

J-groups of lens spaces

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§1. Introduction

The standard lens space mod m is the orbit manifold

$$L^n(m) = S^{2n+1}/Z_m \quad (Z_m = \{z \in S^1 : z^m = 1\})$$

of the $(2n+1)$ -sphere $S^{2n+1}(\subset \mathbb{C}^{n+1})$ by the diagonal action

$$z(z_0, \dots, z_n) = (zz_0, \dots, zz_n).$$

The J -groups of lens spaces were studied by several authors (e.g. [2],[5],[6],[8],[10] and [11]).

Let η_m be the canonical complex line bundle over $L^n(m)$. Then we have the following theorem by making use of the results due to J.F.Adams [1] and D.Quillen [12].

Theorem 1.1. Let p be a prime and let $r(\eta_{p^r}^i - 1) \in \widetilde{KO}(L^n(p^r))$ ($p^r \geq 3$) be the real restriction of the stable class of the i -fold tensor product of η_{p^r} . Then the order of the J -image

$$Jr(\eta_{p^r}^i - 1) \in \widetilde{J}(L^n(p^r))$$

is equal to $p^{f_p(n,r;v)}$,

$$f_p(n,r;v) = \max\{s-v + [n/p^s(p-1)]p^{s-v} : v \leq s < r \text{ and } p^s(p-1) \leq n\},$$

where $v=v_p(1)$ is the exponent of p in the prime power decomposition of i and $\max \phi=0$.

In this lecture, we prove the above theorem only the case $p=2$, since we can prove the above theorem for odd prime p in the similar way (see [10]).

Remark 1.2. By Theorem 1.1 and Proposition 1.3 below, we can determine the order of $Jr(\eta_m^1-1)$ in $\tilde{J}(L^n(m))$ for any m .

Let $L_0^n(m)$ be the $2n$ -skeleton of $L^n(m)$ and $m=\prod p^{r(p)}$ be the prime power decomposition of m . Then we have

Proposition 1.3. (i) The sequence

$$0 \rightarrow \tilde{J}(S^{2n+1}) \rightarrow \tilde{J}(L^n(m)) \xrightarrow{1^*} \tilde{J}(L_0^n(m)) \rightarrow 0$$

is a split extension for odd m .

(ii) There exists a natural isomorphism

$$f = \bigoplus_p \pi_p^* : \tilde{J}(L_0^n(m)) \rightarrow \bigoplus_p \tilde{J}(L_0^n(p^{r(p)})) \quad (m:\text{odd}),$$

$$f = \bigoplus_p (i_p \circ \pi_p)^* \oplus \pi_2^* : \tilde{J}(L^n(m)) \rightarrow \bigoplus_{p:\text{odd prime}} \tilde{J}(L_0^n(p^{r(p)})) \oplus \tilde{J}(L^n(2^{r(2)})) \quad (m:\text{even}),$$

which satisfies

$$f(Jr(\eta_m^1-1)) = \sum_p Jr(\eta_{p^{r(p)}}^1-1),$$

where $\pi_q : L^n(q) \rightarrow L^n(m)$ and $i_p : L_0^n(p^{r(p)}) \rightarrow L^n(p^{r(p)})$ are the natural projection and inclusion, respectively.

Proof. (i) is immediate from Puppe exact sequence in KO -theory and the fact that $\tilde{J}(L_0^n(m))$ is of odd order if m is odd.

(ii) We can show the similar result for $\tilde{KO}$ instead of $\tilde{J}$ by noticing that f is surjective and the both sides groups have the same order (cf. [3, Lemma 2.3 (ii)] and [13, Th.(0.1)]). The

last equality follows from the definitions of f , η_m and $\eta_{pr(p)}$.
q.e.d.

§2. The structure of $\tilde{J}(L^n(2^r))$

Let η be the canonical complex line bundle over $L^n(2^r)$ and ρ be the non-trivial real line bundle over $L^n(2^r)$ and put

$$\sigma(s) = \eta^{2^s} - 1, \quad \sigma(0) = \sigma \in \tilde{K}(L^n(2^r)),$$

$$\kappa = \rho - 1 \in \tilde{KO}(L^n(2^r)).$$

Then $\tilde{KO}(L^n(2^r))$ is generated additively by the elements

$$\kappa \quad \text{and} \quad r(\sigma^d \sigma(s)) \quad (0 \leq s \leq r-2, 0 \leq d < 2^s),$$

and its explicit additive and multiplicative structures are known ([9, Th.1.9]).

The calculation of Adams operations ψ^k on $\tilde{K}(L^n(2^r))$ and the property $r \circ \psi_c^k = \psi_R^k \circ r$ of Adams operations on $\tilde{K}$ and $\tilde{KO}$ imply the following

Lemma 2.1. Let $J : \tilde{KO}(L^n(2^r)) \rightarrow \tilde{J}(L^n(2^r))$ be the J -homomorphism. Then $\text{Ker } J$ is generated additively by the elements

$$r(\sigma^d(1+\sigma)\sigma(s)) \quad (0 \leq s \leq r-1, 0 \leq d < 2^s - 1).$$

From now on, we use the following notation

$$\alpha_s = Jr\sigma(s) \in \tilde{J}(L^n(2^r)).$$

Here, we notice that $\alpha_s = 0$ if $s \geq r$ and $\alpha_{r-1} = 2J\kappa$, since $\eta^{2^s} = 1$ if $s \geq r$ and $\eta^{2^{r-1}} = 2\rho$.

From the above lemma, we see easily the following

Proposition 2.2. $\tilde{J}(L^n(2^r))$ is generated by

$$J\kappa \text{ and } \alpha_s \quad (0 \leq s \leq r-2).$$

Combining the relations of $\tilde{K}O(L^n(2^r))$ given in [9, Th.1.9] and the relations arisen from $\text{Ker } J$ in Lemma 2.1, we have the following theorem on the group structure of the reduced J -group $\tilde{J}(L^n(2^r))$ ($r \geq 2$), where

$$a_s = [n/2^s], \quad b_s = n - 2^s a_s \quad (0 \leq s < r),$$

$$X(d, v) = \sum_{j \in \mathbb{Z}} (-1)^{j(2^v+1)} \binom{2d}{d+2^v j},$$

$$Y(d, v) = \sum_{j \in \mathbb{Z}} \binom{2d-1}{d+2^v(2j+1)}.$$

Theorem 2.3. (i) [5, Th.4.5] $J: \tilde{K}O(L^n(4)) \cong \tilde{J}(L^n(4))$.

(ii) The relations of $\tilde{J}(L^n(2^r))$ for $r \geq 3$ are given as follows:

(a) The case $n \equiv 1 \pmod{4}$:

$$(2.3.1) \quad 2^{1+a_{r-1}} J\kappa = 0, \quad 2^{r-1+2a_1} \alpha_0 = 0, \quad 2^{r-1-s+a_s} \alpha_s = 0 \quad (1 \leq s \leq r-2).$$

$$(2.3.2) \quad 2^{a_{r-1}} J\kappa + \sum_{v=0}^{r-2} 2^{r-1-v(1+a_{r-1})-2} \alpha_v = 0 \quad \text{if } a_1 \geq 2^{r-2}.$$

$$(2.3.3) \quad 2^{r-s-2+a_s} \alpha_s + \sum_{v=0}^{s-1} 2^{r-s-3+2^{s-v}(1+a_s)} \alpha_v = 0 \quad (1 \leq s \leq r-2, 2^s \leq a_1).$$

$$(2.3.4) \quad \sum_{v=0}^s (-1)^{2^{s-v}} 2^{r-s-4+2^{s+1-v}(a_s+1+\delta)} X(d, v) \alpha_v = 0 \\ (1 \leq s \leq r-2, 1 \leq d < 2^s, 2^s + d \leq a_1),$$

where $\delta = 1$ if $2d \leq b_{s+1}$, $= 0$ otherwise.

$$(2.3.5) \quad 2^{2i-2} \alpha_0 - \sum_{v=1}^t Y(i, v) \alpha_v = 0 \quad \text{where } 2^t \leq i < 2^{t+1} (a_1 < i < 2^{r-1}).$$

(b) The case $n \equiv 1 \pmod{4}$: The relations in (a), excluded the

one in (2.3.4) for $s=r-2$, $2d=1+b_{r-1}$ and the one in (2.3.5) for $i=a_1+1$, and in addition,

$$(2.3.6) \quad 2^{2i-2}\alpha_0 - \sum_{v=1}^t Y(i,v)\alpha_v = 0 \text{ where } 2^t \leq a_1+1 < 2^{t+1} \text{ if } a_1 < 2^{r-2}.$$

For the special case that $n=2^{r-1}a$, we can reduce the relations of $\tilde{J}(L^n(2^r))$ in (ii) of the above theorem to more simple ones, and $\tilde{J}(L^n(2^r))$ is given by the following explicit form, where $Z_h(x)$ denotes the cyclic group of order h generated by the element x .

Theorem 2.4. If $n=2^{r-1}a$ ($r \geq 3$, $a \geq 2$), then $\tilde{J}(L^n(2^r))$ is the direct sum

$$\begin{aligned} & Z_{2^{r-1-n}} \langle \alpha_0 \rangle \oplus \bigoplus_{s=1}^{r-2} Z_{2^{a_s-1}} \langle \alpha_s \rangle \oplus Z_{2^{a_{r-1}-1}} \langle \alpha_{r-1} \rangle \\ & \oplus Z_{2^{a_{r-1}}} \langle \alpha_{r-2} \rangle. \end{aligned}$$

By using the above theorem, the known fact about the kernel of $i^*: \tilde{K}O(L^n(2^r)) \rightarrow \tilde{K}O(L^{n-1}(2^r))$ ([9, Prop.4.4]) and the calculation of Adams operation ψ^3 on $\tilde{K}O(L^n(2^r))$, we can determine the kernel of

$$(2.5) \quad i^*: \tilde{J}(L^n(2^r)) \rightarrow \tilde{J}(L^{n-1}(2^r))$$

as follows :

Proposition 2.6. i^* in (2.5) is isomorphic if $n \equiv 3 \pmod{4}$, epimorphic otherwise, and

$$\text{Ker } i^* = \begin{cases} Z_4 \langle 2J\bar{\sigma}^{-2m+1} \rangle & \text{if } n = 4m+2 \\ Z_2 \langle J\bar{\sigma}^{-2m+1} \rangle & \text{if } n = 4m+1 \\ Z_u \langle J\bar{\sigma}^{-2m} \rangle & \text{if } n = 4m > 0, \end{cases}$$

where $\bar{\sigma} = r(n-1) \in \tilde{K}O(L^n(2^r))$ and

$$u = 2^{\min\{r+1, \ell+2\}} \text{ for } n=4m=2^\ell q \text{ with } (2,q)=1.$$

§3. Proof of Theorem 1.1

To prove Theorem 1.1 for $p=2$, we prepare some lemmas.

Lemma 3.1. The following equality holds in $\tilde{J}(L^n(2^r))$ ($r \geq 2$):

$$Jr(\eta^i - 1) = Jr\sigma(v) = \alpha_v \quad \text{for } i \geq 1,$$

where $v = v_2(i)$ is the exponent of 2 in the prime power decomposition of i .

Proof. By Lemma 2.1, we notice that the kernel of J :

$\tilde{K}O(L^n(2^r)) \rightarrow \tilde{J}(L^n(2^r))$ is generated additively by

$$r(\eta^j \sigma(s)) \quad (0 \leq s < r, 1 \leq j < 2^s).$$

If $2^s \leq i < 2^{s+1}$, then $\eta^i - 1 = \eta^i \sigma(s) + \eta^j - 1$ where $j = i - 2^s$. If $j > 0$ in addition, then $Jr(\eta^i - 1) = Jr(\eta^j - 1)$ by the above notice and $\sigma(s) = 0$ ($s \geq r$). By continuing this process, we have the desired equality.

q.e.d.

Now, let $f_2(n, r; v)$ be the non-negative integer such that

$$\#Jr\sigma(v) = \#\alpha_v = 2^{f_2(n, r; v)} \quad \text{in } \tilde{J}(L^n(2^r)) \quad (n \geq 0, r \geq 2),$$

where $\#\alpha$ denotes the order of α . Then by the definition of α_v , we see that

$$(3.2) \quad f_2(n, r; v) = 0 \quad \text{if } n = 0 \text{ or } v \geq r.$$

Lemma 3.3. If $n = 2^{r-1}a$ and $r \geq 3$, then

$$f_2(n, r; v) = r - 1 - v + 2^{r-1-v}a \quad \text{for } n > 0, 0 \leq v < r.$$

Proof. The lemma for $a \geq 2$ is easily seen from Theorem 2.4

and $\alpha_{r-1} = 2J\kappa$.

Consider the case $n=2^{r-1}$. Then, by Proposition 2.6,

$$\#J\bar{\sigma}^{2m} = 2^{r+1} \quad \text{in} \quad \tilde{J}(L^{2^{r-1}}(2^r)) \quad (4m=2^{r-1}).$$

On the other hand, $2^{r-2m}\bar{\sigma} = 2^{r+4m-2}\bar{\sigma}$ in $\tilde{KO}(L^{2^{r-1}}(2^r))$ by [7, Lemma 2.3]. Thus, we obtain

$$(3.4) \quad \#\alpha_0 = \#J\bar{\sigma} = 2^{r-1+2^{r-1}}.$$

Furthermore, we have the following relations in $\tilde{J}(L^{2^{r-1}}(2^r))$ by Theorem 2.3 :

$$(3.5) \quad 2^{a_v}\alpha_v = 2^{a_v-1+1}\alpha_{v-1} \quad (1 \leq v \leq r-3),$$

$$2^2\alpha_{r-2} + 2^5\alpha_{r-3} = 0 = 2J\kappa + 2^2\alpha_{r-2}.$$

The relations (3.4) and (3.5) imply immediately

$$\#\alpha_v = r-1-v+2^{r-1-v} \quad (0 \leq v < r),$$

which is the equality for $a=1$.

q.e.d.

Consider the commutative diagram ($r \geq 3$)

$$(3.6) \quad \begin{array}{ccc} \text{Ker } i^* \subset \tilde{J}(L^n(2^r)) & \xrightarrow{i^*} & \tilde{J}(L^{n-1}(2^r)) \\ \pi^* \downarrow & & \downarrow \pi'^* \\ \text{Ker } i'^* \subset \tilde{J}(L^n(2^{r-1})) & \xrightarrow{i'^*} & \tilde{J}(L^{n-1}(2^{r-1})) \end{array}$$

of the induced homomorphisms, where i and i' are the inclusions and π and π' are the natural projections. Then we have the following

Lemma 3.7. If $n \not\equiv 0 \pmod{2^{r-1}}$ ($r \geq 3$), then

$$\pi^*|_{\text{Ker } i^*} : \text{Ker } i^* \rightarrow \text{Ker } i'^*$$

is isomorphic.

Proof. If $n=4m=2^{\ell}q$ (q :odd), then the assumption $n \not\equiv 0 \pmod{2^{r-1}}$

implies $r-1 > l$ and so $\min\{r+1, l+2\} = l+2 = \min\{r, l+2\}$. Thus, we see immediately the lemma by Proposition 2.6, by noticing that $\pi^* r \eta = r \pi^* \eta = r \eta$ and hence $\pi^* J \bar{\sigma}^{-1} = J \bar{\sigma}^{-1}$. q.e.d.

Lemma 3.8. If $n \not\equiv 0 \pmod{2^{r-1}}$ ($r \geq 3$), then

$$f_2(n, r; v) = \max\{f_2(n-1, r; v), f_2(n, r-1; v)\}.$$

Proof. Consider the diagram (3.6). Then the definition of $f_2(n, r; v)$ implies that

$$f_2(n, r; v) \geq \max\{f_2(n-1, r; v), f_2(n, r-1; v)\},$$

since $i^* \alpha_v = \alpha_v$ and $\pi^* \alpha_v = \alpha_v$. Moreover, if $f_2(n, r; v) > \max\{f_2(n-1, r; v), f_2(n, r-1; v)\}$, then the non-zero element $2^{f_2(n, r; v)-1} \alpha_v$ in $\tilde{J}(L^n(2^r))$ is mapped to 0 by i^* and π^* . This contradicts Lemma 3.7. Thus we have the lemma. q.e.d.

Proof of Theorem 1.2. By (3.2), it is sufficient to show that

$$(3.9) \quad f_2(n, r; v) = \max\{s-v + [n/2^s] 2^{s-v} : v \leq s < r \text{ and } 2^s \leq n\} \quad (0 \leq v < r).$$

(3.9) for $r=2$ is easy consequence of Theorem 2.3 (i) and [4, Th.B]. By Lemma 3.3, (3.9) holds if $r \geq 3$ and $n \equiv 0 \pmod{2^{r-1}}$.

For the case $r \geq 3$ and $2^{r-1}a < n < 2^{r-1}(a+1)$, assume inductively that (3.9) holds for $(n-1, r; v)$ and $(n, r-1; v)$ instead of $(n, r; v)$. Then, we see easily that the right hand side of the equality in Lemma 3.8 is equal to

$$\begin{cases} f_2(n, r-1; v) & \text{if } a=0, \\ \max\{f_2(n, r-1; v), r-1-v + [(n-1)/2^{r-1}] 2^{r-1-v}\} & \text{if } a>0, \end{cases}$$

and hence to the right hand side of (3.9). Thus Lemma 3.8 implies (3.9) by the induction on n and r .

These complete the proof of Theorem 1.2.

q.e.d.

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