepsilon; \mathcal{O}_K)_\alpha$ and $\mathbf{f}(w_0) = f^*$. More precisely, there exists a primitive form $f_w \in S_w^{\text{new}}(N, \varepsilon; \mathcal{O}_K)_\alpha$ satisfying the following conditions:

- (1) $\mathbf{f}(w) = f_w^*$.
- (2) $f_{w_0} = f$.
- (3) $\mathbf{f}(w_1) \in S_{w_1}^{\text{new}}(Np, \varepsilon)_\alpha$ is primitive if there exists an exceptional weight $w_1 \in W(M)$.

In particular, for any positive integer m and $w \in W(M)$, if $w \equiv w_0 \pmod{(p-1)p^{M+m}}$, then

$$(4-1-6) \quad f_w^* \equiv f^* \pmod{p^m \mathcal{O}_K}$$

We refer to the family $\{f_w\}_{w \in W(M)}$ of primitive forms obtained in the theorem above as a *Coleman family* passing through f over K . By the theorem above, for any positive integer m and $w \in W(M)$, if $w \equiv w_0 \pmod{(p-1)p^{M+m}}$, then for any positive integer n with $p \nmid n$,

$$(4-1-7) \quad a_n(f_w) \equiv a_n(f) \pmod{p^m \mathcal{O}_K}.$$

4.2. p -adic families of the D -th Saito-Kurokawa lifts

By Theorem 4.1 and Theorem 3.5, we immediately see that the family $\{F_w := \text{SK}_D(f_w)\}_w$ forms a p -adic family in the sense that the T -eigenvalue $\lambda_{F_w}(T)$ gives a p -adic analytic function $w \mapsto \lambda_{F_w}(T)$ from $W(M)$ into K as follows:

Corollary 4.2. *Let $k_0 \geq 2$ be an even integer, χ a Dirichlet character modulo N with $\chi(-1) = 1$ and D a fundamental discriminant with $D < 0$ and $(D, c_\chi) = 1$. Assume (3-2-2) for $k = k_0$ and $M = Np$. Let $f \in S_{w_0}^{\text{new}}(N, \chi^2)_\alpha$ be a primitive form with $w_0 := 2k_0 - 2 > \alpha + 1$ and K a complete discretely valued subfield of $\mathbb{C}_p$ containing the p -adic completion of the Hecke field $\mathbb{Q}(f^*)$. Assume*

$a_p(f)^2 \neq \chi^2(p)p^{w_0-1}$. Let $\{f_w\}_{w \in W(M)}$ be a Coleman family passing through f over K . Then for any w in

$$(4-2-1) \quad W^{\text{SK}}(M) := \{2k - 2 \in W(M) \mid k \in 2\mathbb{Z}\},$$

if $w \equiv w_0 \pmod{(p-1)p^{M+m}}$, then for any $T \in \mathcal{H}^2(Np)$,

$$(4-2-2) \quad \lambda_{F_w^*}(T) \equiv \lambda_{F_{w_0}^*}(T) \pmod{p^m \mathcal{O}_K},$$

where $F_w^* := \text{SK}_D(f_w^*)$. In particular, for any positive integer n with $p \nmid n$, we have

$$(4-2-3) \quad \lambda_{F_w}(T^2(n)) \equiv \lambda_{F_{w_0}}(T^2(n)) \pmod{p^m \mathcal{O}_K}.$$

Remark 4.3. It is possible that $\{\text{SK}_D(f_w^*)\}_{w \in W^{\text{SK}}(M)}$ and $\{\text{SK}_D(f_w)\}_{w \in W^{\text{SK}}(M)}$ vanishes identically. However, it does follow from $\theta_D(f) \neq 0$ that $\text{SK}_D(f_w^*) \neq 0$ and $\text{SK}_D(f_w) \neq 0$ for any $w \in W^{\text{SK}}(M)$ with sufficiently large M by p -adic interpolation of Fourier coefficients.

5. Cohomological interpretation

5.1. Modular symbols and elliptic cusp forms

Let Δ_0 be a subsemigroup of $M_2(\mathbb{Z}) \cap \text{GL}_2(\mathbb{Q})$ containing $\Gamma_0(M)$. Let $\text{Div}^0(\mathbb{P}^1(\mathbb{Q}))$ be the group of divisors of degree 0 supported on the rational cusps $\mathbb{P}^1(\mathbb{Q}) = \mathbb{Q} \cup \{i\infty\}$ of the complex upper half plane $\mathfrak{H}$. We let Δ_0 act on $\mathfrak{H}$ by fractional linear transformations, i.e.,

$$(5-1-1) \quad \gamma z := \begin{cases} (az + b)(cz + d)^{-1} & \text{if } \det(\gamma) > 0, \\ (a\bar{z} + b)(c\bar{z} + d)^{-1} & \text{if } \det(\gamma) < 0, \end{cases} \quad \left(\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, z \in \mathfrak{H} \right).$$

This induces a natural action of Δ_0 on $\mathfrak{H}^* := \mathfrak{H} \cup \mathbb{P}^1(\mathbb{Q})$ and $\mathbb{P}^1(\mathbb{Q})$. Then Δ_0 acts on $\text{Div}^0(\mathbb{P}^1(\mathbb{Q}))$ by linear fractional transformations. Let R be a commutative ring and E a left $R[\Delta_0]$ -module. We let $\gamma \in \Delta_0$ acts on $\Phi \in \text{Hom}_{\mathbb{Z}}(\text{Div}^0(\mathbb{P}^1(\mathbb{Q})), E)$ by

$$(5-1-2) \quad (\Phi|\gamma)(D) := \gamma\Phi(\gamma D).$$

Then the abstract Hecke algebra $\mathcal{H}_R(\Delta_0, \Gamma_0(M))$ acts on the group of E -valued *modular symbols* over $\Gamma_0(M)$:

$$(5-1-3) \quad \text{Symb}_{\Gamma_0(M)}(E) := \text{Hom}_{\mathbb{Z}}(\text{Div}^0(\mathbb{P}^1(\mathbb{Q})), E)^{\Gamma_0(M)}.$$

Let $\tilde{E}$ be the locally constant sheaf on the open modular curve $Y := \Gamma_0(M) \backslash \mathfrak{H}$ attached to E . Assume that

$$(5-1-4) \quad \text{the orders of the torsion elements of } \Gamma_0(M) \text{ act invertibly on } E.$$

Then by [AS86, Proposition 4.2], there exists a Hecke equivariant canonical isomorphism

$$(5-1-5) \quad H_c^1(Y, \tilde{E}) \xrightarrow{\sim} \text{Symb}_{\Gamma_0(M)}(E).$$

Throughout the paper, we will identify the group of compactly supported cohomology classes with the group of modular symbols under the assumption that (5-1-4). Note that (5-1-4) holds if either E is a vector space over a field of characteristic 0, E is a $\mathbb{Z}_p$ -module with $p \geq 5$, or $\Gamma_0(M)$ is torsion-free. The matrix $\iota := \text{diag}(1, -1)$ induces the natural involution on $\text{Symb}_{\Gamma_0(M)}(E)$ and the decomposition

$$(5-1-6) \quad \text{Symb}_{\Gamma_0(M)}(E) = \text{Symb}_{\Gamma_0(M)}^+(E) \oplus \text{Symb}_{\Gamma_0(M)}^-(E)$$

if 2 acts invertibly on E . Indeed, each element Φ decomposes as $\Phi = \Phi^+ + \Phi^-$, where

$$(5-1-7) \quad \Phi^{\pm} := 2^{-1}(\Phi \pm \Phi|\iota).$$

For a non-negative integer n , let $L(n; R)$ be the R -module of homogeneous polynomials in (X, Y) of degree n with coefficients in R . For an R -valued Dirichlet character ε modulo M , we denote by $L(n, \varepsilon; R)$ the $R[\Gamma_0(M)]$ -module $L(n; R)$ endowed with the following action; for $\gamma \in \Gamma_0(M)$ and $P \in L(n, \varepsilon; R)$,

$$(5-1-8) \quad (\gamma P)(X, Y) = \varepsilon(d_\gamma) P((X, Y)^t \gamma).$$

For each cusp form $f \in S_{w+2}(M, \varepsilon)$, we define the $L(w, \varepsilon; \mathbb{C})$ -valued differential form on $\mathfrak{H}$ by

$$(5-1-9) \quad \omega_f := f(z) \left(\sum_{i=0}^w (-z)^i X^{w-i} Y^i \right) dz.$$

The additive map

$$(5-1-10) \quad \Phi_f : \text{Div}^0(\mathbb{P}^1(\mathbb{Q})) \rightarrow L(w, \varepsilon; \mathbb{C}) ; \{c_2\} - \{c_1\} \mapsto \int_{c_1}^{c_2} \omega_f$$

defines a modular symbol in $\text{Symb}_{\Gamma_0(M)}(L(w, \varepsilon; \mathbb{C}))$ and the map

$$(5-1-11) \quad \Phi : S_{w+2}(M, \varepsilon) \rightarrow \text{Symb}_{\Gamma_0(M)}^-(L(w, \varepsilon; \mathbb{C})) ; f \mapsto \Phi_f^-$$

is an injective $\mathbb{C}[\mathcal{H}^1(M)]$ -homomorphism.

5.2. Cohomological interpretation of θ_D

Proposition 5.1 ([Mak17, Propostion 3.2]). *Let $k \geq 2$ be an integer, M an odd positive integer, χ a Dirichlet character modulo M and D a fundamental discriminant with $\chi(-1)(-1)^{k-1}D > 0$ and $(D, c_\chi) = 1$. Assume (3-2-2). Then we can define a $\mathbb{C}$ -homomorphism Θ_D satisfying the commutative diagram*

$$(5-2-1) \quad \begin{array}{ccc} \text{Symb}_{\Gamma_0(M)}^-(L(2k-4, \chi^2; \mathbb{C})) & \xrightarrow{\Theta_D} & \mathbb{C}[[q]] \\ \uparrow \Phi & & \uparrow \text{Fourier expansion} \\ S_{2k-2}(M, \chi^2) & \xrightarrow{\theta_D} & S_{k-1/2}^+(M, \chi), \end{array}$$

5.3. Algebraic lints θ_D^{alg} and SK_D^{alg}

Let $f \in S_{w+2}^{\text{new}}(N, \varepsilon)$ be a primitive form. Let V_f and V_{f^*} be the representation spaces of Galois representations ρ_f and ρ_{f^*} attached to f and f^* over $\mathbb{Q}_p(f)$ and $\mathbb{Q}_p(f^*)$, respectively. By the Comparison Theorem between étale and Betti induced by a fixed isomorphism $\mathbb{Q}_p \cong \mathbb{C}$, we have

$$(5-3-1) \quad V_f \otimes_{\mathbb{Q}_p(f)} \bar{\mathbb{Q}}_p \cong \text{Symb}(\mathbb{C})[f] := \bigcap_{\ell} \text{Ker} \left((T(\ell) - a_\ell(f)) | \text{Symb}_{\Gamma_0(N)}^-(L(w, \varepsilon; \mathbb{C})) \right),$$

$$(5-3-2) \quad V_{f^*} \otimes_{\mathbb{Q}_p(f^*)} \bar{\mathbb{Q}}_p \cong \text{Symb}(\mathbb{C})[f^*] := \bigcap_{\ell} \text{Ker} \left((T(\ell) - a_\ell(f^*)) | \text{Symb}_{\Gamma_0(Np)}^-(L(w, \varepsilon; \mathbb{C})) \right).$$

Since the left-hand sides of both (5-3-1) and (5-3-2) are isomorphic by the Brauer-Nesbitt Theorem and the Chebotarev Density Theorem, we have

$$(5-3-3) \quad \text{Symb}(\mathbb{C})[f] \cong \text{Symb}(\mathbb{C})[f^*].$$

By [Kit94, Proposition 3.3], these are free of rank one over $\mathbb{C}$, and hence we may assume that $\Phi_f \mapsto \Phi_{f^*}$ gives the isomorphism (5-3-3). By [Kit94, Proposition 3.3], the eigenmodules

$$(5-3-4) \quad \text{Symb}(\mathbb{Z}_{(p)}(f))[f] := \text{Symb}(\mathbb{C})[f] \cap \text{Symb}_{\Gamma_0(N)}^-(L(w, \varepsilon; \mathbb{Z}_{(p)}(f))),$$

$$(5-3-5) \quad \text{Symb}(\mathbb{Z}_{(p)}(f^*))[f^*] := \text{Symb}(\mathbb{C})[f^*] \cap \text{Symb}_{\Gamma_0(Np)}^-(L(w, \varepsilon; \mathbb{Z}_{(p)}(f^*)))$$

are free of rank one over $\mathbb{Z}_{(p)}(f)$ and $\mathbb{Z}_{(p)}(f^*)$, respectively. Let Φ_f° be a generator of $\text{Symb}(\mathbb{Z}_{(p)}(f))[f]$, which is contained in

$$(5-3-6) \quad \text{Symb}(\mathbb{Z}_{(p)}(f^*))[f] := \text{Symb}(\mathbb{C})[f] \cap \text{Symb}_{\Gamma_0(N)}^-(L(w, \varepsilon; \mathbb{Z}_{(p)}(f^*))).$$

Then there exists $\Omega(f) \in \mathbb{C}^\times$ such that

$$(5-3-7) \quad \Phi_f^\circ = \Omega(f)^{-1} \cdot \Phi_f^- \in \text{Symb}(\mathbb{Z}_{(p)}(f))[f].$$

Then the isomorphism (5-3-3) implies

$$(5-3-8) \quad \Omega(f)^{-1} \cdot \Phi_{f^*}^- \in \text{Symb}(\mathbb{Z}_{(p)}(f^*))[f^*].$$

Theorem 5.2. *Let $k \geq 2$ be an integer, χ a Dirichlet character modulo N and D a fundamental discriminant with $\chi(-1)(-1)^{k-1}D > 0$ and $(D, c_\chi p) = p$. Assume (3-2-2). Let $f \in S_{2k-2}^{\text{new}}(N, \chi^2)$ be a primitive form. Then*

$$(5-3-9) \quad (c_D(k, \chi)\Omega(f))^{-1} \theta_D(f) \in S_{k-1/2}^+(N, \chi; \mathbb{Z}_{(p)}(f, \chi)),$$

$$(5-3-10) \quad (c_D(k, \chi)\Omega(f))^{-1} \theta_D(f^*) \in S_{k-1/2}^+(Np, \chi; \mathbb{Z}_{(p)}(f^*, \chi)).$$

Remark 5.3. Note that the values of χ^2 are contained in $\mathbb{Q}(f)$ for a primitive form $f \in S_{2k-2}^{\text{new}}(N, \chi^2)$ but the values of χ are not necessarily.

We fix, once and for all, the complex period $\Omega(f)$ as (5-3-7) and define

$$(5-3-11) \quad \theta_D^{\text{alg}}(f) := \Omega(f)^{-1} \theta_D(f),$$

$$(5-3-12) \quad \theta_D^{\text{alg}}(f^*) := \Omega(f)^{-1} \theta_D(f^*),$$

$$(5-3-13) \quad \text{SK}_D^{\text{alg}}(f) := L\left(\text{EZ}\left(\theta_D^{\text{alg}}(f)\right)\right) = \Omega(f)^{-1} \text{SK}_D(f),$$

$$(5-3-14) \quad \text{SK}_D^{\text{alg}}(f^*) := L\left(\text{EZ}\left(\theta_D^{\text{alg}}(f^*)\right)\right) = \Omega(f)^{-1} \text{SK}_D(f^*).$$

Note that $c_D(k, \chi)^{-1} \text{SK}_D^{\text{alg}}(f) \in \mathbb{Z}_{(p)}(f, \chi)[[q]]$ and $c_D(k, \chi)^{-1} \text{SK}_D^{\text{alg}}(f^*) \in \mathbb{Z}_{(p)}(f^*, \chi)[[q]]$ by (3-3-3). For a Dirichlet character ψ and $j \in [1, k-1] \cap \mathbb{Z}$ with $\psi(-1)(-1)^{j-1} = -1$, we have

$$(5-3-15) \quad L^{\text{alg}}(j, f \otimes \psi) := \frac{G(\psi^{-1})\Gamma(j)L(j, f \otimes \psi)}{(-2\pi\sqrt{-1})^j \Omega(f)} \in \mathbb{Z}_{(p)}(f, \psi)$$

by [Kit94, Lemma 4.1]. If $c_\chi \parallel N$ and $(D, N) = 1$, then

$$(5-3-16) \quad a_{|D|}(\theta_D^{\text{alg}}(f)) = (-1)^{k-1} c_D(k, \chi) (Dc_\chi)^{k-2} R_D(f) L^{\text{alg}}(k-1, f \otimes \chi_D \chi^{-1}).$$

By inner product formula obtained in [Mak], we have the following:

Corollary 5.4. *Let $k \geq 2$ be an even integer, $M \geq 1$ a square-free odd integer, χ a Dirichlet character modulo M with $\chi^2 = \mathbb{1}$ and $\chi(-1) = 1$ and D a fundamental discriminant with $D < 0$ and $(D, M) = 1$. Let $f \in S_{2k-2}^{\text{new}}(M, \mathbb{1})$ be a primitive form. Then*

$$(5-3-17) \quad \frac{\|\text{SK}_D^{\text{alg}}(f)\|^2}{\|f\|^2} = C_D(k, M, \chi) L^{\text{alg}}(k, f) L^{\text{alg}}(k-1, f \otimes \chi_D \chi^{-1})$$

where $\|\mathrm{SK}_D^{\mathrm{alg}}(f)\|^2 := \langle \mathrm{SK}_D^{\mathrm{alg}}(f), \mathrm{SK}_D^{\mathrm{alg}}(f) \rangle$, $\|f\|^2 := \langle f, f \rangle$ and

$$(5-3-18) \quad C_D(k, M, \chi) := \frac{(-2\sqrt{-1})^{2k-1} R_D(f) |D|^{k-3/2} M^2 c_\chi^{2k-3} \mathrm{res}_{s=1} L(s, \chi)}{2^3 3 G(\chi_D \chi)}$$

In particular, if $\chi = \mathbb{1}$, then

$$(5-3-19) \quad \frac{\|\mathrm{SK}_D^{\mathrm{alg}}(f)\|^2}{\|f\|^2} \in \mathbb{Q}(f).$$

Remark 5.5. Since $\mathrm{res}_{s=1} L(s, \chi) \in \bar{\mathbb{Q}}$ if and only if $\chi = \mathbb{1}$ by [MFM, Theorem 3.3.4], we see that

$$(5-3-20) \quad \frac{\|\mathrm{SK}_D^{\mathrm{alg}}(f)\|^2}{\|f\|^2} \in \bar{\mathbb{Q}} \text{ if and only if } \chi = \mathbb{1}.$$

6. p -adic interpolation of Fourier coefficients

In this section, we first present p -adic interpolation of $\{a_n(\theta_D(f_w^*))\}_{w \in W^{\mathrm{SK}}(M)}$. Using this result, we establish p -adic interpolation of $\{a_T(\mathrm{SK}_D(f_w))\}_{w \in W^{\mathrm{SK}}(M)}$ for $T \in \mathcal{L}_{>0}$ following Guerzhoy's method in [Gue00] which is discussed in [Kaw] as well.

6.1. p -adic interpolation of $\theta_D^{\mathrm{alg}}(f^*)$

Theorem 6.1 ([Mak17, Theorem 5.7]). *Let $k_0 \geq 2$ be an integer, χ a Dirichlet character with $c_\chi \mid N$ and D a fundamental discriminant with $\chi(-1)(-1)^{k_0-1}D > 0$ and $(D, c_\chi p) = p$. Assume (3-2-2) for $k = k_0$ and $M = Np$. Let $f \in S_{w_0}^{\mathrm{new}}(N, \chi^2)_\alpha$ be a primitive form with $w_0 := 2k_0 - 2 > \alpha + 1$ and K a complete discretely valued subfield of $\mathbb{C}_p$ containing the p -adic completion of the field obtained by adjoining $c_D(k_0, \chi)$ to $\mathbb{Q}(f^*, \chi)$. Assume $a_p(f)^2 \neq \chi^2(p)p^{w_0-1}$. Let $\{f_w\}_{w \in W(M)}$ be a Coleman family passing through f over K . Then for sufficiently large M , we can define $\theta_D(\mathbf{f}) \in A(B_K[w_0, p^{-M}][[q]])$ such that for any $w \in W^{\mathrm{SK}}(M)$, there exists $e_w \in K^\times$ independent of D satisfying*

$$(6-1-1) \quad \theta_D(\mathbf{f})(w) = e_w \theta_D^{\mathrm{alg}}(f_w^*)$$

and $e_{w_0} = 1$.

6.2. p -adic interpolation of $\theta_D^{\mathrm{alg}}(f)$

Theorem 6.2. *Let $k_0 \geq 2$ be an integer, χ a Dirichlet character with $c_\chi \mid N$ and D_0 a fundamental discriminant with $\chi(-1)(-1)^{k_0-1}D_0 > 0$ and $(D_0, c_\chi p) = p$. Assume (3-2-2) for $k = k_0$ and $M = Np$. Let $f \in S_{w_0}^{\mathrm{new}}(N, \chi^2)_\alpha$ be a primitive form with $w_0 := 2k_0 - 2 > \alpha + 1$ and K a complete discretely valued subfield of $\mathbb{C}_p$ containing the p -adic completion of the field obtained by adjoining $c_{D_0}(k_0, \chi)$ to $\mathbb{Q}(f^*, \chi)$. Assume $a_p(f)^2 \neq \chi^2(p)p^{w_0-1}$. Let $\{f_w\}_{w \in W(M)}$ be a Coleman family passing through f over K . Let D be a fundamental discriminant with $\chi(-1)(-1)^{k-1}D > 0$ and $(D, c_\chi) = 1$. When both $p \mid D$ and $\chi_{D_0 D/p^2}(p) = -1$ holds, we further assume $\chi^2 = \mathbb{1}$, $a_{|D_0|}(\theta_{D_0}(f)) \neq 0$, and the following condition:*

$$(6-2-1) \quad \bigcap_{\ell \nmid N} \mathrm{Ker} \left((T^+(\ell) - a_\ell(f)) |S_{k-1/2}^+(N, \chi) \right) \cong \mathbb{C},$$

where $\ell \nmid N$ runs over all primes $\ell \nmid N$. Then for sufficiently large M , we can define $a_{|D|}(\theta_{D_0}(\mathbf{f}^\circ)) \in A(B_K[w_0, p^{-M}])$ such that for any $w = 2k - 2 \in W^{\mathrm{SK}}(M)$, there exists $e_w \in K^\times$ satisfying

$$(6-2-2) \quad a_{|D|}(\theta_{D_0}(\mathbf{f}^\circ))(w) = e_w \left(1 - \chi_D \chi_0^{-1}(p) p^{k-2} a_p(f_w^*)^{-1} \right) a_{|D|}(\theta_{D_0}^{\mathrm{alg}}(f_w))$$

and $e_{w_0} = 1$. In particular, for any positive integer m and $w \in W(M)$, if $w \equiv w_0 \pmod{(p-1)p^{M+m}}$ and $m > \text{val}_p \left((1 - \chi_D \chi_0^{-1}(p) p^{k_0-2} a_p(f^*)^{-1}) a_{|D|} \left(\theta_{D_0}^{\text{alg}}(f) \right) \right)$, then

$$(6-2-3) \quad \begin{aligned} & \text{val}_p \left(e_w \left(1 - \chi_D \chi_0^{-1}(p) p^{k-2} a_p(f_w^*)^{-1} \right) a_{|D|} \left(\theta_{D_0}^{\text{alg}}(f_w) \right) \right) \\ &= \text{val}_p \left(\left(1 - \chi_D \chi_0^{-1}(p) p^{k_0-2} a_p(f^*)^{-1} \right) a_{|D|} \left(\theta_{D_0}^{\text{alg}}(f) \right) \right). \end{aligned}$$

Remark 6.3. Let the notation be the same as above.

- (1) By (6-2-3), we see that $\theta_D(f_w) \neq 0$ if $\theta_D(f) \neq 0$.
- (2) If N is square-free, then the condition (6-2-1) holds by [Koh82, Theorem 2.ii)].

Combining the theorem above with Theorem 3.3 gives the following:

Corollary 6.4. Let $k_0 \geq 2$ be an integer and χ a Dirichlet character with $c_\chi \parallel N$. Assume (3-2-2) for $k = k_0$ and $M = Np$. Let $f \in S_{w_0}^{\text{new}}(N, \chi^2)_\alpha$ be a primitive form with $w_0 := 2k_0 - 2 > \alpha + 1$ and K a complete discretely valued subfield of $\mathbb{C}_p$ containing the p -adic completion of the field obtained by adjoining $c_{D_0}(k_0, \chi)$ to $\mathbb{Q}(f^*, \chi)$. Assume $a_p(f)^2 \neq \chi^2(p) p^{w_0-1}$. Let $\{f_w\}_{w \in W(M)}$ be a Coleman family passing through f over K . Let D_0 and D be a fundamental discriminant with $\chi_{D_0} \chi(-1)(-1)^{k_0-2} = \chi_D \chi(-1)(-1)^{k_0-2} = -1$ and $(D_0, Np) = (D, Np) = p$. Assume $a_{|D_0|}(\theta_{D_0}(f)) \neq 0$ and $a_{|D|}(\theta_D(f)) \neq 0$. When $\chi_{D_0 D/p^2}(p) = -1$ holds, we further assume $\chi^2 = 1$ and (6-2-1). Then for any $w = 2k - 2 \in W^{\text{SK}}(M)$ with sufficiently large M ,

$$(6-2-4) \quad \text{val}_p \left(\frac{L^{\text{alg}}(k-1, f_w \otimes \chi_{D_0} \chi^{-1})}{L^{\text{alg}}(k-1, f_w \otimes \chi_D \chi^{-1})} \right) = \text{val}_p \left(\frac{L^{\text{alg}}(k_0-1, f \otimes \chi_{D_0} \chi^{-1})}{L^{\text{alg}}(k_0-1, f \otimes \chi_D \chi^{-1})} \right).$$

Remark 6.5.

$$(6-2-5) \quad \frac{G(\chi_{D_0} \chi) L(k-1, f_w \otimes \chi_{D_0} \chi^{-1})}{G(\chi_D \chi) L(k-1, f_w \otimes \chi_D \chi^{-1})} = \frac{L^{\text{alg}}(k-1, f_w \otimes \chi_{D_0} \chi^{-1})}{L^{\text{alg}}(k-1, f_w \otimes \chi_D \chi^{-1})} \in \mathbb{Q}(f_w, \chi)$$

6.3. p -adic interpolation of $\text{SK}_D^{\text{alg}}(f^*)$

Theorem 6.6. Let $k_0 \geq 2$ be an even integer, χ a Dirichlet character with $c_\chi \mid N$ and $\chi(-1) = 1$, and $D < 0$ a fundamental discriminant with $(D, c_\chi p) = p$. Assume (3-2-2) for $k = k_0$ and $M = Np$. Let $f \in S_{w_0}^{\text{new}}(N, \chi^2)_\alpha$ be a primitive form with $w_0 := 2k_0 - 2 > \alpha + 1$ and K a complete discretely valued subfield of $\mathbb{C}_p$ containing the p -adic completion of the field obtained by adjoining $c_D(k_0, \chi)$ to $\mathbb{Q}(f^*, \chi)$. Assume $a_p(f)^2 \neq \chi^2(p) p^{w_0-1}$. Let $\{f_w\}_{w \in W(M)}$ be a Coleman family passing through f over K . Then for sufficiently large M , we can define $\text{SK}_D^{\text{alg}}(\mathbf{f}) \in A(B_K[w_0, p^{-M}])[[q]]_2$ such that for any $w \in W^{\text{SK}}(M)$, there exists $e_w \in K^\times$ satisfying

$$(6-3-1) \quad \text{SK}_D^{\text{alg}}(\mathbf{f})(w) = e_w \text{SK}_D^{\text{alg}}(f_w^*)$$

and $e_{w_0} = 1$.

6.4. p -adic interpolation of $\text{SK}_D^{\text{alg}}(f)$

Theorem 6.7. Let $k_0 \geq 2$ be an even integer, χ a Dirichlet character modulo N and $D_0 < 0$ a fundamental discriminant with $(D_0, c_\chi p) = p$. Assume (3-2-2) for $k = k_0$ and $M = Np$. Let $f \in S_{w_0}^{\text{new}}(N, \chi^2)_\alpha$ be a primitive form with $w_0 := 2k_0 - 2 > \alpha + 1$ and K a complete discretely valued subfield of $\mathbb{C}_p$ containing the p -adic completion of the field obtained by adjoining $c_{D_0}(k_0, \chi)$ to $\mathbb{Q}(f^*, \chi)$. Assume $a_p(f)^2 \neq \chi^2(p) p^{w_0-1}$. Let $\{f_w\}_{w \in W(M)}$ be a Coleman family passing through f over K . Let $T \in \mathcal{L}_{>0}$ such that $D := -\det(2T)$ is a fundamental discriminant with $(D, c_\chi) = 1$ and $D \equiv 1 \pmod{4}$. When both $p \mid D$ and $\chi_{D_0 D/p^2}(p) = -1$ holds, we further assume

$\chi^2 = 1$, $a_{|D_0|}(\theta_{D_0}(f)) \neq 0$, and (6-2-1). Then for sufficiently large M , we can define an element $a_T(\mathrm{SK}_{D_0}(\mathbf{f}^\circ)) \in A(B_K[w_0, p^{-M}])$ such that for any $w = 2k - 2 \in W^{\mathrm{SK}}(M)$, there exists $e_w \in K^\times$ satisfying

$$(6-4-1) \quad a_T(\mathrm{SK}_{D_0}(\mathbf{f}^\circ))(w) = e_w \left(1 - \chi_D \chi_0^{-1}(p) p^{k-2} a_p(f_w^*)^{-1}\right) a_T(\mathrm{SK}_{D_0}^{\mathrm{alg}}(f_w))$$

and $e_{w_0} = 1$.

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