

On geometry of quasi-minimal structures

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Abstract

Itai, Tsuboi and Wakai investigated the geometric properties of quasi-minimal structures by using the countable closure [1]. I considered another closure operator in such structures.

1. Quasi-minimal structure and the countable closure

We recall some definitions.

Definition 1 An uncountable structure M is called *quasi-minimal* if every definable subset of M with parameters is at most countable or co-countable.

I introduce the examples in [1] and [2].

Example 2

1. $M = (\mathcal{Q}^\omega, +, \sigma, 0)$ where σ is the shift function ;
for $x = (x_0, x_1, x_2, \dots)$, $\sigma(x) = (x_1, x_2, x_3, \dots)$
2. $M_0 = (2^\omega, E_i (i < \omega))$ such that $E_i(x, y) \iff x(i) = y(i)$ for $x, y \in 2^\omega$.
Let $M' \prec M_0$ be a countable elementary substructure and fix $a \in M'$. And let $M_1 = (M' \cup B, E_i (i < \omega))$ where $|B| > \omega$ and $stp(b) = stp(a)$ for all $b \in B$. Then M_1 is quasi-minimal.

Definition 3 Let M be quasi-minimal. Then a type $p(x)$ defined by $p(x) = \{\psi(x) \in L(M) : |\psi^M| \geq \omega_1\}$ is a complete type.

We call the type $p(x)$ the *main type* of M .

Definition 4 Let M be an uncountable structure and $A \subset M$.

The n -th countable closure $\text{ccl}_n(A)$ of A is inductively defined as follows :

$\text{ccl}_0(A) = A$ and

$\text{ccl}_{n+1}(A) = \bigcup \{\phi^M : \phi(x) \in L(\text{ccl}_n(A)), \phi^M \text{ is countable}\}$

We put $\text{ccl}(A) = \bigcup_{n \in \omega} \text{ccl}_n(A)$ (the countable closure of A).

Definition 5 Let X be an infinite set and cl a function from $\mathcal{P}(X)$ to $\mathcal{P}(X)$ where $\mathcal{P}(X)$ denotes the set of all subsets of X . If the function cl satisfies the following properties, we say (X, cl) is a *pregeometry*.

- (I) $A \subset B \implies A \subset \text{cl}(A) \subset \text{cl}(B)$,
- (II) $\text{cl}(\text{cl}(A)) = \text{cl}(A)$,
- (III) (Finite character) $b \in \text{cl}(A) \implies b \in \text{cl}(A_0)$ for some finite $A_0 \subset A$,
- (IV) (Exchange axiom)
 $b \in \text{cl}(A \cup \{c\}) - \text{cl}(A) \implies c \in \text{cl}(A \cup \{b\})$.

It is shown that the countable closure is a closure operator in [1].

Fact 6 Let M be a quasi-minimal structure. Then (M, ccl) satisfies the first three properties (I) through (III) of pregeometry.

The exchange axiom (IV) does not hold in (M, ccl) generally. In [1], Itai, Tsuboi and Wakai showed some conditions for M such that (M, ccl) satisfies the exchange axiom.

Theorem 7 Let M be a quasi-minimal structure. Then (M, ccl) satisfies the axioms of pregeometry under some conditions.

And we recall the next theorem from [1].

Theorem 8 Let M be a quasi-minimal structure. And $\text{Th}(M)$ is ω -stable. Then M can be elementarily embedded to an ω -saturated quasi-minimal structure M' .

The notion of quasi-minimal structures is a generalization of minimal structures. Thus the countable closure is the canonical closure operator for quasi-minimal structures. However, I tried to divide the countable closure by some P -closure.

2. P -closure in quasi-minimal structures

First we recall some definitions from [6].

Definition 9 A family P of partial types is *A -invariant* if it is invariant under A -automorphisms (where A is a subset of a sufficiently large saturated model as usual).

Let P be an A -invariant family of partial types.

A partial type q over A is *P -internal* if for every realization a of q , there is $B \downarrow_A a$, types $\bar{p}$ from P based on B , and realizations $\bar{c}$ of $\bar{p}$, such that $a \in \text{dcl}(B\bar{c})$.

A partial type q is *P-analysable* if for any $a \models q$, there are $(a_i : i < \alpha) \in \text{dcl}(A, a)$ such that $\text{tp}(a_i/A, \{a_j : j < i\})$ is *P*-internal for all $i < \alpha$, and $a \in \text{bdd}(A, \{a_i : i < \alpha\})$.

A complete type $q \in S(A)$ is *foreign* to *P* if for all $a \models q$, $B \downarrow_A a$, and realizations $\bar{c}$ of extensions of types in *P* over *B*, we always have $a \downarrow_{AB} \bar{c}$.

Definition 10 Let *P* be an $\emptyset$ -invariant family of types.

A partial type q is *co-foreign* to *P* if every type in *P* is foreign to q .

The *P*-closure $\text{cl}_P(A)$ of a set *A* is the collection of all element a such that $\text{tp}(a/A)$ is *P*-analysable and co-foreign to *P*. (The *P*-analysable assumption could be modified or even omitted, resulting in a larger *P*-closure.)

Fact 11 *P*-closure satisfies the axioms (I) and (II) of pregeometry.

The axiom (III) and the exchange axiom (IV) do not hold in general.

We define *P*-closures in stable quasi-minimal structures. We argue under the assumptions in the following.

Assumptions

M is an ω -saturated quasi-minimal structure such that $\text{Th}(M)$ is ω -stable.

We may assume that the main type $p(x) \in S(M)$ strongly based on $\emptyset$.

The set *P* of types is defined by

$$P = \{q \in S(A) : q \text{ is a conjugate of } p[A \text{ for some finite } A \subset M\}.$$

We can prove the next fact.

Fact 12 Under the assumptions as above, the *P*-closure cl_P is a closure operator in *M*.

(M, cl_P) satisfies the axioms (I) through (III) of pregeometry.

And $\text{acl}(A) \subset \text{cl}_P(A) \subset \text{ccl}(A)$ for $A \subset M$.

If we omit the *P*-analysability assumption from cl_P , then $\text{cl}_P(A) = \text{ccl}(A)$.

Remark 13 In Example 2.1, $\text{cl}_P(A) = \text{ccl}(A)$ for $A \subset M$ under the *P*-analysability assumption. By the argument in [3], we can show the same fact for $(\omega$ -stable) quasi-minimal groups in general.

3. p-closure for regular types p

We recall some definitions from [4].

Definition 14 Let $p(x), q(x)$ be complete types over *A*. We say that *p* is *almost orthogonal* to *q* if whenever *a* realizes *p*, and *b* realizes *q*, then

$\text{tp}(a/Ab)$ does not fork over A .

Let $p(x) \in S(A)$, $q \in S(B)$ are stationary types.

We say that p is *orthogonal to q* if whenever $C \supset A \cup B$, then $p|C$ is almost orthogonal to $q|C$.

And we say that p is *hereditarily orthogonal to q* if every extension of p is orthogonal to q .

Definition 15 Let $p(x) \in S(A)$ be a non-algebraic stationary type.

We say that p is *regular* if for any forking extension q of p , p is orthogonal to q .

In the following, let p be a regular type over some domain.

Definition 16 Let $q(x) \in S(X)$ be a strong type, where p is non-orthogonal to X .

We say that q is *p -simple* if there is a set $B \supset A \cup X$, some realization a of $q|B$ and a set Y of realizations of p such that $\text{stp}(a/BY)$ is hereditarily orthogonal to p .

And we say that q is *p -semi-regular* if q is p -simple and domination equivalent to some non-zero power $p^{(n)}$ of p .

Definition 17 Let $q = \text{stp}(a/X)$ be p -simple. Then the *p -weight of q* , $w_p(q)$ is defined to be

$\min\{\kappa : \text{there is } B \supset A \cup X, \text{ there is } a' \text{ realizing } q|B, \text{ and there is } J, \text{ an independent set of realizations of } p|B, \text{ such that } \text{stp}(a'/BJ) \text{ is hereditarily orthogonal to } p \text{ and } |J| = \kappa\}$

We define the *p -closure of X* , denoted $cl_p(X)$, the set $\{b : \text{stp}(b/X) \text{ is } p\text{-simple and } w_p(b/X) = 0\}$

We try to argue p -closure in quasi-minimal structures.

We can check the next fact easily.

Fact 18 Let M be a quasi-minimal structure. And $\text{Th}(M)$ is ω -stable.

Then we may assume that the main type $p \in S(M)$ is a regular type.

It is well known that for regular types p of stable theory, (p^C, cl_p) is pregeometry (where C is the big model).

For quasi-minimal structure M of stable theory and the main type p of M , we consider cl_p .

We can prove the next fact like Fact. 12.

Fact 19 Let M be a quasi-minimal structure of ω -stable theory. Then cl_p is a closure operator, i.e. (M, cl_p) satisfies the axioms (I) through (III) of pregeometry.

And $\text{acl}(A) \subset cl_p(A) = \text{ccl}(A)$ for $A \subset M$.

4. Further problem

We recall some definitions and theorems from [4] again.

Definition 20 Let (S, cl) be pregeometry.

(S, cl) is *modular* if for any closed sets $X, Y \subset S$, X is independent from Y over $X \cap Y$.

Equivalently, for any finite-dimensional closed sets X, Y ,

$$\dim(X) + \dim(Y) - \dim(X \cap Y) = \dim(X \cup Y).$$

(S, cl) is *locally modular* if for some $a \in S$, $(S, cl_{\{a\}})$ (the localization of S at $\{a\}$) is modular.

The next theorems are well-known.

Theorem 21 Let $p \in S(\emptyset)$ be a stationary, minimal locally modular type. Then p is trivial, or p is non-trivial modular (in which case the geometry on p is projective over a division ring), or p is non-modular in which case the geometry associated to p (over $\emptyset$) is affine geometry over a division ring.

Theorem 22 Let $p \in S(\emptyset)$ be a stationary, regular, locally modular type over $\emptyset$. Then the geometry of (p^C, cl_p) is either trivial, or affine or projective geometry over some division ring.

There are examples of quasi-minimal structures whose main type is locally modular. (See Example 2.1)

Question

Let p be the main type of a (ω) -stable quasi-minimal structure. And let p be a locally modular regular type.

Does its geometry (p^C, cl_p) have characteristics?

Apology and acknowledgement

I did not know the paper [3] by A.Pillay and P.Tanović until Kirishima meeting. Some participants told me about their work. The content of my talk is not shown in their paper on the surface.

References

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