

Three-dimensional hypersurface purely elliptic
singularities of $(0,1)$ -type

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Introduction.

In the theory of normal two-dimensional singularities, simple elliptic singularities and cusp singularities are regarded as the next most reasonable class of singularities after rational double points. They are characterized as two-dimensional purely elliptic singularities of $(0,1)$ -type and of $(0,0)$ -type, respectively. What are natural generalizations in three-dimensional case of simple elliptic singularities. The notion of a simple K3 singularity was defined in (IW1, IW2) as a three-dimensional isolated Gorenstein purely elliptic singularity of $(0,2)$ -type. A simple K3 singularity is characterized as a normal three-dimensional isolated singularity such that the exceptional set of any $\mathbb{Q}$ -factorial terminal modification is a normal K3 surface (see (IW2)). Here we are interested in a three-dimensional hypersurface purely elliptic singularities of $(0,1)$ -type. Let $f \in \mathbb{C}(z_0, z_1, z_2, z_3)$ be a polynomial which is nondegenerate with respect to its Newton boundary $\Gamma(f)$ in the sense of (V), and whose zero locus $X =$

$\{f = 0\}$ in $\mathbb{C}^4$ has an isolated singularity at the origin $0 \in \mathbb{C}^4$. Then the condition for (X, x) to be a purely elliptic singularity of $(0,1)$ -type is given by a property of the Newton boundary of $\Gamma(f)$ of f .

In this paper, we classify the principal parts of defining equations, which define three-dimensional hypersurface purely elliptic singularities of $(0,1)$ -type.

1. Preliminaries. In this section, we recall some definitions and results from (I1), (IW2), (W1) and (W2).

First we define the plurigenera δ_m , $m \in \mathbb{N}$, for normal isolated singularities and define purely elliptic singularities.

Let (X, x) be a normal isolated singularity in an n -dimensional analytic space X , and $\pi : (M, E) \longrightarrow (X, x)$ a good resolution. In the following, we assume that X is a sufficiently small Stein neighbourhood of x .

Definition 1.1 ((W1)). Let (X, x) be a normal isolated singularity. For any positive integer m ,

$$\delta_m(X, x) := \dim_{\mathbb{C}} \Gamma(X - \{x\}, \mathcal{O}(mK)) / L^{2/m}(X - \{x\}),$$

where K is the canonical line bundle on $X - \{x\}$, and $L^{2/m}(X - \{x\})$ is the set of all $L^{2/m}$ -integrable (at x) holomorphic m -ple n -form on $X - \{x\}$.

Then δ_m is finite and does not depend on the choice of a Stein neighbourhood X .

Definition 1.2 ((W1)). A singularity (X, x) is said to be purely elliptic if $\delta_m = 1$ for every $m \in \mathbb{N}$.

When X is a two-dimensional analytic space, purely elliptic singularities are quasi-Gorenstein singularities, i.e., there is a nowhere vanishing holomorphic 2-form on $X - \{x\}$ (see (88)). But in higher dimension, purely elliptic singularities are not always quasi-Gorenstein (see (IW1)).

In the following, we assume that (X, x) is quasi-Gorenstein. Let $E = \bigcup E_i$ be the decomposition of the exceptional set E into its irreducible components, and write

$$K_M = \pi^* K_X + \sum_{i \in I} m_i E_i - \sum_{j \in J} m_j E_j$$

with $m_i \geq 0$, $m_j > 0$. Ishii (I1) defined the essential part of the exceptional set E as $E_J = \sum_{j \in J} m_j E_j$, and showed that if (X, x) is purely elliptic, then $m_j = 1$ for all $j \in J$.

Definition 1.3 (Ishii (I1)). A quasi-Gorenstein purely elliptic singularity (X, x) is of $(0, i)$ -type if $H^{n-1}(E_J, \mathcal{O}_E)$ consists of the $(0, i)$ -Hodge component $H^{0, i}(E_J)$, where

$$\mathbb{C} \simeq H^{n-1}(E_J, \mathcal{O}_E) = \text{Gr}_F^0 H^{n-1}(E_J) = \bigoplus_{i=1}^{n-1} H^{0, i}(E_J).$$

n -dimensional quasi-Gorenstein purely elliptic singularities are classified into $2n$ classes, including the condition that the singularity is Cohen-Macaulay or not.

Next we consider the case where (X, x) is a hypersurface singularity defined by a nondegenerate polynomial $f = \sum a_\nu z^\nu \in$

$\mathbb{C}(z_0, z_1, \dots, z_n)$, and $x = 0 \in \mathbb{C}^{n+1}$. Recall that the Newton boundary $\Gamma(f)$ of f is the union of the compact faces of $\Gamma_+(f)$, where $\Gamma_+(f)$ is the convex hull of $\bigcup_{a_v \neq 0} (v + \mathbb{R}_0^{n+1})$ in $\mathbb{R}^{n+1}$. For any face Δ of $\Gamma_+(f)$, set $f_\Delta := \sum_{v \in \Delta} a_v z^v$. We say f to be nondegenerate, if

$$\frac{\partial f_\Delta}{\partial z_0} = \frac{\partial f_\Delta}{\partial z_1} = \dots = \frac{\partial f_\Delta}{\partial z_n} = 0$$

has no solution in $(\mathbb{C}^*)^{n+1}$ for any face Δ . When f is nondegenerate, the condition for (X, x) to be a purely elliptic singularity of $(0, i)$ -type is given as follows:

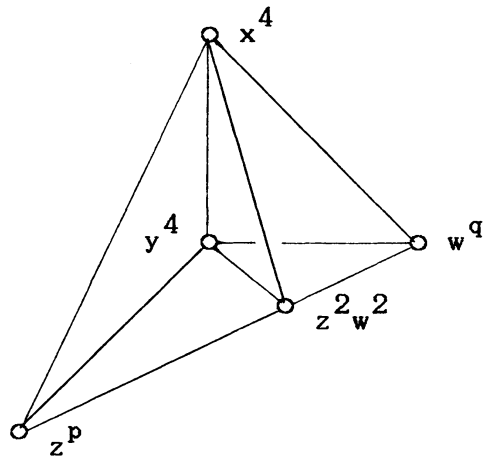
Theorem 1.4. Let f be a nondegenerate polynomial and suppose $X = \{f = 0\}$ has an isolated singularity at $x = 0 \in \mathbb{C}^{n+1}$.

(1) (X, x) is purely elliptic if and only if $(1, 1, \dots, 1) \in \Gamma(f)$.

(2) Let $n = 3$ and let Δ_0 be the face of $\Gamma(f)$ containing the point $(1, 1, 1, 1)$ in the relative interior of Δ_0 . Then (X, x) is a singularity of $(0, 1)$ -type if and only if $\dim_{\mathbb{R}} \Delta_0 = 2$.

Example 1.5. Let f be a polynomial of the form $f = x^4 + y^4 + (zw)^2 + z^p + w^q$ ($p, q \geq 5$). Then $\{f = 0\}$ has a purely elliptic singularity of $(0, 1)$ -type at the origin in $\mathbb{C}^4$. The two-dimensional face, spanned by $(4, 0, 0, 0)$, $(0, 4, 0, 0)$ and $(0, 0, 2, 2)$, contains the point $(1, 1, 1, 1)$ in its interior with respect to the relative topology. The principal parts of the

polynomial is $f_{\Delta_0} = x^4 + y^4 + (zw)^2$. Then the weights of f is $\alpha(f) = (\frac{1}{4}, \frac{1}{4}, \alpha, \frac{1}{2} - \alpha)$, not uniquely determined.



$$f = x^4 + y^4 + (zw)^2 + z^p + w^q$$

$$f_{\Delta_0} = x^4 + y^4 + (zw)^2$$

$$\alpha(f) = (\frac{1}{4}, \frac{1}{4}, r, \frac{1}{2} - r)$$

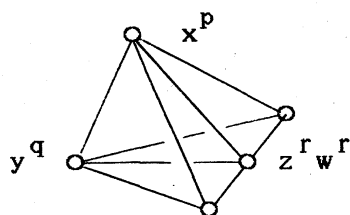
Thus if f is nondegenerate and defines a three-dimensional purely elliptic singularity of $(0,1)$ -type, then f_{Δ_0} is a quasi-homogeneous polynomial of a not uniquely determined set α of weights. Let $\alpha = (\alpha_1, \alpha_2, \alpha_3, \alpha_4) \in \mathbb{Q}_+^4$ be one of such sets of weights. Then $\deg_{\alpha}(v) := \sum_{i=1}^4 \alpha_i v_i = 1$ for any $v \in \Delta_0$. In particular, $\sum_{i=1}^4 \alpha_i = 1$, since $(1,1,1,1)$ is always contained in Δ_0 .

2. Principal parts of hypersurface purely elliptic singularities of $(0,1)$ -type

In this section, we calculate the principal parts of hypersurface purely elliptic singularities of $(0,1)$ -type defined by nondegenerate polynomials.

In the following can be found the complete list of the principal parts of defining equations:

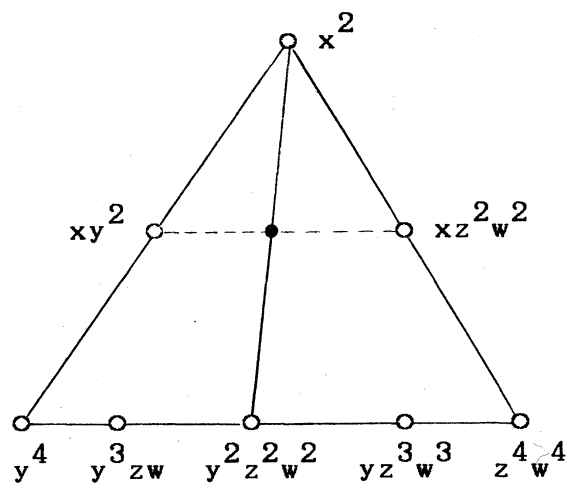
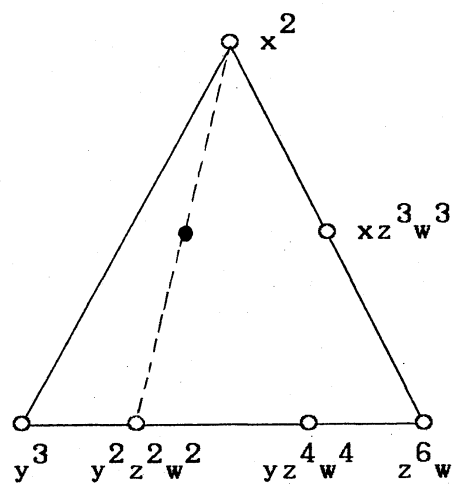
I.



$$\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1 \quad (p \leq q)$$

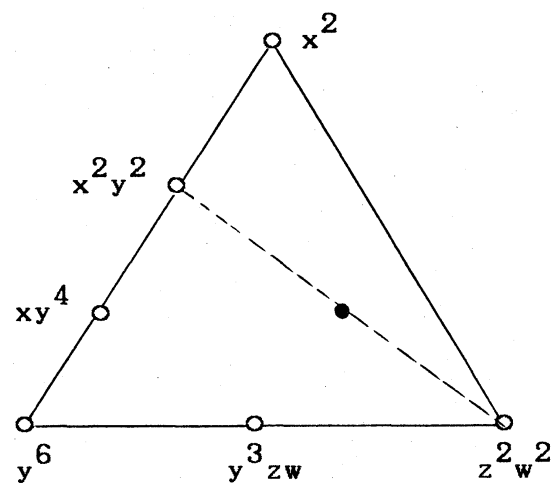
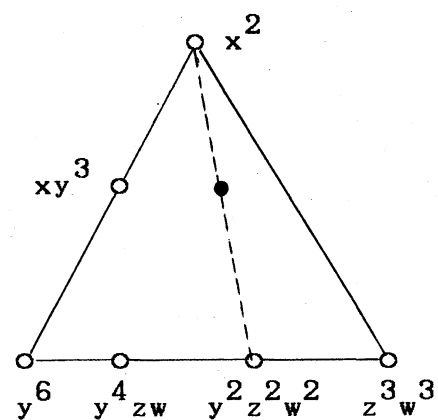
$$1. \left(\frac{1}{2}, \frac{1}{3}, \alpha, \frac{1}{6} - \alpha \right)$$

$$2. \left(\frac{1}{2}, \frac{1}{4}, \alpha, \frac{1}{4} - \alpha \right)$$

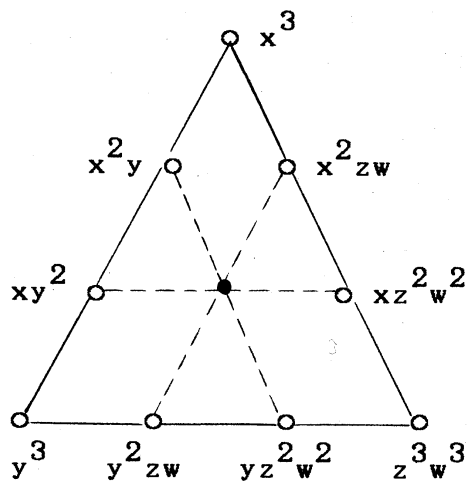


$$3. \left(\frac{1}{2}, \frac{1}{6}, \alpha, \frac{1}{3} - \alpha \right)$$

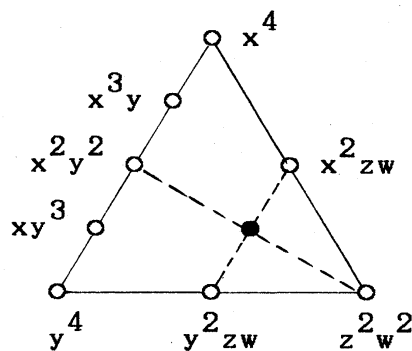
$$4. \left(\frac{1}{3}, \frac{1}{6}, \alpha, \frac{1}{2} - \alpha \right)$$



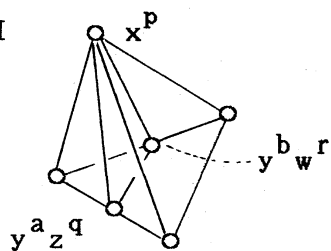
5. $(\frac{1}{3}, \frac{1}{3}, \alpha, \frac{1}{3} - \alpha)$



6. $(\frac{1}{4}, \frac{1}{4}, \alpha, \frac{1}{2} - \alpha)$

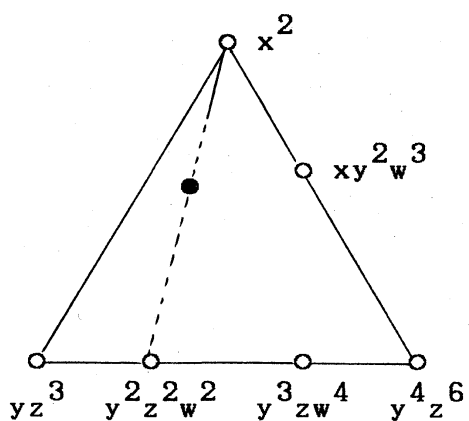


II

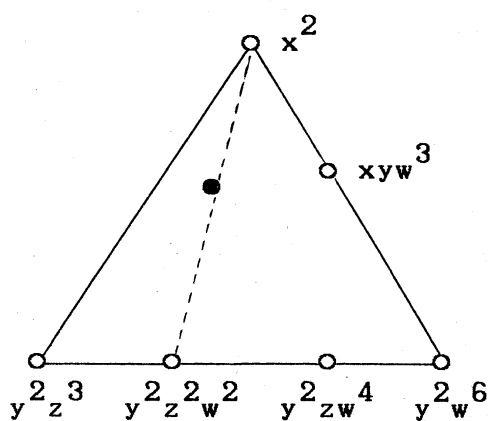


$$\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1, \quad \frac{a}{q} + \frac{b}{r} = 1 \quad (a \leq b)$$

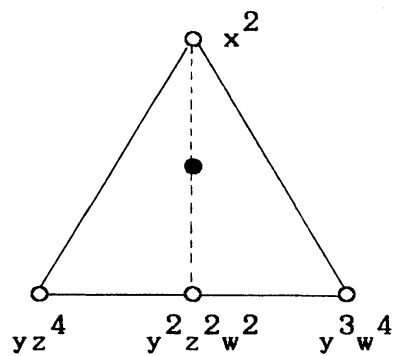
1. $(\frac{1}{2}, \alpha, \frac{1-\alpha}{3}, \frac{1-4\alpha}{6})$



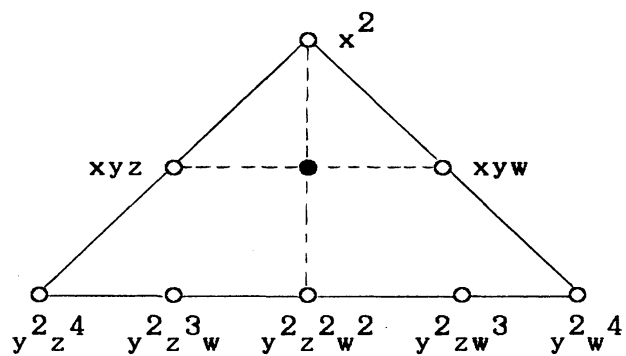
2. $(\frac{1}{2}, \alpha, \frac{1-2\alpha}{3}, \frac{1-2\alpha}{6})$



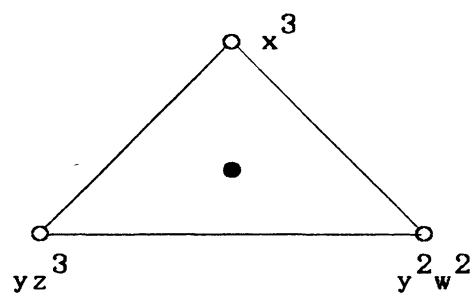
3. $(\frac{1}{2}, \alpha, \frac{1-\alpha}{4}, \frac{1-3\alpha}{4})$



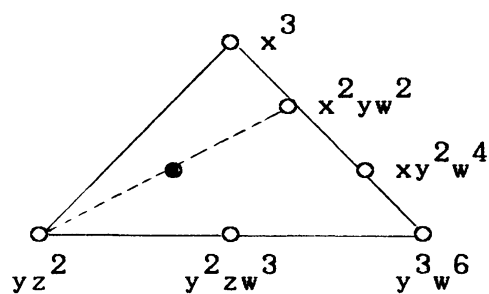
4. $(\frac{1}{2}, \alpha, \frac{1-2\alpha}{4}, \frac{1-2\alpha}{4})$



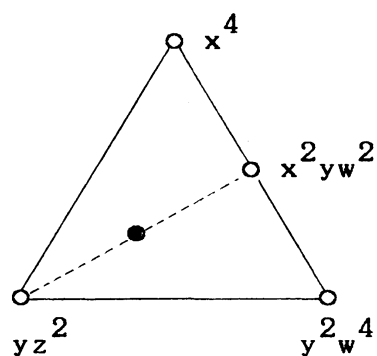
5. $(\frac{1}{3}, \alpha, \frac{1-\alpha}{3}, \frac{1-2\alpha}{3})$



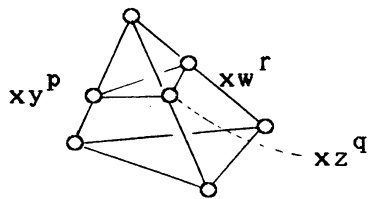
6. $(\frac{1}{3}, \alpha, \frac{1-\alpha}{2}, \frac{1-2\alpha}{6})$



7. $(\frac{1}{4}, \alpha, \frac{1-\alpha}{2}, \frac{1-2\alpha}{4})$

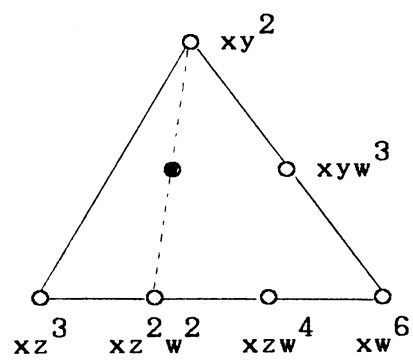


III

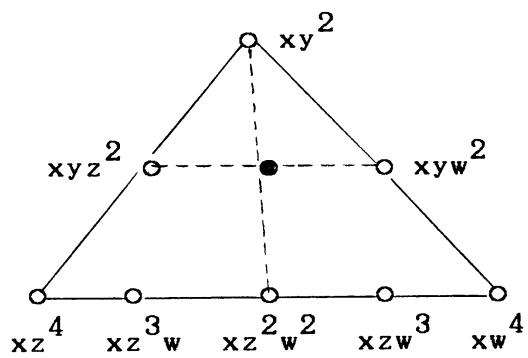


$$\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1 \quad (p \leq q \leq r)$$

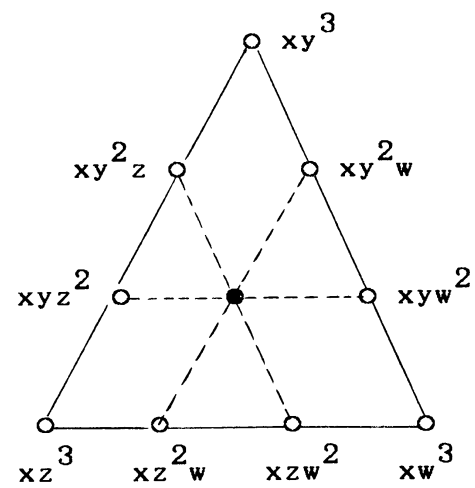
$$1. \left(\alpha, \frac{1-\alpha}{2}, \frac{1-\alpha}{3}, \frac{1-\alpha}{6} \right)$$



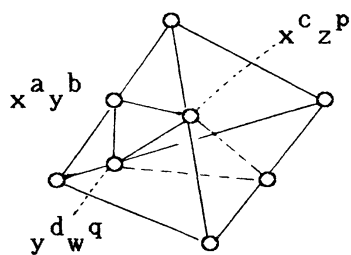
$$2. \left(\alpha, \frac{1-\alpha}{2}, \frac{1-\alpha}{4}, \frac{1-\alpha}{4} \right)$$



$$3. \left(\alpha, \frac{1-\alpha}{3}, \frac{1-\alpha}{3}, \frac{1-\alpha}{3} \right)$$

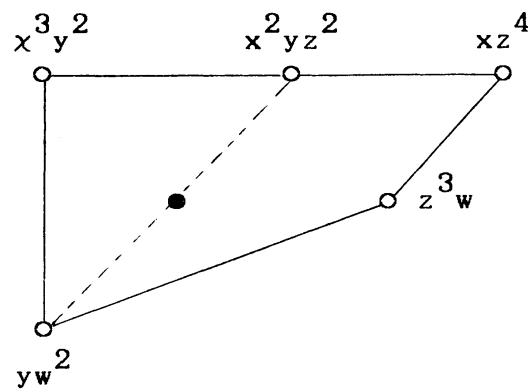
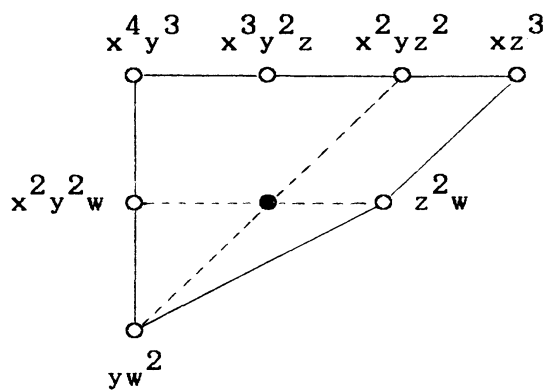


IV



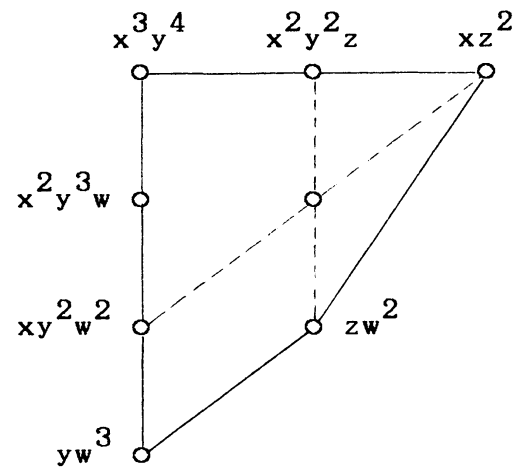
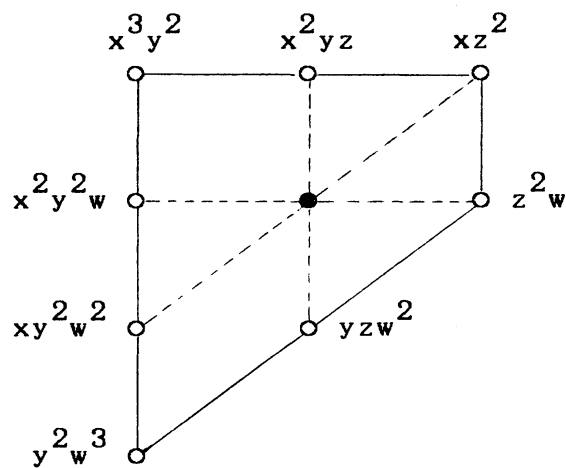
1. $(\alpha, \frac{1-4\alpha}{3}, \frac{1-\alpha}{3}, \frac{1+2\alpha}{3})$

2. $(\alpha, \frac{1-3\alpha}{2}, \frac{1-\alpha}{4}, \frac{1-3\alpha}{4})$

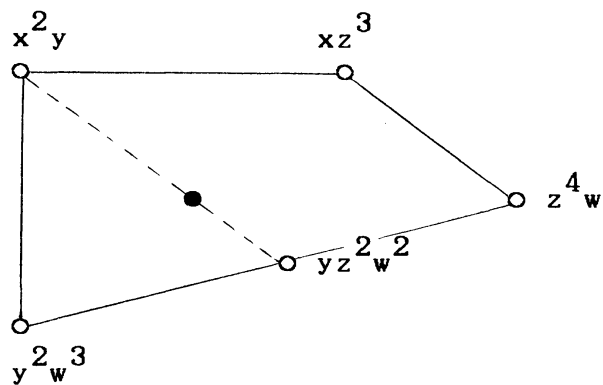


3. $(\alpha, \frac{1-3\alpha}{2}, \frac{1-\alpha}{2}, \frac{1+2\alpha}{3})$

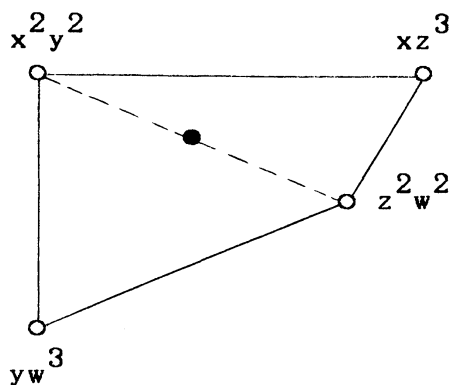
4. $(\alpha, \frac{1-3\alpha}{4}, \frac{1-\alpha}{2}, \frac{1+\alpha}{4})$



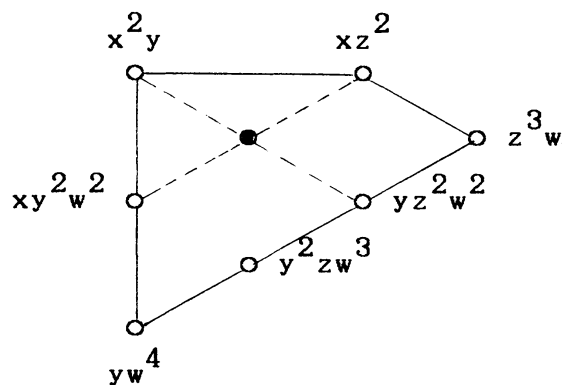
5. $(\alpha, 1-2\alpha, \frac{1-\alpha}{3}, \frac{4\alpha-1}{3})$



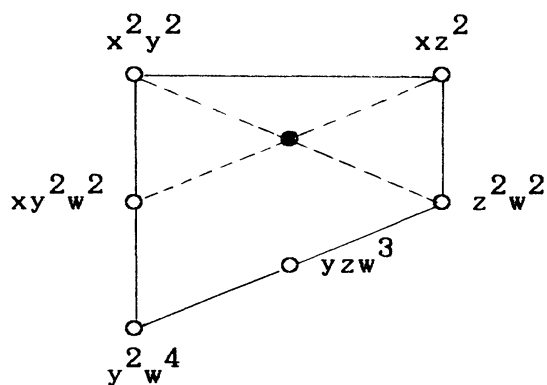
6. $(\alpha, \frac{1-2\alpha}{2}, \frac{1-\alpha}{3}, \frac{1+2\alpha}{6})$



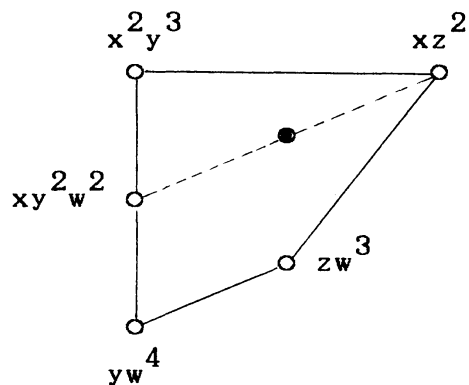
7. $(\alpha, 1-2\alpha, \frac{1-\alpha}{2}, \frac{3\alpha-1}{2})$



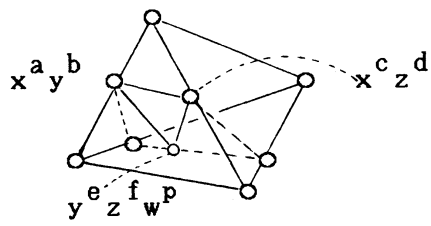
8. $(\alpha, \frac{1-2\alpha}{2}, \frac{1-\alpha}{2}, \frac{\alpha}{2})$



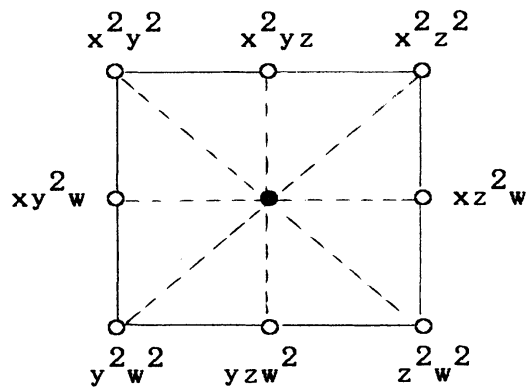
9. $(\alpha, \frac{1-2\alpha}{3}, \frac{1-\alpha}{2}, \frac{\alpha+1}{2})$



V



$$1. \left(\alpha, \frac{1-2\alpha}{2}, \frac{1-2\alpha}{2}, 2 \right)$$



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