

On solutions of initial-boundary

problem for $u_t = u_{xx} + \frac{1}{1-u}$

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§1. Introduction and Theorem

Various works^{1), 2), 3)} have been published on the blowing-up of solutions of the Cauchy problem and the initial-boundary value problem of nonlinear partial differential equations.

Blowing-up means that the solutions of these problems become infinite in a finite time.

The objective of the present paper is to introduce the concept of quenching which has more general sense than blowing-up and to find some sufficient conditions for quenching of the solutions of the following initial-boundary value problem for

$$u = u(t, x), \quad t > 0, \quad x \in (0, \ell),$$

$$(1.1a) \quad u_t = u_{xx} + \frac{1}{1-u}, \quad t > 0, \quad x \in (0, \ell),$$

$$(1.1b) \quad u(t, 0) = u(t, \ell) = 0, \quad t > 0,$$

$$(1.1c) \quad u(0, x) = 0, \quad x \in (0, \ell),$$

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where ℓ is a positive constant. The above initial-boundary value problem (1.1a-c) is denoted by IVP. Our study may be said to be more illustrative than general, since we restrict ourselves to one-space-dimensional mixed problems of semilinear heat equations. Nevertheless, we hope that our results will give an insight into a more general situation. The nonlinear perturbation $\frac{1}{1-u}$ ($u \neq 1$) in (1.1a) is a locally Lipschitz continuous. Thus IVP has a unique solution which may be local in t .

The present problems came to our attention in connection with the diffusion equation generated by a polarization phenomena in ionic conductors⁴⁾.

We shall define quenching for the solutions of the initial value problems.

Definition 1. Let $u = u(t, x)$ be the solution of the initial value problems which are defined in $t > 0, x \in \Omega$. Ω means R^m which stands for the m -dimensional Euclidean space or the bounded domain in R^m .

We shall say that u quenches if $\|u_t\|_C$ becomes infinite in a finite time where $\|\cdot\|_C$ denotes the maximum norm over Ω .

In order to clarify the nature of quenching, let us take some examples.

Example 2. α being constant, the solution of the initial value problem for $u = u(t)$, $t > 0$,

$$\begin{cases} \frac{du}{dt} = \frac{1}{1-u}, & t > 0 \\ u(0) = \alpha, \end{cases}$$

is $u = 1 + \sqrt{(1-\alpha)^2 - 2t}$, if $\alpha > 1$ and $u = 1 - \sqrt{(1-\alpha)^2 - 2t}$, if $\alpha < 1$. In both cases, we see quenching at $t = \frac{(1-\alpha)^2}{2}$.

Example 3. Let α be as above. The solution of the initial-boundary value problem for $u = u(t, x)$, $t > 0$, $x \in (0, \ell)$,

$$\begin{cases} u_t = u_{xx} + \frac{1}{1-u}, & t > 0, \quad x \in (0, \ell), \\ u_x(t, 0) = u_x(t, \ell) = 0, & t > 0 \\ u(0, x) = \alpha, & x \in (0, \ell) \end{cases}$$

is the same as above.

Example 4. Blowing-up in the initial value problems means quenching. As our main result, we have

Theorem. In the IVP, suppose $\ell > 2\sqrt{2}$. Then the solution of the IVP quenches.

The present paper has two sections apart from this section. In §2, we shall give a Lemma. §3 is devoted to the proof of our Theorem.

§2. Lemma

As a preparation for the proof of Theorem we state the following lemma. Henceforce, let $u = u(t, x)$ be the solution of IVP.

Lemma. In the IVP, suppose $\ell > 2\sqrt{2}$. Then u reaches 1 in a finite time at $x = \frac{\ell}{2}$.

Proof:

1st Step. We show that $u(t, x)$ is increasing in t for every x in $(0, \ell)$ as long as u exists. In fact, putting $v = u_t$, we have

$$(2.1) \quad v_t = v_{xx} + \frac{1}{(1-u)^2} \cdot v, \quad x \in (0, \ell),$$

$$v(t, 0) = v(t, \ell) = 0,$$

and

$$v(0, x) = 1, \quad x \in (0, \ell) \text{ as long as } u \text{ exists.}$$

We notice that v is a solution of the linear parabolic equation (2.1) and is non-negative on the "parabolic boundary". Thus v is non-negative everywhere, which implies the required

monotonicity of u .

2nd Step. The solution $u_1 = u_1(t, x)$ of the initial-boundary value problem for $u = u(t, x)$,

$$\begin{cases} u_t = u_{xx} + 1, & t > 0, \quad x \in (0, \ell), \\ u(t, 0) = u(t, \ell) = 0, & t > 0, \\ u(0, x) = 0, & x \in (0, \ell) \end{cases}$$

converges its stationary solution $\psi(x) = \frac{1}{2}\ell(\ell-x)$ ($0 < x < \ell$) as $t \rightarrow +\infty$. Thus u_1 crosses 1 in a finite time if $\ell > 2\sqrt{2}$.

Suppose that u does not reach 1 in a finite time if $\ell > 2\sqrt{2}$. Then IVP has a global solution, i.e., u satisfies $0 \leq u \leq 1$ in $(0, \ell) \times [0, +\infty)$ by virtue of the monotonicity of u .

Comparing u with u_1 , we get $u \geq u_1$ in $(0, \ell) \times [0, +\infty)$

since $\frac{1}{1-\lambda} \geq 1$ in $0 \leq \lambda \leq 1$. This contradicts the assumption.

We shall denote the time when u reaches 1 by $t = T_0$.

3rd Step. u satisfies (i) $u_x(t, 0) > 0$ by virtue of positivity of u ; (ii) $u_x(t, \frac{\ell}{2}) = 0$ since u is an even function with respect to $x = \frac{\ell}{2}$. Putting $\pi = u_x$, we have

$$\begin{aligned} \pi_t &= \pi_{xx} + \frac{1}{(1-u)^2} \cdot \pi, & t \in [0, T_0), \quad x \in (0, \frac{\ell}{2}), \\ \pi(t, 0) &> 0, \quad \pi(t, \frac{\ell}{2}) &= 0, \quad t \in [0, T_0), \end{aligned}$$

and

$$\pi(0, x) = 0, \quad x \in (0, \frac{\ell}{2}).$$

Repeating the same argument as in 1st Step, we see that

$$(2.2) \quad \pi = u_x(t, x) > 0, \quad t \in [0, T_0), \quad x \in (0, \frac{\ell}{2}).$$

Combining (2.2) and (ii), we get that u takes its maximum

at $x = \frac{\ell}{2}$ for any $t \in [0, T_0)$. This completes the proof.

§3. Proof of Theorem

1st Step.

1.a) Put $\mu = \mu(t) = u(t, \frac{\ell}{2})$ in $[0, T_0)$. μ satisfies

$$(3.1) \quad \frac{d\mu}{dt} \leq \frac{1}{1-\mu} \quad \text{in } [T_0 - \varepsilon, T_0)$$

for sufficiently small $\varepsilon (> 0)$ since $u_{xx}(t, \frac{\ell}{2}) \leq 0$ in $[0, T_0)$.

Put $T_1 = T_0 - \varepsilon$ and $\Omega_\varepsilon = (0, \ell) \times [T_1, T_0)$. Comparing $\mu(t)$

with $v = v(t) = 1 - \sqrt{2\sqrt{T_0 - t}}$ in $[T_1, T_0)$, we get

$$(3.2) \quad \mu \geq v, \quad \text{in } [T_1, T_0)$$

since v satisfies (see Example 2)

$$\frac{dv}{dt} = \frac{1}{1-v}, \quad t \in [T_1, T_0)$$

and

$$\lim_{t \rightarrow T_0} v(t) = 1.$$

(3.2) implies that there exists the domain D_ε in which

u satisfies

$$u(t, x) \geq v(t) .$$

Denote the complement of D_ε by E_ε and put $E_\varepsilon^{(1)} = E_\varepsilon \cap \{(0, \frac{\ell}{2}) \times [T_1, T_0)\}$ and $E_\varepsilon^{(2)} = E_\varepsilon \cap \{(\frac{\ell}{2}, \ell) \times [T_1, T_0)\}$.

For D_ε , there may be two cases:

Case (a) D_ε has no interior points; i.e., there holds

$$u_{xx}(t, \frac{\ell}{2}) = 0 \quad \text{in } [T_1, T_0) .$$

Case (b) D_ε has interior points.

For the case (a), u quenches obviously. Henceforce we consider only the case (b).

1.b) Denote the boundary between D_ε and $E_\varepsilon^{(i)}$ by $x = s^{(i)}(t)$ ($t \in [T_1, T_0)$) for $i=1, 2$. Then $x = s^{(i)}(t)$ satisfies

$$(i) \quad \lim_{t \rightarrow T_0} s^{(i)}(t) = \frac{\ell}{2} ;$$

$$(ii) \quad u_x(t, s^{(i)}(t)) \cdot \dot{s}^{(i)}(t) = -u_{xx}(t, s^{(i)}(t)), \quad t \in [T_1, T_0)$$

where $\dot{s}^{(i)}(t)$ means $\frac{ds^{(i)}(t)}{dt}$ for $i=1, 2$. In fact, there holds

$$(3.3) \quad u = v \quad \text{on } x = s^{(i)}(t) , \quad t \in [T_1, T_0) .$$

Differentiating both sides of (3.3) and using (3.3), we get

$$(3.4) \quad u_t(t, s^{(i)}(t)) + u_x(t, s^{(i)}(t)) \cdot \dot{s}^{(i)}(t) = \frac{1}{1 - u(t, s^{(i)}(t))} .$$

By virtue of (1.1a) on $x = s^{(i)}(t)$ and (3.3) we have (ii).

1.c) Obviously we have the following inequalities

$$(3.5a) \quad \frac{1}{1-u} \geq \frac{1}{\sqrt{2\sqrt{T_0}-t}} \quad \text{in } D_\varepsilon ,$$

and

$$(3.5b) \quad \frac{1}{1-u} < \frac{1}{\sqrt{2\sqrt{T_0}-t}} \quad \text{in } E_\varepsilon.$$

2nd Step.

2.a) Let $p = p(t, x)$ be $\frac{1}{2(T_0-t)}$ in D_ε and $\frac{1}{(1-u)^2}$ in E_ε .

Then the solution $v_1 = v_1(t, x)$ of the initial-boundary value problem for $v = v(t, x)$ in Ω_ε ,

$$\begin{cases} v_t = v_{xx} + p \cdot v & \text{in } \Omega_\varepsilon \\ v(t, 0) = v(t, \ell) = 0, & t \in [T_1, T_0), \\ v(T_1, x) = \beta(x) = u_t(T_1, x), & x \in (0, \ell), \end{cases}$$

exists and satisfies $v_1 \leq v$ in Ω_ε by virtue of (3.5a).

2.b) Put $W = W(t, x) = \sqrt{T_0-t} \cdot v_1$. Denoting W in D_ε by $W^{(1)}$, we have $W_t^{(1)} = W_{xx}^{(1)}$ in D_ε .

3rd Step.

3.a) We shall deal with the following initial-boundary value problem for $V = V(t, x)$ in $(-\infty, +\infty) \times [T_1, T_0)$.

$$(3.6a) \quad V_t = V_{xx} \quad \text{in } (-\infty, +\infty) \times [T_1, T_0)$$

$$(3.6b) \quad V = W^{(1)} \quad \text{in } D_\varepsilon$$

$$(3.6c) \quad V = \sqrt{\varepsilon} \cdot \beta(x), \quad x \in [0, s^{(1)}(T_1)) \cup (s^{(2)}(T_1), \ell]$$

$$(3.6d) \quad V = 0, \quad x \in (-\infty, 0) \cup (l, +\infty).$$

In what follows we impose on the solution $V(t, x)$ the following conditions at infinity: $V(t, x)$ and $V_x(t, x)$ are bounded as $x \rightarrow \pm\infty$ uniformly with respect to t in $[T_1, T_0)$. We see the solution $\hat{W} = \hat{W}(t, x)$ of (3.6) uniquely exists. Uniqueness of $\hat{W}$ is shown by Holmgren's theorem. Using the Green's function

$$K(t, x; \tau, \xi) = \frac{1}{2\sqrt{\pi(t-\tau)}} \exp\left\{-\frac{(x-\xi)^2}{4(t-\tau)}\right\},$$

$\hat{W}$ is represented by

$$\begin{aligned} \hat{W}(t, x) = & \int_{T_1}^t [K(t, x; \tau, s^{(1)}(\tau)) W_\xi^{(1)}(\tau, s^{(1)}(\tau)) \\ & - W^{(1)}(\tau, s^{(1)}(\tau)) K_\xi(t, x; \tau, s^{(1)}(\tau))] d\tau \\ (3.7) \quad & + \int_0^{s^{(1)}(T_1)} K(t, x; T_1, \xi) \sqrt{\varepsilon} \cdot \beta(\xi) d\xi \\ & + \int_{T_1}^t K(t, 0; \tau, s^{(1)}(\tau)) W^{(1)}(\tau, s^{(1)}(\tau)) \cdot \dot{s}^{(1)}(\tau) d\tau, \\ & -\infty < x < s^{(1)}(t), \quad t \in [T_1, T_0). \end{aligned}$$

Also in $s^{(2)}(t) < x < +\infty$, $t \in [T_1, T_0)$, we have the similar expression as (3.7).

3.b) Using the positivity of β , W and maximum principle, we have

$$\hat{W}(t, x) \geq 0 \quad \text{in } (-\infty, +\infty) \times [T_1, T_0).$$

Thus from (3.6) and (3.5b) we see

$$\hat{W}(t, x) \geq W(t, x) \quad \text{in } \Omega_\varepsilon.$$

4th Step. We claim that

$$\lim_{t \rightarrow T_0} \hat{W}(t, \frac{\ell}{2}) > 0.$$

On the contrary, we suppose that

$$\lim_{t \rightarrow T_0} \hat{W}(t, \frac{\ell}{2}) = 0,$$

which implies that $0 \equiv \hat{W}(t, x) \geq W(t, x) \geq 0$ in Ω_ε by the strong maximum principle⁵⁾. This is a contradiction. Thus we

get that

$$\lim_{t \rightarrow T_0} \frac{du(t)}{dt} = \lim_{t \rightarrow T_0} v(t, x) \geq \lim_{t \rightarrow T_0} v_1(t, \frac{\ell}{2}) = \lim_{t \rightarrow T_0} \frac{\hat{W}(t, \frac{\ell}{2})}{\sqrt{t-T_0}} = +\infty$$

This completes the proof.

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