

Pulling back continuously varying shortest paths through maps to 1-dimensional spaces

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Abstract

Following a brief survey of the main aspects of generalized universal covering space theory, as developed in [11], and of some spaces to which it applies, we discuss a particular application: the selection of continuously varying shortest representatives for the path homotopy classes of a subset Y of a $\text{CAT}(0)$ -nonpositively curved metric space Z , given a map $f : Y \rightarrow X$ to a 1-dimensional space X with convex null fibers.

§1. Generalized universal covering spaces

The classical concept of a (simply-connected) universal covering of a (locally path-connected) space is a useful tool in geometric topology, as it decouples the geometry of a space from its fundamental group, while encoding the interplay between the two in the group of covering transformations. A significant drawback of this procedure, though, is the fact that it does not apply to spaces with high local complexity: the space at hand must be semilocally simply-connected and, in the separable metric case, its fundamental group must be countable.

In [11], the concept of a universal covering is generalized to include many non-semilocally simply-connected spaces with uncountable fundamental group. This is achieved by abandoning the requirement of *local homeomorphism*, while maintaining virtually all other features of the classical theory. Specifically, we recall the following definition.

Definition. A continuous function $p : \tilde{X} \rightarrow X$, from a path-connected, locally path-connected and simply-connected topological space $\tilde{X}$ onto a topological space X , is called a *generalized universal covering* of X if for every path-connected and locally path-connected topological space Y , for every continuous function $f : (Y, y) \rightarrow (X, x)$ with $f_{\#}(\pi_1(Y, y)) = 1$, and for every $\tilde{x}$ in $\tilde{X}$ with $p(\tilde{x}) = x$, there exists a *unique* continuous function $g : (Y, y) \rightarrow (\tilde{X}, \tilde{x})$ with $p \circ g = f$.

$$\begin{array}{ccc} & & (\tilde{X}, \tilde{x}) \\ & \nearrow g & \downarrow p \\ (Y, y) & \xrightarrow{f} & (X, x) \end{array}$$

It follows immediately from this definition that if a generalized universal covering $p : \tilde{X} \rightarrow X$ exists, then it is uniquely determined. Furthermore, it has the following properties:

- (a) The group $\text{Aut}(\tilde{X} \xrightarrow{p} X)$ of covering transformations is isomorphic to $\pi_1(X)$ and it acts freely and transitively on the (not necessarily homeomorphic) fibers;

- (b) $p : \tilde{X} \rightarrow X$ is a Serre fibration with unique path lifting;
- (c) $\pi_k(\tilde{X}) \cong \pi_k(X)$ for all $k \geq 2$;
- (d) if X is locally path-connected and metrizable (first-countable is enough), then the map $p : \tilde{X} \rightarrow X$ is open, so that the quotient $\tilde{X}/\pi_1(X)$ is homeomorphic to X ;
- (e) if X is locally path-connected and semilocally simply-connected then $p : \tilde{X} \rightarrow X$ agrees with the *classical* universal covering.

If one attempts to build a generalized universal covering $p : \tilde{X} \rightarrow X$ of an arbitrary path-connected topological space X by means of the standard construction (using homotopy classes of paths, which start at a fixed base point, endowed with the standard topology described in [15]), then the following condition of X being “homotopically Hausdorff” is quickly observed to be necessary: we call X *homotopically Hausdorff* if for every $x \in X$, the only element of $\pi_1(X, x)$, which can be represented by arbitrarily small loops, is the trivial element.

In general, the property of being homotopically Hausdorff is considerably weaker than semilocal simple-connectivity, as can be seen with the example of the Hawaiian Earring (whose definition is reviewed in Section 4 below). In combination with a countable fundamental group, however, it does imply the existence of the generalized universal covering by means of the standard construction. This is not too surprising, since for metric spaces (first-countable is enough) with countable fundamental group, being homotopically Hausdorff is equivalent to semilocal simple-connectivity. In turn, separable metric spaces which allow for a *classical* universal covering have countable fundamental groups.

In summary, we note that generalized universal covering space theory is mainly concerned with spaces that have *uncountable* fundamental group.

Before stating a sufficient condition for the existence of generalized universal coverings of spaces with uncountable fundamental group, we recall from [11] that if X is a metric space (paracompact Hausdorff is enough) for which the *classical* universal covering exists, then its fundamental group $\pi_1(X, x_0)$ is naturally isomorphic to the first Čech homotopy group $\tilde{\pi}_1(X, x_0)$. The main result of [11] then features a partial converse to this fact:

Theorem 1 (F.–Zastrow [11]). *Let X be a path-connected topological space and let $x_0 \in X$. Suppose the natural homomorphism $\pi_1(X, x_0) \rightarrow \tilde{\pi}_1(X, x_0)$ is injective. Then the generalized universal covering $p : \tilde{X} \rightarrow X$ exists and can be built by the standard construction via homotopy classes of paths starting at x_0 , when given the topology of [15]. If X is metrizable, then so is $\tilde{X}$, with $\pi_1(X, x_0) \cong \text{Aut}(\tilde{X} \xrightarrow{p} X)$ acting by isometry.*

§2. Examples of spaces with generalized universal coverings

We list three important classes of spaces to which Theorem 1 can be applied.

Theorem 2 (Eda–Kawamura [6]). *If X is either a 1-dimensional compact Hausdorff space or if X is a 1-dimensional separable metric space, then the natural homomorphism $\pi_1(X, x_0) \rightarrow \tilde{\pi}_1(X, x_0)$ is injective for all $x_0 \in X$.*

As shown in [11], if X is a 1-dimensional path-connected separable metrizable space, then its generalized universal covering space $\tilde{X}$ is an $\mathbb{R}$ -tree (i.e. a uniquely arcwise connected geodesic metric space) with “exotic” action by $\pi_1(X)$; in general not by isometry.

Theorem 3 (F.–Zastrow [12]). *If M^2 is a closed surface and if $X \subseteq M^2$ is any subset, then the natural homomorphism $\pi_1(X, x_0) \rightarrow \tilde{\pi}_1(X, x_0)$ is injective for all $x_0 \in X$.*

Theorem 4 (F.–Guilbault [10]). *If X is a tree of n -manifolds (well-balanced if $n = 2$), then the natural homomorphism $\pi_1(X, x_0) \rightarrow \tilde{\pi}_1(X, x_0)$ is injective for all $x_0 \in X$.*

We recall that an inverse limit

$$X = \varprojlim \left(N_1 \xleftarrow{f_1} N_1 \#_{D_1} N_2 \xleftarrow{f_2} (N_1 \# N_2) \#_{D_2} N_3 \xleftarrow{f_3} (N_1 \# N_2 \# N_3) \#_{D_3} N_4 \xleftarrow{f_4} \dots \right)$$

of connected sums of closed PL n -manifolds N_i , formed along corresponding attaching disks $D_i \subseteq N_1 \# N_2 \# \dots \# N_i$ and $D'_i \subseteq N_{i+1}$, is called a *tree of manifolds* if

- (i) every f_i is the identity on $N_1 \# N_2 \# \dots \# N_i \setminus \text{int}(D_i)$;
- (ii) $f_i(N_{i+1} \setminus D'_i) \subseteq \text{int}(D_i)$ for all i ;
- (iii) the image $D_{i,j} = f_i \circ f_{i+1} \circ \dots \circ f_{j-1}(D_j)$ does not intersect ∂D_i for any $i < j$;
- (iv) for each fixed i , $\text{diam}(D_{i,j}) \rightarrow 0$ as $j \rightarrow \infty$.

See Figure 1. The tree of manifolds is said to be *well-balanced* if, for each i , the set $N_i \cap (D_i \cup \bigcup \{D_{i,j} \mid i < j\})$ is either dense in $N_i \setminus D'_{i-1}$ or has finitely many components.

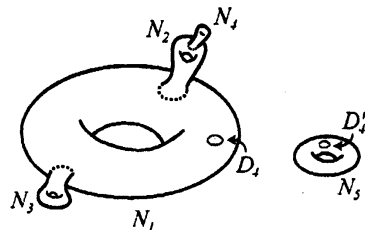


Figure 1: A finite stage of a tree of manifolds N_i

§3. More on trees of manifolds

Trees of manifolds arise naturally in the study of boundaries of Coxeter groups. A *right-angled Coxeter group* Γ is a group with finite presentation

$$\Gamma = \langle V \mid v^2 = 1, (uv)^{m(u,v)} = 1 \ \forall u, v \in V \rangle,$$

where $m(u, v) = m(v, u) \in \{2, \infty\}$ for $u \neq v$. Every right-angled Coxeter group Γ has a well-defined *nerve*, which is the abstract simplicial complex

$$N(\Gamma) = \{\Delta \mid \emptyset \neq \Delta \subseteq V \text{ and } \Delta \text{ generates a finite subgroup of } \Gamma\}.$$

The nerve of every right-angled Coxeter group is a flag complex, that is, it contains every simplex whose edges it contains. Conversely, every finite flag complex (for example, the barycentric subdivision of any finite simplicial complex) is the nerve of a unique right-angled Coxeter group: identify V with the vertex set, declare $v^2 = 1$ for all $v \in V$ and put $m(u, v) = 2$ if $\{u, v\}$ spans an edge.

There is a canonical CAT(0) cubical complex $X(\Gamma)$, the so-called *Davis-Vinberg complex*, on which Γ acts properly discontinuously and cocompactly by isometry. The space of geodesic rays of $X(\Gamma)$, emanating from a fixed base point, in the compact-open topology, is called the *boundary* of Γ , and is denoted by $\text{bdy}(\Gamma)$. See [5] and [1].

Trees of manifolds arise in this context as follows:

Theorem 5 (F. [9]). *If Γ is a right-angled Coxeter group, whose nerve $N(\Gamma)$ is a closed PL-manifold, then $\text{bdy}(\Gamma)$ is a (well-balanced) tree of manifolds $N_i = N(\Gamma)$.*

Combining Theorem 4 with Theorem 5 and the fact that $\pi_1(\text{bdy}(\Gamma)) = \pi_1^\infty(X(\Gamma)) = \pi_1^\infty(\Gamma)$ is an invariant of Γ , one obtains the following corollary.

Corollary 6 (F.–Guilbault [10]). *Let Γ be a right-angled Coxeter group whose nerve $N(\Gamma)$ is a closed PL-manifold. Then the homomorphism $\pi_1(\text{bdy } \Gamma) \rightarrow \pi_1^\infty(\Gamma)$ is injective.*

In [4] M.W. Davis constructs exotic open contractible manifolds $M(\Gamma)$, in all dimensions 4 and higher, based on any given right-angled Coxeter group Γ , whose nerve $N(\Gamma)$ is a nonsimply-connected PL-homology sphere, in such a way that Γ acts on $M(\Gamma)$ properly discontinuously and cocompactly (generated by reflections). These open contractible manifolds $M(\Gamma)$ are exotic in that they cover closed manifolds, although they are not homeomorphic to Euclidean space—they were the first such examples.

In fact, no $M(\Gamma)$ can be the interior of any compact manifold with boundary, because its π_1 -system is not stable at infinity. In contrast, trees of manifolds provide a natural “near-manifold compactification”:

Theorem 7 (F. [9]). *The Davis-manifolds $M(\Gamma)$ can be equivariantly Z -compactified by homogeneous cohomology manifolds, namely by the trees of manifolds $\text{bdy}(\Gamma)$.*

We refer the reader to [9] for additional features of this compactification and to [13] for another kind of analysis of Davis’ examples. Finally, we mention that it is shown in [10] how to realize trees of manifolds, which are based on homology spheres N_i of dimension greater than three, as boundaries of $\text{CAT}(\kappa)$ spaces with $\kappa < 0$.

§4. The generalized pullback construction

In preparation for the application in the next section, we now have a look at the pullback construction for generalized universal coverings.

Suppose X is a path-connected topological space such that $p : \tilde{X} \rightarrow X$ is a generalized universal covering, built by the standard construction. Let $f : Y \rightarrow X$ be a continuous map from a path-connected topological space Y . Consider the pullback diagram

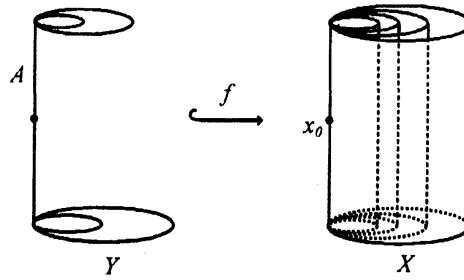
$$\begin{array}{ccc} f^*\tilde{X} & \xrightarrow{\tilde{f}} & \tilde{X} \\ f^*p \downarrow & & \downarrow p \\ Y & \xrightarrow{f} & X \end{array}$$

where $f^*\tilde{X} = \{(y, \tilde{x}) \in Y \times \tilde{X} \mid f(y) = p(\tilde{x})\} \subseteq Y \times \tilde{X}$ with projections $f^*p : f^*\tilde{X} \rightarrow Y$ given by $f^*p(y, \tilde{x}) = y$ and $\tilde{f} : f^*\tilde{X} \rightarrow \tilde{X}$ given by $\tilde{f}(y, \tilde{x}) = \tilde{x}$.

It is reasonable to expect that for a path-component $\tilde{Y}$ of $f^*\tilde{X}$, the restriction $f^*p : \tilde{Y} \rightarrow Y$ should be the generalized universal covering of Y , provided the homomorphism $f_\# : \pi_1(Y) \rightarrow \pi_1(X)$ is injective, because the corresponding statement would be true in the classical theory. The fact that this is not the case in the generalized setting, not even for Peano continua Y and X , is illustrated in [7] by the following example.

Example. Consider the Hawaiian Earring $\mathbb{H}$, that is, consider the planar continuum $\mathbb{H} = \bigcup_{n \in \mathbb{N}} L_n$, where $L_n = \{(x, y) \in \mathbb{R}^2 \mid x^2 + (y - \frac{1}{n})^2 = (\frac{1}{n})^2\}$. Define $X = \mathbb{H} \times [0, 1]$, with base point $x_0 = ((0, 0), \frac{1}{2})$. Put $A = \{(0, 0)\} \times [0, 1]$ and define the subset $Y \subseteq X$ by

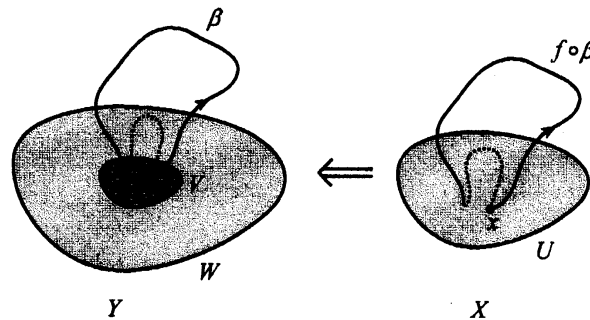
$$Y = \left(\left[\bigcup_{k \geq 0} L_{2k+1} \right] \times \{0\} \right) \cup A \cup \left(\left[\bigcup_{k \geq 1} L_{2k} \right] \times \{1\} \right).$$

Figure 2: The map $f : Y \rightarrow X$

Let $f : Y \rightarrow X$ be inclusion. See Figure 2. Then $f_{\#} : \pi_1(Y) \rightarrow \pi_1(X)$ is injective, because, from a homotopy point of view, $f : Y \rightarrow X$ is equivalent to the quotient map $q : Y \rightarrow Y/A$, which identifies the vertical arc A to a point. The effect of this on the topology of $\bar{Y}$ is that it folds on itself along the preimages of A , very much like the path-components of the Knaster continuum, rendering $\bar{Y}$ non-locally path-connected and, ultimately, non-contractible. On the other hand, the generalized universal covering space $\bar{Y}$ of the 1-dimensional continuum Y is contractible, since it is an $\mathbb{R}$ -tree.

What goes wrong in the above example is that, although $f : Y \rightarrow X$ is π_1 -injective, loops in the domain Y cannot be shrunk to a size which is in accordance with how small they can be made in the range X . This prompts the following definition.

Definition. We call the map $f : Y \rightarrow X$ *gradually π_1 -injective* if for every $y \in Y$ and every open subset W of Y with $y \in W$ there exist open subsets $V \subseteq Y$ and $U \subseteq X$ with $y \in V \subseteq W$ and $f(V) \subseteq U$, such that the kernel of the homomorphism $f_{\#} : \pi_1(Y, V, y) \rightarrow \pi_1(X, U, x)$ is contained in the kernel of the inclusion induced homomorphism $i_{\#} : \pi_1(Y, V, y) \rightarrow \pi_1(Y, W, y)$.

Figure 3: Gradual π_1 -injectivity

Put differently, if $\beta : [0, 1] \rightarrow Y$ is a path with $\beta(0) = y$ and $\beta(1) \in V$ such that $f \circ \beta$ can be homotoped into U , while fixing the endpoints, we require that β can be homotoped into W , while fixing its endpoints. This gradual contractive relationship is depicted in Figure 3. For a space which is homotopically Hausdorff, it is the appropriate strengthening of π_1 -injectivity, since we have the following criterion.

Theorem 8 (F. [7]). *Let $f : Y \rightarrow X$ be a map between path-connected topological spaces. Suppose that $p : \tilde{X} \rightarrow X$ is a standardly constructed generalized universal covering and let $\tilde{Y}$ be a path-component of the pullback $f^*\tilde{X}$. Then $f^*p|_{\tilde{Y}} : \tilde{Y} \rightarrow Y$ is a generalized universal covering if and only if Y is homotopically Hausdorff and $f : Y \rightarrow X$ is gradually π_1 -injective.*

§5. An asphericity proof technique: pulling back shortest paths

A topological space is called *aspherical* if its homotopy groups are trivial in all dimensions greater than one. Aspherical spaces are important, for their topology is intimately tied to the algebra of their fundamental group. Deciding whether a given space is aspherical can be surprisingly difficult, though, especially if it exhibits high local complexity. If we happen to know that the space is homotopy equivalent to a 1-dimensional separable metric space, then it is aspherical because of the following classical result.

Theorem 9 (Curtis–Fort [3]). *1-dimensional separable metric spaces are aspherical.*

However, even planar Peano continua need not be homotopy equivalent to any 1-dimensional spaces [14]. Still, all planar sets are known to be aspherical. Proofs of this nontrivial fact can be found in [16] and [2]. For subsets of higher-dimensional Euclidean spaces or, more generally, of CAT(0) spaces, we have the following theorem, which actually establishes a slightly stronger result than asphericity. (See [1] for a comprehensive discussion of CAT(0) spaces.)

Theorem 10 (F. [8]). *Let Y be a path-connected subset of a CAT(0) space Z . Suppose there is a map $f : Y \rightarrow X$ to a 1-dimensional separable metric space X , such that the nontrivial point preimages of f form a null sequence of convex subsets of Z . Then Y has an arc-smooth generalized universal covering space. In particular, Y is aspherical.*

For a detailed proof of Theorem 10, we refer the reader to [8]. Below, we only sketch the main idea.

Recall that a metric space W is called *arc-smooth* if there is a continuous function $A : (W, w_0) \rightarrow (C(W), \{w_0\})$ to the space $C(W)$ of nonempty compact connected subsets of W , endowed with the Hausdorff metric, such that $A(w)$ is a (possibly degenerate) arc from w_0 to w and $A(w') \subseteq A(w)$ for all $w \in W$ and $w' \in A(w)$. Arc-smooth spaces have trivial homotopy groups: given any compact subset B , such as the image of a sphere, B lies in the arc-smooth continuum $D = \bigcup \{A(b) \mid b \in B\}$, and arc-smooth continua are known to be contractible.

We should note that the map $f : Y \rightarrow X$ in Theorem 10 need not be a homotopy equivalence. This is best demonstrated by [14, Example 3], where a dense pattern of holes in the planar Sierpiński carpet is filled, while another dense pattern of holes remains. What results is an object, which is not homotopy equivalent to any 1-dimensional space. Indeed, in [14] it is shown that the resulting space is “everywhere homotopically 2-dimensional”. However, if we consider the quotient X of this space Y , in which all filled squares are identified to points, then the quotient map $f : Y \rightarrow X$ is as in Theorem 10.

The proof of Theorem 10 hinges on the fact that X has a generalized universal covering $p : \tilde{X} \rightarrow X$ by an $\mathbb{R}$ -tree $\tilde{X}$. The assumptions ensure that f is π_1 -injective, although not necessarily gradually π_1 -injective. Still, combining the following commutative square with

Theorems 1 and 2 guarantees the existence of a generalized universal covering $q : \tilde{Y} \rightarrow Y$.

$$\begin{array}{ccc} \pi_1(Y, y_0) & \xrightarrow{f_{\#}} & \pi_1(X, x_0) \\ \downarrow & & \downarrow \\ \tilde{\pi}_1(\tilde{Y}, y_0) & \xrightarrow{f_{\#}} & \tilde{\pi}_1(\tilde{X}, x_0) \end{array}$$

This enables one to pull back the canonical arc-structure of the $\mathbb{R}$ -tree $\tilde{X}$ to the generalized universal covering space $\tilde{Y}$, through the following commutative diagram, in which the map $r : \tilde{Y} \rightarrow \tilde{Y}$ is a continuous bijection, induced by the underlying lifting properties.

$$\begin{array}{ccccccc} & & [0, 1] & & & & \\ & \swarrow \tilde{\beta} & \downarrow g & \searrow \tilde{\alpha} & & & \\ \tilde{Y} & \xrightarrow{r} & \tilde{Y} & \xrightarrow{\text{incl}} & f^*\tilde{X} & \xrightarrow{\tilde{f}} & \tilde{X} \\ & \searrow q & & & \downarrow f^*p & & \downarrow p \\ & & Y & \xrightarrow{f} & X & & \end{array}$$

More specifically, since $\tilde{Y}$ is path-connected and Hausdorff, any two points of $\tilde{Y}$ can be joined with an arc, parametrized by some $g = (\beta, \tilde{\alpha}) : [0, 1] \rightarrow \tilde{Y} \subseteq Y \times \tilde{X}$. It follows from the assumptions of Theorem 10 and the unique arcwise connectivity of $\tilde{X}$ that the component $\tilde{\alpha} : [0, 1] \rightarrow \tilde{X}$ must be monotone, tracing out the appropriate arc of $\tilde{X}$. This allows a tightening of the component $\beta : [0, 1] \rightarrow Y$ to a local geodesic of unit speed, within every convex point preimage of f over which $\tilde{\alpha}$ is constant. While a thus tightened path g need not be rectifiable, its image is the unique shortest possible arc in $\tilde{Y}$ between its endpoints, and this is equally true for each of its subarcs. These shortest arcs of $\tilde{Y}$ need not vary continuously with their endpoints, because $\tilde{Y}$ is, in general, not locally path-connected. However, by lifting $\beta : [0, 1] \rightarrow Y$ to $\tilde{\beta} : [0, 1] \rightarrow \tilde{Y}$, we can transfer the entire collection of arcs to $\tilde{Y}$, where we obtain continuous dependence on the endpoints. Finally, fixing one endpoint renders $\tilde{Y}$ arc-smooth, as desired.

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