

Integral Means of Analytic Functions

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Abstract

For analytic functions $f(z)$ and $g(z)$ which satisfy the subordination $f(z) \prec g(z)$, J. E. Littlewood (Proc. London Math. Soc. **23**(1925), 481-519) has shown some interesting results for integral means of $f(z)$ and $g(z)$. The object of the present paper is to derive some applications of integral means by J. E. Littlewood. We also show interesting examples for our theorems.

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1. Introduction

Let $\mathcal{A}_n$ denote the class of functions $f(z)$ of the form

$$f(z) = z + \sum_{k=n+1}^{\infty} a_k z^k \quad (n \in \mathbb{N} := \{1, 2, 3, \dots\}) \quad (1.1)$$

that are analytic in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} \mid |z| < 1\}$. Let $\mathcal{S}_n^*(\alpha)$ be the subclass of $\mathcal{A}_n$ consisting of all functions $f(z)$ satisfying

$$\operatorname{Re} \left(\frac{zf'(z)}{f(z)} \right) > \alpha \quad (z \in \mathbb{U}) \quad (1.2)$$

for some $\alpha (0 \leq \alpha < 1)$. A function $f(z)$ in $\mathcal{S}_n^*(\alpha)$ is said to be starlike of order α in $\mathbb{U}$. Let $\mathcal{K}_n(\alpha)$ denote the subclass of $\mathcal{A}_n$ consisting of functions $f(z)$ which satisfy

$$\operatorname{Re} \left(1 + \frac{zf''(z)}{f'(z)} \right) > \alpha \quad (z \in \mathbb{U}) \quad (1.3)$$

for some $\alpha (0 \leq \alpha < 1)$. A function $f(z)$ belonging to $\mathcal{K}_n(\alpha)$ is called as a convex function of order α in $\mathbb{U}$. Note that $f(z) \in \mathcal{K}_n(\alpha)$ if and only if $zf'(z) \in \mathcal{S}_n^*(\alpha)$.

For the classes $\mathcal{S}_n^*(\alpha)$ and $\mathcal{K}_n(\alpha)$, Chatterjea [1] (also see Srivastava, Owa and Chatterjea [9]) has given the following results.

Theorem A. *If a function $f(z) \in \mathcal{A}_n$ satisfies*

$$\sum_{k=n+1}^{\infty} (k - \alpha) |a_k| \leq 1 - \alpha \quad (1.4)$$

for some $\alpha(0 \leq \alpha < 1)$, then $f(z) \in \mathcal{S}_n^*(\alpha)$.

Theorem B. If a function $f(z) \in \mathcal{A}_n$ satisfies

$$\sum_{k=n+1}^{\infty} k(k-\alpha) |a_k| \leq 1-\alpha. \quad (1.5)$$

for some $\alpha(0 \leq \alpha < 1)$, then $f(z) \in \mathcal{K}_n(\alpha)$.

When $n = 1$ in Theorem A and Theorem B, the results for $\mathcal{S}_1^*(\alpha)$ and $\mathcal{K}_1(\alpha)$ above were given by Silverman [7].

For analytic functions $f(z)$ and $g(z)$, the function $f(z)$ is said to be subordinate to $g(z)$ in $\mathbb{U}$ if there exists a function $w(z)$ analytic in $\mathbb{U}$ with $w(0) = 0$ and $|w(z)| < 1$, such that $f(z) = g(w(z))$. We denote this subordination by

$$f(z) \prec g(z) \quad (\text{cf. Duren}[2]).$$

For subordinations, Littlewood [4] has given the following integral mean.

Theorem C. If $f(z)$ and $g(z)$ are analytic in $\mathbb{U}$ with $f(z) \prec g(z)$, then, for $\mu > 0$ and $z = re^{i\theta}$ ($0 < r < 1$)

$$\int_0^{2\pi} |f(z)|^\mu d\theta \leq \int_0^{2\pi} |g(z)|^\mu d\theta.$$

Applying the Theorem C by Littlewood [4] above, Silvermann [8], Kim and Choi [3], Sekine, Tsurumi and Srivastava [6], and Owa, Tsurumi, Nunokawa and Sekine [5] have considered some interesting properties for integral means of analytic functions. In the present paper, we discuss some conditions of coefficients for integral means.

2. Integral means for $f(z)$ and $g(z)$

In this section, we discuss the integral means for $f(z) \in \mathcal{A}_n$ and $g(z)$ defined by

$$g(z) = z + b_j z^j + b_{2j-1} z^{2j-1} \quad (j \geq n+1). \quad (2.1)$$

Our first result for integral means is contained in

Theorem 2.1 Let $f(z) \in \mathcal{A}_n$ and $g(z)$ be given by (2.1). If $f(z)$ satisfies

$$\sum_{k=n+1}^{\infty} |a_k| \leq |b_{2j-1}| - |b_j| \quad (|b_j| < |b_{2j-1}|), \quad (2.2)$$

then, for $\mu > 0$ and $z = re^{i\theta}$ ($0 < r < 1$),

$$\int_0^{2\pi} |f(z)|^\mu d\theta \leq \int_0^{2\pi} |g(z)|^\mu d\theta. \quad (2.3)$$

Proof. By putting $z = re^{i\theta}$ ($0 < r < 1$), we see that

$$\int_0^{2\pi} |f(z)|^\mu d\theta = r^\mu \int_0^{2\pi} \left| 1 + \sum_{k=n+1}^{\infty} a_k z^{k-1} \right|^\mu d\theta$$

and

$$\int_0^{2\pi} |g(z)|^\mu d\theta = r^\mu \int_0^{2\pi} |1 + b_j z^{j-1} + b_{2j-1} z^{2j-2}|^\mu d\theta.$$

Applying Theorem C, we have to show that

$$1 + \sum_{k=n+1}^{\infty} a_k z^{k-1} \prec 1 + b_j z^{j-1} + b_{2j-1} z^{2j-2}.$$

Let us define the function $w(z)$ by

$$1 + \sum_{k=n+1}^{\infty} a_k z^{k-1} = 1 + b_j w(z)^{j-1} + b_{2j-1} w(z)^{2j-2},$$

or, by

$$b_{2j-1} w(z)^{2j-2} + b_j w(z)^{j-1} = \sum_{k=n+1}^{\infty} a_k z^{k-1}. \quad (2.4)$$

Since, for $z = 0$,

$$w(0)^{j-1} (b_{2j-1} w(0)^{j-1} + b_j) = 0,$$

there exists an analytic function $w(z)$ in $\mathbb{U}$ such that $w(0) = 0$.

Next, we prove the analytic function $w(z)$ satisfies $|w(z)| < 1$ ($z \in \mathbb{U}$) for

$$\sum_{k=n+1}^{\infty} |a_k| \leq |b_{2j-1}| - |b_j| \quad (|b_j| < |b_{2j-1}|).$$

By the equality (2.4), we know that

$$|b_{2j-1} w(z)^{2j-2} + b_j w(z)^{j-1}| \leq \left| \sum_{k=n+1}^{\infty} a_k z^{k-1} \right| < \sum_{k=n+1}^{\infty} |a_k|,$$

for $z \in \mathbb{U}$, hence,

$$|b_{2j-1}| |w(z)|^{2j-2} - |b_j| |w(z)|^{j-1} - \sum_{k=n+1}^{\infty} |a_k| < 0. \quad (2.5)$$

Letting $t = |w(z)|^{j-1}$ ($t \geq 0$) in (2.5), we define the function $G(t)$ by

$$G(t) = |b_{2j-1}| t^2 - |b_j| t - \sum_{k=n+1}^{\infty} |a_k| \quad (t \geq 0).$$

If $G(1) \geq 0$, then we have $t < 1$ for $G(t) < 0$. Therefore, for $|w(z)| < 1$ ($z \in \mathbb{U}$), we need

$$G(1) = |b_{2j-1}| - |b_j| - \sum_{k=n+1}^{\infty} |a_k| \geq 0,$$

that is,

$$\sum_{k=n+1}^{\infty} |a_k| \leq |b_{2j-1}| - |b_j|.$$

Consequently, if the inequality (2.2) holds true, there exists an analytic function $w(z)$ with $w(0) = 0$, $|w(z)| < 1$ ($z \in \mathbb{U}$) such that $f(z) = g(w(z))$. This completes the proof of Theorem 2.1.

Corollary 2.1. *Let $f(z) \in \mathcal{A}_n$ and $g(z)$ be given by (2.1). If $f(z)$ satisfies (2.2), then, for $0 < \mu \leq 2$ and $z = re^{i\theta}$ ($0 < r < 1$),*

$$\begin{aligned} \int_0^{2\pi} |f(z)|^\mu d\theta &\leq 2\pi r^\mu \left\{ 1 + |b_j|^2 r^{2(j-1)} + |b_{2j-1}|^2 r^{4(j-1)} \right\}^{\frac{\mu}{2}} \\ &< 2\pi \left\{ 1 + |b_j|^2 + |b_{2j-1}|^2 \right\}^{\frac{\mu}{2}}. \end{aligned} \quad (2.6)$$

Further, we have that $f(z) \in \mathcal{H}^p(\mathbb{U})$ for $0 < p \leq 2$, where $\mathcal{H}^p$ denotes the Hardy space (cf. Duren [2]).

Proof. Since,

$$\int_0^{2\pi} |g(z)|^\mu d\theta = \int_0^{2\pi} |z|^\mu |1 + b_j z^{j-1} + b_{2j-1} z^{2j-2}|^\mu d\theta,$$

applying Hölder inequality for $0 < \lambda < 2$, we obtain that

$$\begin{aligned} &\int_0^{2\pi} |g(z)|^\mu d\theta \\ &\leq \left(\int_0^{2\pi} (|z|^\mu)^{\frac{2}{2-\mu}} d\theta \right)^{\frac{2-\mu}{2}} \left\{ \int_0^{2\pi} (|1 + b_j z^{j-1} + b_{2j-1} z^{2j-2}|^\mu)^{\frac{2}{\mu}} d\theta \right\}^{\frac{\mu}{2}} \\ &= \left(r^{\frac{2\mu}{2-\mu}} \int_0^{2\pi} d\theta \right)^{\frac{2-\mu}{2}} \left(\int_0^{2\pi} |1 + b_j z^{j-1} + b_{2j-1} z^{2j-2}|^2 d\theta \right)^{\frac{\mu}{2}} \\ &= \left(2\pi r^{\frac{2\mu}{2-\mu}} \right)^{\frac{2-\mu}{2}} \left\{ 2\pi \left(1 + |b_j|^2 r^{2(j-1)} + |b_{2j-1}|^2 r^{4(j-1)} \right) \right\}^{\frac{\mu}{2}} \\ &= 2\pi r^\mu \left(1 + |b_j|^2 r^{2(j-1)} + |b_{2j-1}|^2 r^{4(j-1)} \right)^{\frac{\mu}{2}} \\ &< 2\pi \left(1 + |b_j|^2 + |b_{2j-1}|^2 \right)^{\frac{\mu}{2}}. \end{aligned}$$

Further, it is easy to see that, for $\mu = 2$,

$$\begin{aligned} \int_0^{2\pi} |f(z)|^2 d\theta &\leq 2\pi r^2 \left(1 + |b_j|^2 r^{2j-1} + |b_{2j-1}|^2 r^{4(j-1)} \right) \\ &< 2\pi \left(1 + |b_j|^2 + |b_{2j-1}|^2 \right). \end{aligned}$$

From the above, we also have that, for $0 < \mu \leq 2$,

$$\sup_{z \in \mathbb{U}} \frac{1}{2\pi} \int_0^{2\pi} |f(z)|^\mu d\theta < \left(1 + |b_j|^2 + |b_{2j-1}|^2\right)^{\frac{\mu}{2}} < \infty,$$

which observe that $f(z) \in \mathcal{H}^2(\mathbb{U})$. Noting that $\mathcal{H}^q \subset \mathcal{H}^p$ ($0 < p < q < \infty$), we complete the proof.

Example 2.1. Let $f(z) \in \mathcal{A}_n$ satisfy the coefficient inequality (1.4) in Theorem A and

$$g(z) = z + \frac{n}{n+1-\alpha} \epsilon z^j + \delta z^{2j-1} \quad (|\epsilon| = |\delta| = 1) \quad (2.7)$$

with $0 \leq \alpha < 1$. Then $b_j = (n\epsilon)/(n+1-\alpha)$ and $b_{2j-1} = \delta$.

By virtue of (1.4), we observe that

$$\sum_{k=n+1}^{\infty} |a_k| \leq \frac{1-\alpha}{n+1-\alpha} = 1 - \frac{n}{n+1-\alpha} = |b_{2j-1}| - |b_j|.$$

Therefore, $f(z)$ and $g(z)$ satisfy the conditions in Theorem 2.1. Thus, we have, for $0 < \mu \leq 2$ and $z = re^{i\theta}$ ($0 < r < 1$),

$$\begin{aligned} & \int_0^{2\pi} |f(z)|^\mu d\theta \\ &= 2\pi r^\mu \left\{ 1 + \left(\frac{n}{n+1-\alpha} \right)^2 r^{2(j-1)} + r^{4(j-1)} \right\}^{\frac{\mu}{2}} \\ &< 2\pi \left\{ 2 + \left(\frac{n}{n+1-\alpha} \right)^2 \right\}^{\frac{\mu}{2}}. \end{aligned}$$

Using the same technique as in the proof of Theorem 2.1, we derive the following theorem.

Theorem 2.2. Let $f(z) \in \mathcal{A}_n$ and $g(z)$ be given by (2.1). If $f(z)$ satisfies

$$\sum_{k=n+1}^{\infty} k |a_k| \leq (2j-1)|b_{2j-1}| - j|b_j| \quad (j|b_j| < (2j-1)|b_{2j-1}|), \quad (2.8)$$

then, for $\mu > 0$ and $z = re^{i\theta}$ ($0 < r < 1$),

$$\int_0^{2\pi} |f'(z)|^\mu d\theta \leq \int_0^{2\pi} |g'(z)|^\mu d\theta. \quad (2.9)$$

Further, with the help of Hölder inequality, we have

Corollary 2.2. Let $f(z) \in \mathcal{A}_n$ and $g(z)$ be given by (2.1). If $f(z)$ satisfies (2.8), then, for $0 < \mu \leq 2$ and $z = re^{i\theta}$ ($0 < r < 1$),

$$\begin{aligned} \int_0^{2\pi} |f'(z)|^\mu d\theta &\leq 2\pi \left\{ 1 + j^2 |b_j|^2 r^{2(j-1)} + (2j-1)^2 |b_{2j-1}|^2 r^{4(j-1)} \right\}^{\frac{\mu}{2}} \\ &< 2\pi \left\{ 1 + j^2 |b_j|^2 + (2j-1)^2 |b_{2j-1}|^2 \right\}^{\frac{\mu}{2}}. \end{aligned} \quad (2.10)$$

Example 2.2. Let $f(z) \in \mathcal{A}_n$ satisfy the coefficient inequality (1.5) in Theorem B and

$$g(z) = z + \frac{n\epsilon}{j(n+1-\alpha)}z^j + \frac{\delta}{2j-1}z^{2j-1} \quad (|\epsilon| = |\delta| = 1) \quad (2.11)$$

with $0 \leq \alpha < 1$. Then,

$$b_j = \frac{n\epsilon}{j(n+1-\alpha)} \quad \text{and} \quad b_{2j-1} = \frac{\delta}{2j-1}.$$

Since

$$\sum_{k=n+1}^{\infty} k|a_k| \leq \frac{1-\alpha}{n+1-\alpha} = 1 - \frac{n}{n+1-\alpha} = (2j-1)|b_{2j-1}| - j|b_j|,$$

$f(z)$ and $g(z)$ satisfy the conditions in Theorem 2.2. Thus, by Corollary 2.2, we have, for $0 < \mu \leq 2$ and $z = re^{i\theta}$ ($0 < r < 1$),

$$\begin{aligned} \int_0^{2\pi} |f'(z)|^\mu d\theta &= 2\pi \left\{ 1 + \left(\frac{n}{n+1-\alpha} \right)^2 r^{2(j-1)} + r^{4(j-1)} \right\}^{\frac{\mu}{2}} \\ &< 2\pi \left\{ 2 + \left(\frac{n}{n+1-\alpha} \right)^2 \right\}^{\frac{\mu}{2}}. \end{aligned}$$

3. Integral means for $f(z)$ and $h(z)$

In this section, we introduce an analytic function $h(z)$ given by

$$h(z) = z + b_j z^j + b_{2j-1} z^{2j-1} + b_{3j-2} z^{3j-2} \quad (j \geq n+1) \quad (3.1)$$

Theorem 3.1. Let $f(z) \in \mathcal{A}_n$ and $h(z)$ be given by (3.1), if $f(z)$ satisfies

$$\sum_{k=n+1}^{\infty} |a_k| \leq |b_{3j-2}| - |b_{2j-1}| - |b_j| \quad (|b_j| + |b_{2j-1}| < |b_{3j-2}|), \quad (3.2)$$

then, for $\mu > 0$ and $z = re^{i\theta}$ ($0 < r < 1$),

$$\int_0^{2\pi} |f(z)|^\mu d\theta \leq \int_0^{2\pi} |h(z)|^\mu d\theta \quad (\mu > 0). \quad (3.3)$$

Proof. In a same way with the proof of Theorem 2.1, we have to show that there exists an analytic function $w(z)$ with $w(0) = 0$ and $|w(z)| < 1$ ($z \in \mathbb{U}$) such that $f(z) = h(w(z))$. Note that this function $w(z)$ is defined by

$$b_{3j-2}w(z)^{3j-3} + b_{2j-1}w(z)^{2j-2} + b_jw(z)^{j-1} = \sum_{k=n+1}^{\infty} a_k z^{k-1}. \quad (3.4)$$

Since, for $z = 0$,

$$w(0)^{j-1} (b_{3j-2}w(0)^{2j-2} + b_{2j-1}w(0)^{j-1} + b_j) = 0,$$

we consider $w(z)$ such as $w(0) = 0$.

On the other hand, we have that

$$|b_{3j-2}| |w(z)|^{3(j-1)} - |b_{2j-1}| |w(z)|^{2(j-1)} - |b_j| |w(z)|^{j-1} - \sum_{k=n+1}^{\infty} |a_k| < 0. \quad (3.5)$$

Putting $t = |w(z)|^{j-1}$ ($t \geq 0$), we define the function $H(t)$ by ,

$$H(t) = |b_{3j-2}| t^3 - |b_{2j-1}| t^2 - |b_j| t - \sum_{k=n+1}^{\infty} |a_k| \quad (t \geq 0).$$

It follows that $H(0) \leq 0$, and

$$H'(t) = 3|b_{3j-2}| t^2 - 2|b_{2j-1}| t - |b_j|.$$

Since the discriminant of $H'(t) = 0$ is greater than 0, if $H'(1) \geq 0$, then $t < 1$ for $H(t) < 0$. Therefore, we need the following inequality

$$H(1) = |b_{3j-2}| - |b_{2j-1}| - |b_j| - \sum_{k=n+1}^{\infty} |a_k| \geq 0,$$

or

$$\sum_{k=n+1}^{\infty} |a_k| \leq |b_{3j-2}| - |b_{2j-1}| - |b_j|.$$

This completes the proof of Theorem 3.1.

Corollary 3.1. Let $f(z) \in \mathcal{A}_n$ and $h(z)$ be given by (3.1). If $f(z)$ satisfies (3.2), then , for $0 < \mu \leq 2$ and $z = re^{i\theta}$ ($0 < r < 1$),

$$\begin{aligned} \int_0^{2\pi} |f(z)|^\mu d\theta &\leq 2\pi r^\mu \left(1 + |b_j|^2 r^{2(j-1)} + |b_{2j-1}|^2 r^{4(j-1)} + |b_{3j-2}|^2 r^{6(j-1)} \right)^{\frac{\mu}{2}} \\ &< 2\pi \left(1 + |b_j|^2 + |b_{2j-1}|^2 + |b_{3j-2}|^2 \right)^{\frac{\mu}{2}}. \end{aligned} \quad (3.6)$$

Further, we have that $f(z) \in \mathcal{H}^p(\mathbb{U})$ for $0 < p \leq 2$.

Example 3.1. Let $f(z) \in \mathcal{A}_n$ satisfy the coefficient inequality (1.4) in Theorem A and $h(z)$ be given by

$$h(z) = z + \frac{nt}{n+1-\alpha} \epsilon z^j + \frac{n(1-t)}{n+1-\alpha} \delta z^{2j-1} + \sigma z^{3j-2} \quad (0 \leq t \leq 1, |\epsilon| = |\delta| = |\sigma| = 1) \quad (3.7)$$

with $0 \leq \alpha < 1$. Then

$$b_j = \frac{nt}{n+1-\alpha} \epsilon, \quad b_{2j-1} = \frac{n(1-t)}{n+1-\alpha} \delta, \quad \text{and} \quad b_{3j-2} = \sigma.$$

In view of (1.4), we see that

$$\begin{aligned} \sum_{k=n+1}^{\infty} |a_k| &\leq \frac{1-\alpha}{n+1-\alpha} = 1 - \frac{n(1-t)}{n+1-\alpha} - \frac{nt}{n+1-\alpha} \\ &= |b_{3j-2}| - |b_{2j-1}| - |b_j|. \end{aligned}$$

This shows us that $f(z)$ and $h(z)$ satisfy the conditions in Theorem 3.1. Therefore, applying Corollary 3.1, we have, for $0 < \mu \leq 2$ and $z = re^{i\theta}$ ($0 < r < 1$),

$$\begin{aligned} &\int_0^{2\pi} |f(z)|^\mu d\theta \\ &= 2\pi r^\mu \left\{ 1 + \left(\frac{nt}{n+1-\alpha} \right)^2 r^{2(j-1)} + \frac{n(1-t)}{(n+1-\alpha)} r^{4(j-1)} + r^{6(j-1)} \right\}^{\frac{\mu}{2}} \\ &< 2\pi \left\{ 2 + (2t^2 - 2t + 1) \left(\frac{n}{n+1-\alpha} \right)^2 \right\}^{\frac{\mu}{2}}. \end{aligned}$$

Finally, for the integral means of $f'(z)$ and $h'(z)$, we derive the following theorem.

Theorem 3.2. Let $f(z) \in \mathcal{A}_n$ and $h(z)$ be given by (3.1). If $f(z)$ satisfies

$$\begin{aligned} \sum_{k=n+1}^{\infty} k |a_k| &\leq (3j-2)|b_{3j-2}| - (2j-1)|b_{2j-1}| - j|b_j| \\ &\quad (j|b_j| + (2j-1)|b_{2j-1}|(2j-1) < (3j-2)|b_{3j-2}|), \end{aligned} \quad (3.8)$$

then, for $\mu > 0$ and $z = re^{i\theta}$ ($0 < r < 1$),

$$\int_0^{2\pi} |f'(z)|^\mu d\theta \leq \int_0^{2\pi} |h'(z)|^\mu d\theta. \quad (3.9)$$

The proof of this theorem is similar to one of Theorem 2.2. Therefore, we omit the proof of the theorem.

Corollary 3.2. Let $f(z) \in \mathcal{A}_n$ and $h(z)$ be given by (3.1). If $f(z)$ satisfies (3.8), then, for $0 < \mu \leq 2$ and $z = re^{i\theta}$ ($0 < r < 1$),

$$\begin{aligned} &\int_0^{2\pi} |f'(z)|^\mu d\theta \\ &\leq 2\pi \left\{ 1 + j^2 |b_j|^2 r^{2j-1} + (2j-1)^2 |b_{2j-1}|^2 r^{4(j-1)} + (3j-2)^2 |b_{3j-2}|^2 r^{6(j-1)} \right\}^{\frac{\mu}{2}} \\ &< 2\pi \left\{ 1 + j^2 |b_j|^2 + (2j-1)^2 |b_{2j-1}|^2 + (3j-2)^2 |b_{3j-2}|^2 \right\}^{\frac{\mu}{2}}. \end{aligned} \quad (3.10)$$

Example 3.2. Let $f(z) \in \mathcal{A}_n$ satisfy the coefficient inequality (1.5) in Theorem B and $h(z)$ be given by

$$\begin{aligned} h(z) &= z + \frac{nt}{j(n+1-\alpha)} \epsilon z^j + \frac{n(1-t)}{n+1-\alpha} \delta z^{2j-1} + \frac{\sigma}{3j-2} z^{3j-2} \\ &\quad (0 \leq t \leq 1, |\epsilon| = |\delta| = |\sigma| = 1) \end{aligned} \quad (3.11)$$

with $0 \leq \alpha < 1$. It follows that

$$b_j = \frac{nte}{j(n+1-\alpha)}, \quad b_{2j-1} = \frac{n(1-t)\delta}{(2j-1)(n+1-\alpha)}, \quad \text{and} \quad b_{3j-2} = \frac{\sigma}{3j-2}.$$

By the coefficient inequality (1.5), we obtain that

$$\begin{aligned} \sum_{k=n+1}^{\infty} k|a_k| &\leq \frac{1-\alpha}{n+1-\alpha} = 1 - \frac{n}{n+1-\alpha} \\ &= (3j-2)|b_{3j-2}| - (2j-1)|b_{2j-1}| - j|b_j|. \end{aligned}$$

This gives us that $f(z)$ and $h(z)$ satisfy the conditions in Theorem 3.2. Thus, applying Corollary 3.2, we see, for $0 < \mu \leq 2$ and $z = re^{i\theta}$ ($0 < r < 1$),

$$\begin{aligned} &\int_0^{2\pi} |f'(z)|^\mu d\theta \\ &= 2\pi r^\mu \left\{ 1 + \left(\frac{nt}{n+1-\alpha} \right)^2 r^{2(j-1)} + \frac{n(1-t)}{(n+1-\alpha)} r^{4(j-1)} + r^{6(j-1)} \right\}^{\frac{\mu}{2}} \\ &< 2\pi \left\{ 2 + (2t^2 - 2t + 1) \left(\frac{n}{n+1-\alpha} \right)^2 \right\}^{\frac{\mu}{2}}. \end{aligned}$$

4. Appendix

Applying the Hölder inequality for analytic functions $F(z)$ and $G(z)$, we obtain, for $z = re^{i\theta}$ ($0 < r < 1$),

$$\int_a^b |F(z)G(z)| d\theta \leq \left(\int_a^b |F(z)|^p d\theta \right)^{\frac{1}{p}} \left(\int_a^b |G(z)|^q d\theta \right)^{\frac{1}{q}}. \quad (4.1)$$

with $p > 1$ and $1/p + 1/q = 1$. Note that the inequality (4.1) gives

$$\int_0^{2\pi} |F(z)|^p d\theta \geq \frac{\left(\int_0^{2\pi} |F(z)G(z)| d\theta \right)^p}{\left(\int_0^{2\pi} |G(z)|^q dz \right)^{\frac{p}{q}}}. \quad (4.2)$$

Considering $p = \mu/2$, $q = \mu/(\mu - 2)$, and $\mu > 2$ in in (4.2), we have, for $f(z)$ in the class $\mathcal{A}_n$,

$$\begin{aligned}
\int_0^{2\pi} |f(z)|^\mu d\theta &= \int_0^{2\pi} \left(|f(x)|^2 \right)^{\frac{\mu}{2}} d\theta \\
&\geq \frac{\left(\int_0^{2\pi} |f(z)|^2 d\theta \right)^{\frac{\mu}{2}}}{\left(\int_0^{2\pi} d\theta \right)^{\frac{\mu-2}{2}}} \\
&= (2\pi)^{\frac{2-\mu}{2}} \left\{ 2\pi \left(r^2 + \sum_{k=n+1}^{\infty} |a_k|^2 r^{2k} \right) \right\}^{\frac{\mu}{2}} \\
&= 2\pi r^\mu \left(1 + \sum_{k=n+1}^{\infty} |a_k|^2 r^{2(k-1)} \right)^{\frac{\mu}{2}}.
\end{aligned}$$

When $\mu = 2$, we also have that, for $z = re^{i\theta}$ ($0 < r < 1$),

$$\begin{aligned}
\int_0^{2\pi} |f(z)|^2 d\theta &= 2\pi r^2 \left(1 + \sum_{k=n+1}^{\infty} |a_k|^2 r^{2(k-1)} \right) \\
&< 2\pi \left(1 + \sum_{k=n+1}^{\infty} |a_k|^2 \right).
\end{aligned}$$

Thus, we conclude that

Theorem 4.1 Let $f(z) \in \mathcal{A}_n$ and $\mu \geq 2$. Then, for $z = re^{i\theta}$ ($0 < r < 1$),

$$\int_0^{2\pi} |f(z)|^\mu d\theta \geq 2\pi r^\mu \left(1 + \sum_{k=n+1}^{\infty} |a_k|^2 r^{2(k-1)} \right)^{\frac{\mu}{2}}.$$

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