

Solutions to The Nonhomogeneous Chebyshev's Equation by Means of N- Fractional Calculus Operator

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Abstract

In this article, solutions to the nonhomogeneous Chebyshev's equations

$$L[\varphi; z; \nu] = \varphi_2 \cdot (z^2 - 1) + \varphi_1 \cdot z - \varphi \cdot \nu^2 = f \quad (z^2 - 1 \neq 0, f \neq 0)$$

$$(\varphi_\alpha = d^\alpha \varphi / dz^\alpha \text{ for } \alpha > 0, \varphi_0 = \varphi = \varphi(z), f = f(z))$$

are discussed by means of N-fractional calculus operator (NFCO- Method).

By our method, some particular solutions to the above equations are given as below for example, in fractional differintegrated forms.

Group I.

$$(i) \quad \varphi = (G(\nu) \cdot H(\nu))_{-(1+\nu)} \equiv \varphi_{[1](z, \nu)}^* \quad (\text{denote})$$

$$(ii) \quad \varphi = (H(\nu) \cdot G(\nu))_{-(1+\nu)} \equiv \varphi_{[2](z, \nu)}^*$$

$$(iii) \quad \varphi = (G(-\nu) \cdot H(-\nu))_{\nu-1} \equiv \varphi_{[3](z, \nu)}^*$$

$$(iv) \quad \varphi = (H(-\nu) \cdot G(-\nu))_{\nu-1} \equiv \varphi_{[4](z, \nu)}^*$$

where

$$G(\nu) = (f_\nu \cdot (z^2 - 1)^{\nu-1/2})_{-1}, \quad H(\nu) = (z^2 - 1)^{-(\nu+1/2)}.$$

§ 0. Introduction (Definition of Fractional Calculus)

and

§ 1. Preliminary

are omitted, then refer to the previous paper for the Homogeneous Chebyshev's equation.

§ 2. Solutions to The Nonhomogeneous Chebyshev's Equations by Means of N-Fractional Calculus Operator

Theorem 1. Let $\varphi = \varphi(z) \in F$ and $f = f(z) \in F$, then the nonhomogeneous Chebyshev's equation

$$L[\varphi; z; \nu] = \varphi_2 \cdot (z^2 - 1) + \varphi_1 \cdot z - \varphi \cdot \nu^2 = f, \quad (\nu \in R, z^2 - 1 \neq 0) \quad (1)$$

has particular solutions of the forms (fractional differintegrated form);

Group I.

$$(i) \quad \varphi(z) = (G(\nu) \cdot H(\nu))_{-(1+\nu)} \equiv \varphi_{[1](z, \nu)}^* \quad (\text{denote}) \quad (2)$$

$$(ii) \quad \varphi(z) = (H(\nu) \cdot G(\nu))_{-(1+\nu)} \equiv \varphi_{[2](z, \nu)}^* \quad (3)$$

$$(iii) \quad \varphi(z) = (G(-\nu) \cdot H(-\nu))_{\nu-1} \equiv \varphi_{[3](z, \nu)}^* \quad (4)$$

$$(iv) \quad \varphi(z) = (H(-\nu) \cdot G(-\nu))_{\nu-1} \equiv \varphi_{[4](z, \nu)}^* \quad (5)$$

where

$$G(\nu) = (f_\nu \cdot (z^2 - 1)^{\nu-1/2})_{-1}, \quad H(\nu) = (z^2 - 1)^{-(\nu+1/2)}. \quad (6)$$

Group II.

$$(i) \quad \varphi(z) = (z^2 - 1)^{1/2} (P(\nu) \cdot H(\nu))_{-\nu} \equiv \varphi_{[5](z, \nu)}^* \quad (7)$$

$$(ii) \quad \varphi(z) = (z^2 - 1)^{1/2} (H(\nu) \cdot P(\nu))_{-\nu} \equiv \varphi_{[6](z, \nu)}^* \quad (8)$$

$$(iii) \quad \varphi(z) = (z^2 - 1)^{1/2} (P(-\nu) \cdot H(-\nu))_\nu \equiv \varphi_{[7](z, \nu)}^* \quad (9)$$

$$(iv) \quad \varphi(z) = (z^2 - 1)^{1/2} (H(-\nu) \cdot P(-\nu))_\nu \equiv \varphi_{[8](z, \nu)}^* \quad (10)$$

where

$$P(\nu) = ((f \cdot (z^2 - 1)^{-1/2})_{\nu-1} \cdot (z^2 - 1)^{\nu-1/2})_{-1}, \quad H(\nu) = (z^2 - 1)^{-(\nu+1/2)} \dots (11)$$

Group III.

$$(i) \quad \varphi(z) = (z-1)^{1/2} (Q(\nu) \cdot S(\nu))_{-(\nu+1/2)} \equiv \varphi^*_{[9](z, \nu)} \quad (12)$$

$$(ii) \quad \varphi(z) = (z-1)^{1/2} (S(\nu) \cdot Q(\nu))_{-(\nu+1/2)} \equiv \varphi^*_{[10](z, \nu)} \quad (13)$$

$$(iii) \quad \varphi(z) = (z-1)^{1/2} (Q(-\nu) \cdot S(-\nu))_{\nu-1/2} \equiv \varphi^*_{[11](z, \nu)} \quad (14)$$

$$(iv) \quad \varphi(z) = (z-1)^{1/2} (S(-\nu) \cdot Q(-\nu))_{\nu-1/2} \equiv \varphi^*_{[12](z, \nu)} \quad (15)$$

where

$$Q(\nu) = ((f \cdot (z-1)^{-1/2})_{\nu-1/2} \cdot (z-1)^\nu (z+1)^{\nu-1})_{-1}, \quad S(\nu) = ((z-1)^{-(\nu+1)} (z+1)^{-\nu}) \dots (16)$$

Group IV.

$$(i) \quad \varphi(z) = (z+1)^{1/2} (T(\nu) \cdot Y(\nu))_{-(\nu+1/2)} \equiv \varphi^*_{[13](z, \nu)} \quad (17)$$

$$(ii) \quad \varphi(z) = (z+1)^{1/2} (Y(\nu) \cdot T(\nu))_{-(\nu+1/2)} \equiv \varphi^*_{[14](z, \nu)} \quad (18)$$

$$(iii) \quad \varphi(z) = (z+1)^{1/2} (T(-\nu) \cdot Y(-\nu))_{\nu-1/2} \equiv \varphi^*_{[15](z, \nu)} \quad (19)$$

$$(iv) \quad \varphi(z) = (z+1)^{1/2} (Y(-\nu) \cdot T(-\nu))_{\nu-1/2} \equiv \varphi^*_{[16](z, \nu)} \quad (20)$$

where

$$T(\nu) = ((f \cdot (z+1)^{-1/2})_{\nu-1/2} \cdot (z-1)^{\nu-1} (z+1)^\nu)_{-1}, \quad Y(\nu) = ((z-1)^{-\nu} (z+1)^{-(\nu+1)}) \dots (21)$$

Proof of Group I ;

Operate N^α to the both sides of (1), we have then

$$\varphi_{2+\alpha} \cdot (z^2 - 1) + \varphi_{1+\alpha} \cdot z(2\alpha + 1) + \varphi_\alpha \cdot (\alpha^2 - \nu^2) = f_\alpha, \quad (f_\alpha \neq 0). \quad (22)$$

(Refer to the proof of **Group I** in § 2. in the previous paper for Homogeneous one.)

(I) Case $\alpha = \nu$;

In this case we obtain

$$\varphi_{2+\nu} \cdot (z^2 - 1) + \varphi_{1+\nu} \cdot z(2\nu + 1) = f_\nu, \quad (23)$$

from (22), then letting

$$\varphi_{1+\nu} = \psi = \psi(z) \quad (\varphi = \psi_{-(1+\nu)}), \quad (24)$$

yields

$$\psi_1 \cdot (z^2 - 1) + \psi \cdot z(2\nu + 1) = f_\nu, \quad (25)$$

from (24). A particular solution to this linear first order equation is given by

$$\psi = (f_\nu \cdot (z^2 - 1)^{\nu-1/2})_{-1} \cdot (z^2 - 1)^{-(\nu+1/2)}. \quad (26)$$

Therefore, we obtain

$$\varphi = \psi_{-(1+\nu)} = ((f_\nu \cdot (z^2 - 1)^{\nu-1/2})_{-1} \cdot (z^2 - 1)^{-(\nu+1/2)})_{-(1+\nu)} \quad (27)$$

$$= (G(\nu) \cdot H(\nu))_{-(1+\nu)} \equiv \varphi_{[1](z, \nu)}^* \quad (2)$$

from (24) and (26).

Inversely, we have

$$\begin{aligned} \varphi_{2+\nu} = \psi_1 &= (f_\nu \cdot (z^2 - 1)^{\nu-1/2}) \cdot (z^2 - 1)^{-(\nu+1/2)} \\ &\quad - (f_\nu \cdot (z^2 - 1)^{\nu-1/2})_{-1} \cdot (\nu + \frac{1}{2}) \cdot 2z(z^2 - 1)^{-(\nu+1/2+1)} \end{aligned} \quad (28)$$

and

$$\varphi_{1+\nu} = \psi = (f_\nu \cdot (z^2 - 1)^{\nu-1/2})_{-1} \cdot (z^2 - 1)^{-(\nu+1/2)} \quad (29)$$

from (2) respectively. Then we obtain

$$\begin{aligned} \text{LHS of (23)} &= \{ (f_\nu \cdot (z^2 - 1)^{\nu-1/2}) \cdot (z^2 - 1)^{-(\nu+1/2)} \\ &\quad - (f_\nu \cdot (z^2 - 1)^{\nu-1/2})_{-1} \cdot (\nu + \frac{1}{2}) \cdot 2z(z^2 - 1)^{-(\nu+1/2+1)} \} (z^2 - 1) \\ &\quad + z(2\nu+1) (f_\nu \cdot (z^2 - 1)^{\nu-1/2})_{-1} \cdot (z^2 - 1)^{-(\nu+1/2)} = f_\nu \end{aligned} \quad (30)$$

applying (28) and (29).

That is, the function shown by (2) satisfies equation (23).

Moreover, we have (1) operating $N^{-\nu}$ to the both sides of (23).

Therefore, the function given by (2) satisfies equation (1).

Next changing the order $G(\nu)$ and $H(\nu)$ in parenthesis $(\cdot)_{-(1+\nu)}$ in (2) we obtain

$$\varphi = (H(\nu) \cdot G(\nu))_{-(1+\nu)} \equiv \varphi_{[2](z, \nu)}^* \quad (3)$$

where

$$\varphi_{[1](z, \nu)}^* \neq \varphi_{[2](z, \nu)}^* \quad (\text{for } -(1+\nu) \notin \mathbb{Z}_0^+) \quad (31)$$

(II) Case $\alpha = -\nu$;

In the same way as (I) above, setting $-\nu$ instead of ν in the solutions (2) and (3), we obtain

$$\varphi = (G(-\nu) \cdot H(-\nu))_{\nu-1} \equiv \varphi_{[3](z, \nu)}^* \quad (4)$$

$$\varphi = (H(-\nu) \cdot G(-\nu))_{\nu-1} \equiv \varphi_{[4](z, \nu)}^* \quad (5)$$

where

$$\varphi_{[3](z, \nu)}^* \neq \varphi_{[4](z, \nu)}^* \quad (\text{for } (\nu-1) \notin \mathbb{Z}_0^+) \quad (32)$$

And

$$\varphi_{[3](z, \nu)}^* = \varphi_{[1](z, -\nu)}^*, \quad \varphi_{[4](z, \nu)}^* = \varphi_{[2](z, -\nu)}^* \quad (33)$$

Note 1. When $f_v = 0$, we have

$$\varphi_{[1](z,v)}^* = ((0)_{-1} \cdot (z^2 - 1)^{-(v+1/2)})_{-(1+v)} = (K \cdot (z^2 - 1)^{-(v+1/2)})_{-(1+v)} = 0 \quad (34-1)$$

(for $(1+v) \notin \mathbb{Z}_0^+$)

(K ; arbitrary constant for integration)

$$\begin{aligned} \varphi_{[2](z,v)}^* &= ((z^2 - 1)^{-(v+1/2)} \cdot (0)_{-1})_{-(1+v)} = ((z^2 - 1)^{-(v+1/2)} \cdot K)_{-(1+v)} \\ &= ((z^2 - 1)^{-(v+1/2)})_{-(1+v)} K \end{aligned} \quad (34-2)$$

from (2) and (3), respectively, by Lemmas (iv) and (i).

And we have

$$((z^2 - 1)^{-(v+1/2)} K)_{-(1+v)} = (K(z^2 - 1)^{-(v+1/2)})_{-(1+v)} = K((z^2 - 1)^{-(v+1/2)})_{-(1+v)} \quad (34-3)$$

by our definition § 0. (1), for N- Fractional Calculus.

Proof of Group I I ;

Set

$$\varphi = (z^2 - 1)^\lambda \phi, \quad \phi = \phi(z), \quad (35)$$

we have then

$$\phi_2 \cdot (z^2 - 1) + \phi_1 \cdot z(4\lambda + 1) + \phi \cdot \left\{ (4\lambda^2 - v^2) + \frac{2\lambda(2\lambda-1)}{z^2-1} \right\} = f \cdot (z^2 - 1)^{-\lambda} \quad (36)$$

from (1), applying (35).

(Refer to the proof of **Group II** in § 2. in the previous paper for Homogeneous one.)

When $\lambda = 0$, (36) is reduced to (1). We have then the same particular solutions as Group I

When $\lambda = 1/2$, we have

$$\phi_2 \cdot (z^2 - 1) + \phi_1 \cdot 3z + \phi \cdot (1 - v^2) = f \cdot (z^2 - 1)^{-1/2} \quad (37)$$

from (36).

Operate N^α to the both sides of (37), then yields

$$\phi_{2+\alpha} \cdot (z^2 - 1) + \phi_{1+\alpha} \cdot z(2\alpha + 3) + \phi_\alpha \cdot ((\alpha + 1)^2 - v^2) = (f \cdot (z^2 - 1)^{-1/2})_\alpha \quad (38)$$

(I) **Case** $\alpha = v - 1$;

In this case ,letting

$$\phi_v = V = V(z) \quad (\phi = V_{-v}), \quad (39)$$

we obtain

$$V_1 \cdot (z^2 - 1) + V \cdot z(2\nu + 1) = (f \cdot (z^2 - 1)^{-1/2})_{\nu-1} , \quad (40)$$

from (38). A particular solution to this linear first order equation is given by

$$V = ((f \cdot (z^2 - 1)^{-1/2})_{\nu-1} \cdot (z^2 - 1)^{\nu-1/2})_{-1} \cdot (z^2 - 1)^{-(\nu+1/2)} = P(\nu)H(\nu) . \quad (41)$$

Therefore, we obtain

$$\varphi = (z^2 - 1)^{1/2} \left(((f \cdot (z^2 - 1)^{-1/2})_{\nu-1} \cdot (z^2 - 1)^{\nu-1/2})_{-1} \cdot (z^2 - 1)^{-(\nu+1/2)} \right)_{-\nu} \quad (42)$$

$$= (z^2 - 1)^{1/2} (P(\nu) \cdot H(\nu))_{-\nu} \equiv \varphi_{[5](z, \nu)}^* , \quad (7)$$

from (35), applying (39) and (41), for $\lambda = 1/2$.

Next changing the order $P(\nu)$ and $H(\nu)$ in parenthesis ($\cdot$) _{$-\nu$} in (7) we obtain

$$\varphi = (z^2 - 1)^{1/2} (H(\nu) \cdot P(\nu))_{-\nu} \equiv \varphi_{[6](z, \nu)}^* . \quad (8)$$

where

$$\varphi_{[5](z, \nu)}^* \neq \varphi_{[6](z, \nu)}^* \quad (\text{for } (-\nu) \notin \mathbb{Z}_0^+) . \quad (43)$$

(II) Case $\alpha = -\nu - 1$;

In the same way as (I) above, setting $-\nu$ instead of ν in the solutions (7) and (8), we obtain

$$\varphi = (z^2 - 1)^{1/2} (P(-\nu) \cdot H(-\nu))_{\nu} \equiv \varphi_{[7](z, \nu)}^* , \quad (9)$$

$$\varphi = (z^2 - 1)^{1/2} (H(-\nu) \cdot P(-\nu))_{\nu} \equiv \varphi_{[8](z, \nu)}^* . \quad (10)$$

where

$$\varphi_{[7](z, \nu)}^* \neq \varphi_{[8](z, \nu)}^* \quad (\text{for } \nu \notin \mathbb{Z}_0^+) . \quad (44)$$

And

$$\varphi_{[7](z, \nu)}^* = \varphi_{[5](z, -\nu)}^* , \quad \varphi_{[8](z, \nu)}^* = \varphi_{[6](z, -\nu)}^* . \quad (45)$$

Proof of Group I II ;

Set

$$\varphi = (z - 1)^{\lambda} \phi, \quad \phi = \phi(z), \quad (46)$$

we have then

$$\phi_2 \cdot (z^2 - 1) + \phi_1 \cdot \{z(2\lambda + 1) + 2\lambda\} + \phi \cdot \{(\lambda^2 - \nu^2) + \frac{\lambda(2\lambda - 1)}{z - 1}\} = f \cdot (z - 1)^{-\lambda} \quad (47)$$

from (1), applying (46)

(Refer to the proof of **Group III** in § 2. in the previous paper for Homogeneous one)

When $\lambda = 0$, (47) is reduced to (1). We have then the same particular solutions as Group I.

When $\lambda = 1/2$, we have

$$\phi_2 \cdot (z^2 - 1) + \phi_1 \cdot (2z + 1) + \phi \cdot (1/4 - v^2) = f \cdot (z - 1)^{-1/2} \quad (48)$$

from (47).

Operate N^α to the both sides of (48), then yields

$$\phi_{2+\alpha} \cdot (z^2 - 1) + \phi_{1+\alpha} \cdot \{z(2\alpha + 2) + 1\} + \phi_\alpha \cdot \{(\alpha + 1/2)^2 - v^2\} = (f \cdot (z - 1)^{-1/2})_\alpha. \quad (49)$$

(I) Case $\alpha = v - 1/2$;

In this case letting

$$\phi_{v+1/2} = V = V(z) \quad (\phi = V_{-(v+1/2)}), \quad (50)$$

we obtain

$$V_1 \cdot (z^2 - 1) + V \cdot \{z(2v + 1) + 1\} = (f \cdot (z - 1)^{-1/2})_{v-1/2}, \quad (51)$$

from (49).

A particular solution to this equation is given by

$$V = ((f \cdot (z - 1)^{-1/2})_{v-1/2} \cdot \frac{z-1}{(z^2-1)^{1-v}})_{-1} \cdot \frac{(z^2-1)^{-v}}{z-1} = Q(v)S(v). \quad (52)$$

Therefore, we obtain

$$\varphi = (z - 1)^{1/2} \left(((f \cdot (z - 1)^{-1/2})_{v-1/2} \cdot (z - 1)^v (z + 1)^{v-1})_{-1} \cdot ((z - 1)^{-(v+1)} (z + 1)^{-v}) \right)_{-(v+1/2)} \quad (53)$$

$$= (z - 1)^{1/2} (Q(v) \cdot S(v))_{-(v+1/2)} \equiv \varphi_{[9](z, v)}^* \quad (12)$$

from (46) and (50), applying (52), for $\lambda = 1/2$.

Next changing the order $Q(v)$ and $S(v)$ in parenthesis $(\cdot)_{-(v+1/2)}$ in (12) we obtain

$$\varphi = (z - 1)^{1/2} (S(v) \cdot Q(v))_{-v} \equiv \varphi_{[10](z, v)}^* \quad (13)$$

where

$$\varphi_{[9](z, v)}^* \neq \varphi_{[10](z, v)}^* \quad (\text{for } -(v + 1/2) \notin \mathbb{Z}_0^+). \quad (54)$$

(II) Case $\alpha = -v - 1/2$;

In the same way as (I) above, setting $-v$ instead of v in the solutions (12) and (13), we obtain

$$\varphi = (z - 1)^{1/2} (Q(-v) \cdot S(-v))_v \equiv \varphi_{[11](z, v)}^* \quad (14)$$

$$\varphi = (z-1)^{1/2} (S(-v) \cdot Q(-v))_v \equiv \varphi_{[12](z,v)}^* \quad (15)$$

where

$$\varphi_{[11](z,v)}^* \neq \varphi_{[12](z,v)}^* \quad (\text{for } (v-1/2) \notin Z_0^+) \quad (55)$$

And

$$\varphi_{[11](z,v)}^* = \varphi_{[9](z,-v)}^*, \quad \varphi_{[12](z,v)}^* = \varphi_{[10](z,-v)}^* \quad (56)$$

Proof of Group I V ;

Set

$$\varphi = (z+1)^\lambda \phi, \quad \phi = \phi(z), \quad (57)$$

we have then

$$\phi_2 \cdot (z^2 - 1) + \phi_1 \cdot \{z(2\lambda + 1) - 2\lambda\} + \phi \cdot \{(\lambda^2 - v^2) - \frac{\lambda(2\lambda-1)}{z+1}\} = f \cdot (z+1)^{-\lambda} \quad (58)$$

from (1) applying (57),

(Refer to the proof of **Group IV** in § 2. in the previous paper for Homogeneous one)

When $\lambda = 0$, (58) is reduced to (1). We have then the same particular solutions as Group I.

When $\lambda = 1/2$, we have

$$\phi_2 \cdot (z^2 - 1) + \phi_1 \cdot (2z - 1) + \phi \cdot (1/4 - v^2) = f \cdot (z+1)^{-1/2} \quad (59)$$

from (58).

Operate N^α to the both sides of (59), then yields

$$\phi_{2+\alpha} \cdot (z^2 - 1) + \phi_{1+\alpha} \cdot \{z(2\alpha + 2) - 1\} + \phi_\alpha \cdot \{(\alpha + 1/2)^2 - v^2\} = (f \cdot (z+1)^{-1/2})_\alpha \quad (60)$$

(I) **Case** $\alpha = v - 1/2$;

Letting

$$\phi_{v+1/2} = V = V(z) \quad (\phi = V_{-(v+1/2)}), \quad (61)$$

we obtain

$$V_1 \cdot (z^2 - 1) + V \cdot \{z(2v + 1) - 1\} = (f \cdot (z+1)^{-1/2})_{v-1/2}, \quad (62)$$

from (60).

A particular solution to this equation is given by

$$V = ((f \cdot (z+1)^{-1/2})_{v-1/2} \cdot (z-1)^{v-1} (z+1)^v)_{-1} \cdot (z-1)^{-v} (z+1)^{-(v+1)} = T(v) \cdot Y(v) \quad (63)$$

Therefore, we obtain

$$\varphi = (z+1)^{1/2} \left(((f \cdot (z+1)^{-1/2})_{v-1/2} \cdot (z-1)^{v-1} (z+1)^v)_{-1} \cdot ((z-1)^{-v} (z+1)^{-(v+1)})_{-(v+1/2)} \right) \quad (64)$$

$$= (z+1)^{1/2} (T(\nu) \cdot Y(\nu))_{-(\nu+1/2)} \equiv \varphi_{[13](z, \nu)}^* \quad , \quad (17)$$

from (57) and (61), applying (63), for $\lambda = 1/2$.

Next changing the order $T(\nu)$ and $Y(\nu)$ in parenthesis $(\cdot)_{-(\nu+1/2)}$ in (17) we obtain

$$\varphi = (z+1)^{1/2} (Y(\nu) \cdot T(\nu))_{-(\nu+1/2)} \equiv \varphi_{[14](z, \nu)}^* \quad . \quad (18)$$

where

$$\varphi_{[13](z, \nu)}^* \neq \varphi_{[14](z, \nu)}^* \quad (\text{for } -(\nu+1/2) \notin \mathbb{Z}_0^+) \quad . \quad (65)$$

(II) Case $\alpha = -\nu - 1/2$;

In the same way as (I) above, setting $-\nu$ instead of ν in the solutions (17) and (18), we obtain

$$\varphi = (z+1)^{1/2} (T(-\nu) \cdot Y(-\nu))_{\nu-1/2} \equiv \varphi_{[15](z, \nu)}^* \quad , \quad (19)$$

$$\varphi = (z+1)^{1/2} (Y(-\nu) \cdot T(-\nu))_{\nu-1/2} \equiv \varphi_{[16](z, \nu)}^* \quad . \quad (20)$$

where

$$\varphi_{[15](z, \nu)}^* \neq \varphi_{[16](z, \nu)}^* \quad (\text{for } (\nu-1/2) \notin \mathbb{Z}_0^+) \quad . \quad (66)$$

And

$$\varphi_{[15](z, \nu)}^* = \varphi_{[13](z, -\nu)}^* \quad , \quad \varphi_{[16](z, \nu)}^* = \varphi_{[14](z, -\nu)}^* \quad . \quad (67)$$

§3. Some Example

(i) When $\nu = -1$ and $f = (z-1)^{-1}$ we have

$$\varphi_2 \cdot (z^2 - 1) + \varphi_1 \cdot z - \varphi = (z-1)^{-1} \quad (15)$$

and

$$\varphi = \varphi_{[1](z, -1)}^* = \left((f_{-1} \cdot (z^2 - 1)^{-3/2})_{-1} (z^2 - 1)^{1/2} \right)_0 \quad (16)$$

$$= \left(\log(z-1) \cdot (z^2 - 1)^{-3/2} \right)_{\neq 1} (z^2 - 1)^{1/2} \quad (17)$$

from §2. (1) and §2. (2) respectively.

Hence

$$\varphi_1 = \log(z-1) \cdot (z^2 - 1)^{-1} + \left(\log(z-1) \cdot (z^2 - 1)^{-3/2} \right)_{-1} (z^2 - 1)^{-1/2} z \quad (18)$$

and

$$\begin{aligned}\varphi_2 &= (z-1)^{-1}(z^2-1)^{-1} - \log(z-1) \cdot 2z(z^2-1)^{-2} + \log(z-1) \cdot z(z^2-1)^{-2} \\ &\quad + \left(\log(z-1) \cdot (z^2-1)^{-3/2} \right)_{-1} \{ -z^2(z^2-1)^{-3/2} + (z^2-1)^{-1/2} \}\end{aligned}\quad (19)$$

respectively.

Then applying (17), (18) and (19), we obtain

$$\text{LHS of (15)} = (z-1)^{-1}. \quad (20)$$

The function shown by (17) satisfies equation (15) clearly.

(ii) When $\nu = -1/2$ and $f = (z-1)^{-1/2}$ we have

$$\varphi_2 \cdot (z^2-1) + \varphi_1 \cdot z - \varphi \cdot (1/4) = (z-1)^{-1/2} \quad (21)$$

and

$$\varphi = \varphi^*_{[9](z, -1/2)} = (z+1)^{1/2} \left(\log(z-1) \cdot (z-1)^{-1/2} (z+1)^{-3/2} \right)_{-1} \quad (22)$$

from §2.(1) and §2.(9) respectively.

Hence we have

$$\varphi_1 = \frac{1}{2}(z+1)^{-1/2} (W)_{-1} + \log(z-1) \cdot (z-1)^{-1/2} (z+1)^{-1} \quad (23)$$

$$(W = \log(z-1) \cdot (z-1)^{-1/2} (z+1)^{-3/2})$$

and

$$\begin{aligned}\varphi_2 &= -\frac{1}{4}(z+1)^{-3/2} (W)_{-1} + (z-1)^{-3/2} (z+1)^{-1} + \left\{ \frac{1}{2}(z-1)^{-1/2} (z+1)^{-2} \right. \\ &\quad \left. - \frac{1}{2}(z-1)^{-3/2} (z+1)^{-1} - (z-1)^{-1/2} (z+1)^{-2} \right\} \log(z-1)\end{aligned}\quad (24)$$

from (22), respectively.

Then applying (22), (23) and (24), we obtain

$$\begin{aligned}\text{LHS of (21)} &= (W)_{-1} (z+1)^{-1/2} \left\{ -\frac{1}{4}(z-1) + \frac{1}{2}z - \frac{1}{4}(z+1) \right\} + (z-1)^{-1/2} \\ &\quad + (z+1)^{-1} (z-1)^{-1/2} \left\{ \frac{1}{2}(z-1) - \frac{1}{2}(z+1) - (z-1) + z \right\} \log(z-1) \\ &= (z-1)^{-1/2}.\end{aligned}\quad (25)$$

The function shown by (22) satisfies equation (21) clearly.

References

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