

On periodicity of geodesic continued fractions: a precis

By

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Abstract

This article is a research announcement of the paper [2] and is based on the author’s talk at RIMS Workshop: Algebraic Number Theory and Related Topics 2016. In the paper, we present some generalizations of Lagrange’s theorem in the theory of continued fractions.

§ 1. Introduction

In [2], we present a geodesic generalization of continued fractions motivated by a geometric interpretation of the classical continued fractions in terms of geodesics on the modular curves. As a result, for an extension F/F' of number fields with rank one relative unit group, we construct a geodesic multi-dimensional continued fraction algorithm to “expand” any basis of F over $\mathbb{Q}$, and prove its periodicity. Furthermore we show that its period gives a non-torsion unit in the relative unit group. By an adelic argument, we also obtain p -adelic continued fractions and its periodicity for imaginary quadratic irrationals. The aim of this article is to explain the main ideas of these results.

The central motivation for the study is to consider generalizations of the following classical theorem of Lagrange, which claims:

- (1) the continued fraction expansion of a real number becomes periodic if and only if it is a real quadratic irrational;
- (2) its period “gives” a fundamental unit of the real quadratic order associated to the given real quadratic irrational. (This is also known as a solution of the Pell equations.)

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Our starting point is to observe an analogy between this Lagrange's theorem and the following fact which has appeared, for example, in the works of E. Artin [1], Siegel [6], and Sarnak [5]:

- (1)' a geodesic on the Poincaré upper half plane $\mathfrak{h}$ becomes a closed geodesic on $SL_2(\mathbb{Z}) \backslash \mathfrak{h}$ if and only if its two end points in $\mathbb{R} \cup \{\infty\}$ are conjugate real quadratic irrationals;
- (2)' the length of the closed geodesic equals to twice the regulator of the real quadratic order associated to the real quadratic irrationals.

In the following, we first generalize this geometric framework to the homogeneous space of GL_n , and then apply it to present some generalizations of Lagrange's theorem; Theorems 3.3 and 4.3.

§ 2. Heegner objects

Here we explain an idea of the definition of Heegner objects, which generalize geodesics on the Poincaré upper half plane with conjugate real quadratic end points. See [2, Section 2] for details. First, note that the geodesic ϖ on the Poincaré upper half plane $\mathfrak{h}$ connecting two conjugate real quadratic irrationals α and $\bar{\alpha}$ can be obtained by the left translation of the imaginary axis $I = i\mathbb{R}_{>0} \subset \mathfrak{h}$ by a matrix $W = \begin{pmatrix} \alpha & \bar{\alpha} \\ 1 & 1 \end{pmatrix} \in GL_2(\mathbb{R})$, i.e., $\varpi = WI$. Therefore, in the following, we define Heegner objects by generalizing each of the above objects $\mathfrak{h}, I, W$ and ϖ .

Definition 2.1 (The generalized upper half space). We define the generalized upper half space for $GL_n(\mathbb{R})$ as

$$(2.1) \quad \mathfrak{h}^n := GL_n(\mathbb{R}) / \mathbb{R}^\times O(n).$$

For $g \in GL_n(\mathbb{R})$, we denote by $[g]$ the class of g in $\mathfrak{h}^n$.

Remark. In the original paper [2], we treat Heegner objects S -adelically, where S is a finite set of places of $\mathbb{Q}$ containing ∞ . However, in this article, we only treat the case where $S = \{\infty\}$ for simplicity.

Let $F/\mathbb{Q}$ be a number field of degree n , and let $w_1, \dots, w_n \in F$ be a basis of F over $\mathbb{Q}$. Set $w := {}^t(w_1, \dots, w_n)$. We denote by $|F|$ the set of all archimedean places of F .

Let $F_\infty := F \otimes_{\mathbb{Q}} \mathbb{R}$. We have $F_\infty \simeq \prod_{v \in |F|} F_v$, where F_v is the completion of F at v (fixed in $\mathbb{C}$). Then F_v is either $\mathbb{R}$ or $\mathbb{C}$, and we fix an isomorphism $F_v \simeq \mathbb{R}^{n_v}$ ($n_v := [F_v : \mathbb{R}]$) of $\mathbb{R}$ -vector spaces using $\mathbb{R} = \mathbb{R}$ and $\mathbb{C} \simeq \mathbb{R}^2; x + iy \mapsto (y, x)$. Thus we obtain $\iota : F_\infty \simeq \prod_{v \in |F|} F_v \xrightarrow{\sim} \prod_{v \in |F|} \mathbb{R}^{n_v} \simeq \mathbb{R}^n$ as $\mathbb{R}$ -vector spaces. Now, let $T^1 :=$

$\{t = (t_v)_v \in \prod_{v \in |F|} \mathbb{R}_{>0} \mid \prod_{v \in |F|} t_v^{n_v} = 1\}$. Then T^1 acts on $\prod_{v \in |F|} F_v \simeq F_\infty (\simeq \mathbb{R}^n)$ by component-wise multiplication, and this induces a group homomorphism:

$$(2.2) \quad I : T^1 \rightarrow GL_n(\mathbb{R}).$$

On the other hand, the basis $w = {}^t(w_1, \dots, w_n)$ of F over $\mathbb{Q}$ defines an isomorphism $w : \mathbb{Q}^n \xrightarrow{\sim} F; x \mapsto x \cdot w$ of $\mathbb{Q}$ -vector spaces. Then, by tensoring $\mathbb{R}$, we obtain $w : \mathbb{R}^n \xrightarrow{\sim} F_\infty$. We define $W \in GL_n(\mathbb{R})$ so that $\iota \circ w$ maps any row vector $x \in \mathbb{R}^n$ to xW , i.e.,

$$(2.3) \quad \iota \circ w : \mathbb{R}^n \xrightarrow{\sim} F_\infty \xrightarrow{\sim} \mathbb{R}^n; x \mapsto xW.$$

Definition 2.2 ([2, Definition 2.1.1]). We define the Heegner object $\varpi = \varpi_w$ associated to the basis w of F over $\mathbb{Q}$ as

$$(2.4) \quad \varpi : T^1 \rightarrow \mathfrak{h}^n : t \mapsto [WI(t)].$$

Now, the discrete group $\Gamma := SL_n(\mathbb{Z})$ acts naturally on $\mathfrak{h}^n$ from the left. We are interested in the “periodicity” of the map:

$$(2.5) \quad \pi \circ \varpi : T^1 \rightarrow \Gamma \backslash \mathfrak{h}^n,$$

where $\pi : \mathfrak{h}^n \rightarrow \Gamma \backslash \mathfrak{h}^n$ is the natural projection. In the following, we define the “period groups” which geometrically capture the periods of ϖ , and relate them to the unit group of F .

Definition 2.3 ([2, Definition 2.2.1]). A pair $(A, \rho) \in \Gamma \times T^1$ is said to be a period of ϖ (with respect to Γ) if it satisfies

$$(2.6) \quad A\varpi(t) = \varpi(\rho t) \text{ for all } t \in T^1.$$

Set $\Gamma_\varpi := \{A \in \Gamma \mid \exists \rho \in T^1 \text{ such that } (A, \rho) \text{ is a period of } \varpi\} \subset \Gamma$, and $\Lambda_\varpi := \{\rho \in T^1 \mid \exists A \in \Gamma \text{ such that } (A, \rho) \text{ is a period of } \varpi\} \subset T^1$. They are both subgroups.

Then we have proved the following:

Theorem 2.4 ([2, Theorem 2.2.3]).

- (1) The map $\pi \circ \varpi : T^1 \rightarrow \Gamma \backslash \mathfrak{h}^n$ factors through $\bar{\varpi} : T^1 / \Lambda_\varpi \rightarrow \Gamma \backslash \mathfrak{h}^n$, and T^1 / Λ_ϖ is compact.
- (2) If $A \in \Gamma_\varpi$, then w is an eigenvector of A , and we denote by $\varphi_w(A)$ the corresponding eigenvalue, i.e., $Aw = \varphi_w(A)w$. Then, the map

$$(2.7) \quad \varphi_w : \Gamma_\varpi \rightarrow F^\times; A \mapsto \varphi_w(A)$$

is an injective group homomorphism, and its image coincides with the norm one unit group $\mathcal{O}_w^1$ of the order $\mathcal{O}_w$ associated to w . Here $\mathcal{O}_w := \{x \in F \mid x(\mathbb{Z}w_1 + \dots + \mathbb{Z}w_n) \subset \mathbb{Z}w_1 + \dots + \mathbb{Z}w_n\}$ is an order in F , and $\mathcal{O}_w^1 := \{u \in \mathcal{O}_w^\times \mid N_{F/\mathbb{Q}}u = 1\}$.

Remark.

1. (1) of the above theorem is essentially Dirichlet's unit theorem.
2. We have also proved a “ χ -component” version of this theorem in order to treat relative unit groups. cf. [2, Theorem 2.2.6]

§ 3. Geodesic continued fractions

By Theorem 2.4, we can now study the unit group $\mathcal{O}_w^1$ through the period group Γ_ϖ , which can be examined geometrically. In this section we present a method to capture the periods in Γ_ϖ using the geometry of $\mathfrak{h}^n$ in the case where ϖ is one dimensional.

We keep the notations in Section 2. In the following, we assume the Heegner object ϖ is one dimensional, i.e., $\#|F| = 2$. We fix an isomorphism $T^1 \simeq \mathbb{R}_{>0}$. Then $\varpi : T^1 \simeq \mathbb{R}_{>0} \rightarrow \mathfrak{h}^n$ is a geodesic with respect to the natural Riemann metric on $\mathfrak{h}^n$.

Let $\mathcal{F} \subset \mathfrak{h}^n$ be a “good” fundamental set for $\Gamma = SL_n(\mathbb{Z})$, e.g. the Siegel set for $SL_n(\mathbb{Z})$. We will not reproduce the definition of a “good” fundamental set here. See [2, Definition 3.1.2, Theorem 3.2.5].

Proposition 3.1 ([2, Algorithm 1, Proposition 3.1.3, Theorem 3.1.4.(1)]). *For the Heegner object ϖ , we can take a sequence $\{(A_k, B_k, \varpi_k)\}_{k \in \mathbb{Z}}$ as follows:*

- Set $u_0 := 1 \in \mathbb{R}_{>0} \simeq T^1$, and take $B_0 \in \Gamma$ so that $\varpi(u_0) \in B_0\mathcal{F}$. Set $\varpi_0 := B_0^{-1}\varpi$.
- Suppose $u_k \in \mathbb{R}_{>0}$, $B_k \in \Gamma$ and ϖ_k ($k \geq 0$) such that $\varpi_k(u_k) \in \mathcal{F}$ are given. Let $t_k \in \mathbb{R}_{>0}$ be the supremum element satisfying $\varpi_k([u_k, t_k]) \subset \mathcal{F}$. Take $u_{k+1} > t_k$ and $A_{k+1} \in \Gamma$ so that $\varpi_k([t_k, u_{k+1}]) \subset A_{k+1}\mathcal{F}$. Set $B_{k+1} := B_k A_{k+1}$ and $\varpi_{k+1} := A_{k+1}^{-1}\varpi_k$.
- Suppose $u_k \in \mathbb{R}_{>0}$, $B_k \in \Gamma$ and ϖ_k ($k \leq 0$) such that $\varpi_k(u_k) \in \mathcal{F}$ are given. Let $s_k \in \mathbb{R}_{>0}$ be the infimum element satisfying $\varpi_k([s_k, u_k]) \subset \mathcal{F}$. Take $u_{k-1} < s_k$ and $A_k \in \Gamma$ so that $\varpi_k([u_{k-1}, s_k]) \subset A_k\mathcal{F}$. Set $B_{k-1} := B_k A_k^{-1}$ and $\varpi_{k-1} := A_k\varpi_k$.

Definition 3.2. We call this sequence $\{(A_k, B_k, \varpi_k)\}_{k \in \mathbb{Z}}$ a geodesic continued fraction (GCF) expansion of the Heegner object ϖ , or of the basis w .

Remark.

- (1) The geodesic continued fraction expansion may not be unique in general.
- (2) The idea of geodesic continued fractions was originally considered by Lagarias [4] and Beukers [3] to find good simultaneous approximations of real numbers in the theory of Diophantine approximations, while any Lagrange type periodicity theorem

was not known. Our definitions and arguments in [2, Section 3] are largely based on those in [4] and [3]. However, their original arguments could not be applied to our Heegner objects, and we generalize their formulation in the paper [2].

Now, using Theorem 2.4, we obtain the following theorem.

Theorem 3.3 ([2, Theorem 3.3.1], Rank one generalized Lagrange’s theorem). *Let $F/\mathbb{Q}$ be an extension of degree n with rank one unit group, and let w be a basis of F over $\mathbb{Q}$. Let $\{(A_k, B_k, \varpi_k)\}_{k \in \mathbb{Z}}$ be a GCF expansion of w as above.*

- (1) *There exist $k_0, k_1 \in \mathbb{N}$, $k_0 < k_1$, and $\rho \in \mathbb{R}_{>1}$ such that $\varpi_{k_0}(\rho t) = \varpi_{k_1}(t)$ for all $t \in T^1 \simeq \mathbb{R}_{>0}$.*
- (2) *Then, the period $B_{k_1} B_{k_0}^{-1} \in \Gamma_\varpi \stackrel{\varphi_w}{\simeq} \mathcal{O}_w^1$ gives a non-torsion unit.*

Remark. Using the “ χ -component” version of Theorem 2.4, we can extend this theorem to the case for relative extensions of number fields with rank one relative unit groups. See [2, Theorem 3.3.2].

§ 4. p -adelic continued fractions

In [2, Section 4], we extend the above idea of Heegner objects and geodesic continued fractions to the setting:

$$(4.1) \quad SL_2(\mathbb{Z}[1/p]) \curvearrowright \mathfrak{h} \times \mathfrak{h}_p^2,$$

where $\mathfrak{h}$ is the Poincaré upper half plane and $\mathfrak{h}_p^2 := GL_2(\mathbb{Q}_p)/\mathbb{Q}_p^\times GL_2(\mathbb{Z}_p)$. We first define p -adelic version of the geodesic continued fraction algorithm [2, Algorithm 2]. Then develop it into a more intuitive form [2, Algorithm 3]. In this article, we omit Algorithm 2 and we directly present Algorithm 3 and its periodicity .

In the following, we fix an isomorphism $\mathbb{C} \simeq \overline{\mathbb{Q}_p}$ of fields. Let $\mathcal{D} := \{z \in \mathfrak{h} \mid -1/2 \leq \Re(z) < 1/2, |z| > 1\} \cup \{e^{i\theta} \mid \theta \in [\pi/2, 2\pi/3]\}$ be a fundamental domain for $SL_2(\mathbb{Z})$.

Proposition 4.1 ([2, Algorithm 3]). *Let $z \in \mathfrak{h} \cap \mathbb{Q}_p \subset \mathbb{C} = \overline{\mathbb{Q}_p}$. Set $x_1 = z$. For $x_k \in \mathfrak{h} \cap \mathbb{Q}_p$ ($k \geq 1$), we can define x_{k+1} as follows:*

Initialization: Define $\delta_k \in \{\pm 1\}$ so that $x_k^{\delta_k} \in \mathbb{Z}_p$. Set $z_k := \delta_k x_k^{\delta_k} \in \mathfrak{h} \cap \mathbb{Q}_p$.

p -adic reduction: Define $a_k \in \{0, \dots, p^2 - 1\}$ so that $z_k - a_k \equiv 0 \pmod{p^2}$. Set $z_{k0} := (z_k - a_k)/p^2$.

∞ -adic reduction: For z_{kj} ($j \geq 0$), if $z_{kj} \in \mathcal{D}$, then set $x_{k+1} := z_{kj} \in \mathfrak{h} \cap \mathbb{Q}_p$ and set $l_k := j$. Otherwise, define $b_{kj} \in \mathbb{Z}$ so that $-1/2 \leq \Re(z_{kj} - b_{kj}) < 1/2$ and set $z_{kj+1} := -1/(z_{kj} - b_{kj})$.

Definition 4.2. We call the above procedure the $\{\infty, p\}$ -continued fraction expansion of z , and formally express this as

$$(4.2) \quad z = \delta_1 \left(a_1 + p^2 b_{10} - \frac{p^2}{b_{11} - \frac{1}{\ddots - \frac{1}{b_{1l_1} - \frac{1}{\delta_2(a_2 + p^2 b_{20} - \frac{p^2}{b_{21} - \frac{1}{\ddots}})}}}} \right)^{\delta_1}$$

$$= [\delta_1; a_1; b_{10}, \dots, b_{1l_1}; \delta_2; a_2; b_{20}, \dots, b_{2l_2}; \dots].$$

Moreover, we associate each step of the above algorithm with an elements of $SL_2(\mathbb{Z}[1/p])$ which corresponds to the linear fractional transformation of the step. That is, for $k \geq 1$, we define notations as follows.

initialization: Set $D_k = \begin{pmatrix} 1 & \\ & 1 \end{pmatrix}$ if $\delta_k = 1$, $D_k = \begin{pmatrix} & -1 \\ 1 & \end{pmatrix}$ if $\delta_k = -1$.

p-adic reduction: Set $P_k = \begin{pmatrix} 1/p & \\ & p \end{pmatrix} \begin{pmatrix} 1-a_k & \\ 0 & 1 \end{pmatrix}$.

∞ -adic reduction: Set $Q_k = \begin{pmatrix} 0 & -1 \\ 1 & -b_{kl_k} \end{pmatrix} \cdots \begin{pmatrix} 0 & -1 \\ 1 & -b_{k0} \end{pmatrix}$.

Set $A_k^{-1} = Q_k P_k D_k$ and $B_k = A_1 \cdots A_k$. Then we have $z_k = D_k x_k$, $z_{k0} = P_k z_k$, $x_{k+1} = Q_k z_{k0}$, and $x_{k+1} = A_k^{-1} x_k = B_k^{-1} x_1$, where the action is a linear fractional transformation.

Then, using Theorem 2.4, we obtain the following theorem.

Theorem 4.3 ([2, Theorems 4.3.1 and 4.4.1]). *Let $z \in \mathfrak{h} \cap \mathbb{Q}_p$, and we use the same notations as above, e.g., x_k, A_k, B_k , etc.*

- (1) *If z is imaginary quadratic irrational, then $\{\infty, p\}$ -continued fraction expansion of z becomes periodic, i.e., there exist $k_0, l \geq 1$ such that $x_{k_0} = x_{k_0+l}$.*
- (2) *In this case, if l is the minimal period length, then an eigenvalue of $B_{k_0+l} B_{k_0}^{-1}$ gives a generator of $\mathcal{O}_w[1/p]^1$, where $w := {}^t(z, 1)$ is a basis of $F = \mathbb{Q}(z)$ over $\mathbb{Q}$ and $\mathcal{O}_w[1/p]^1 := \{u \in \mathcal{O}_w[1/p]^\times \mid N_{F/\mathbb{Q}} u = 1\}$.*

Remark. In fact, in [2], we prove (2) of this theorem only in some special cases (cf. [2, Theorem 4.4.1]). However we can also prove this similarly.

§ 5. Examples

In this section, we illustrate the main theorems by presenting some examples.

§ 5.1. Examples of Theorem 3.3

Here we present two examples of Theorem 3.3 contained in [2]. We use the same notations as in Theorem 3.3.

Example 5.1 (Real quadratic field).

Heegner object: $\left(F = \mathbb{Q}(\sqrt{2}), w = {}^t(\sqrt{2}, 1), \varpi(t) = \begin{bmatrix} \sqrt{2}t - \sqrt{2} \\ t & 1 \end{bmatrix} \right)$

GCF expansion: $(B_0, \{A_k\}_{k \geq 1}) =$

$$\left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \overline{\left\{ \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \right\}} \right).$$

Here the overlined part denotes the period as the usual continued fractions.

Period and unit: We obtain a period $B_{k_1} B_{k_0}^{-1} = \begin{pmatrix} -3 & -4 \\ -2 & -3 \end{pmatrix}$, and unit $\epsilon = -2\sqrt{2} - 3$.

See Figure 1 below.

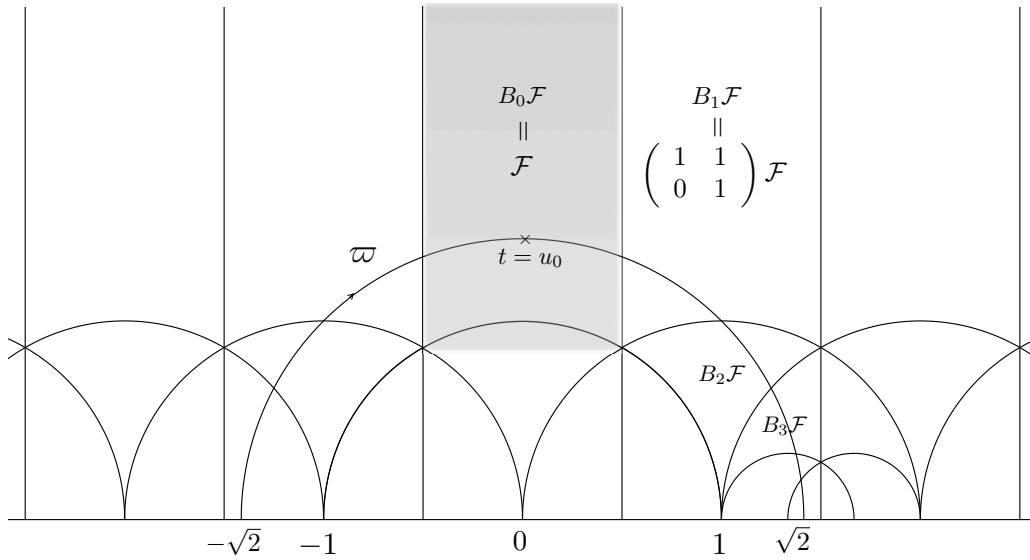


Figure 1. The Heegner object associated to $w = {}^t(\sqrt{2}, 1)$

Example 5.2 (Complex cubic field).

Heegner object: $\left(F = \mathbb{Q}(3^{1/3}), w = {}^t(3^{2/3}, 3^{1/3}, 1), \varpi(t) = \begin{bmatrix} 3^{2/3}t - 3^{2/3}\sqrt{3}/2 & -3^{2/3}/2 \\ 3^{1/3}t & 3^{1/3}\sqrt{3}/2 & -3^{1/3}/2 \\ t & 0 & 1 \end{bmatrix} \right)$

GCF expansion: $(B_0, \{A_k\}_{k \geq 1}) =$

$$\left(\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \right.$$

$$\left. \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\}.$$

Period and unit: We obtain a period $B_{k_1} B_{k_0}^{-1} = \begin{pmatrix} 4 & 6 & 9 \\ 3 & 4 & 6 \\ 2 & 3 & 4 \end{pmatrix}$, and unit $\epsilon = 4 + 3 \cdot 3^{1/3} + 2 \cdot 3^{2/3}$.

§ 5.2. Examples of Theorem 4.3

Here we present some examples of Theorem 4.3. Fix an isomorphism $\mathbb{C} \simeq \overline{\mathbb{Q}}_p$ of fields. We use the same notations as in Section 4.

Example 5.3. Let $p = 5$, and let $z = \sqrt{-1} \in \mathbb{Q}_5 \subset \mathbb{C}$ be the imaginary quadratic irrational such that $\Im(z) > 0$ and $z \equiv 2 \pmod{5}$. Then its $\{\infty, 5\}$ -continued fraction expansion is as follows:

$$\begin{aligned} \sqrt{-1} &= [+1; \overline{7; 0, 4, 1;}] \\ &= 7 - \frac{25}{4 - \frac{1}{1 - \frac{1}{7 - \frac{25}{4 - \frac{1}{\ddots}}}}} \end{aligned}$$

where $[-; \overline{\cdot};] := [-; \sim; \sim; \cdots]$, and we simplify the continued fraction form by deleting the zero-terms as $-\frac{1}{0 - \frac{1}{a}} = a$. Then, from the period, we obtain a fundamental 5-unit $\epsilon = (3\sqrt{-1} + 4)/5$ of $\mathbb{Z}[\sqrt{-1}]$.

Example 5.4. Let $p = 3$, and let $z = \sqrt{-14} \in \mathbb{Q}_3 \subset \mathbb{C}$ be the imaginary quadratic irrational such that $\Im(z) > 0$ and $z \equiv 2 \pmod{3}$. Then its $\{\infty, 3\}$ -continued fraction expansion is as follows:

$$\begin{aligned} \sqrt{-14} &= [+1; \overline{2; 0, 1, 0; +1; 1; 0, 2, 0;}] \\ &= 2 - \frac{9}{2 - \frac{9}{4 - \frac{9}{2 - \frac{9}{4 - \frac{9}{\ddots}}}}} \end{aligned}$$

Then, from the period, we obtain a fundamental 3-unit $\epsilon = (-2\sqrt{-14} - 5)/9$ of $\mathbb{Z}[\sqrt{-14}]$.

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