

# The Borel hierarchy and the complete metrizability of spaces of metrics

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This article is a résumé of the paper [13] with some additional remarks. In [13], the Borel hierarchy and the complete metrizability of spaces of metrics are investigated, which are important issues for recognizing topological types of function spaces in infinite-dimensional topology. Given a metrizable space  $X$ , denote by  $C(X^2)$  the space of continuous bounded real-valued functions on  $X^2$  with the sup-metric. Let  $PM(X)$  be the subspace consisting of continuous bounded pseudometrics on  $X$ , and  $AM(X)$  be the subspace consisting of bounded admissible metrics on  $X$ . Recall that  $C(X^2)$  and  $PM(X)$  are complete metric spaces, where  $PM(X)$  is closed in  $C(X^2)$ . By virtue of the result in [8], we can prove that  $AM(X)$  is always a Baire space. It is well-known that completely metrizable spaces are Baire (the Baire category theorem, cf. [11, Theorem 8.4]), but the inverse does not hold. It is interesting to ask the following:

**Problem.** When is  $AM(X)$  completely metrizable?

Due to the study of Y. Ishiki [7], if  $X$  is separable and locally compact, then  $AM(X)$  is a  $G_\delta$  set in  $C(X^2)$ , and hence it is completely metrizable. Moreover, the author [12] improved this result: the complete metrizability of  $AM(X)$  follows from the  $\sigma$ -compactness of  $X$ . Now we shall consider the inverse of it.

Denote by  $\mathfrak{A}_n$  (respectively,  $\mathfrak{M}_n$ ) the additive (respectively, multiplicative) absolute Borel class with respect to a natural number  $n$ , refer to [5, Problems 4.5.7 and 4.5.8] and [15, Section 5.11]. Note that  $\mathfrak{A}_0 = \{\emptyset\}$  and  $\mathfrak{M}_0$  is consisting of compact metrizable spaces. Furthermore,

- $\mathfrak{M}_1$  is the class of completely metrizable spaces,
- $\mathfrak{A}_1$  is the class of  $\sigma$ -locally compact metrizable spaces ( $\sigma$ -compact metrizable spaces in the separable case), see [16].

The author [13] showed that the Borel hierarchy of  $AM(X)$  is restricted by the one of  $X$ .

**Theorem.** *Let  $X$  be a metrizable space and  $n \geq 1$  be a natural number. If  $AM(X) \in \mathfrak{A}_n$  (respectively,  $\mathfrak{M}_n$ ), then  $X \in \mathfrak{M}_n$  (respectively,  $\mathfrak{A}_n$ ).*

Applying the above theorem and [12, Proposition 3], we can give an equivalent condition on the complete metrizable of  $AM(X)$ .

**Corollary.** *Suppose that  $X$  is a separable metrizable space. Then  $X \in \mathfrak{A}_1$  if and only if  $AM(X) \in \mathfrak{M}_1$ .*

Two key ideas play significant roles in the proof of Theorem. The following example shows us the one of them.

**Example.** Let  $Q = [0, 1] \cap \mathbb{Q}$  and  $P = [0, 1] \cap \mathbb{P}$  with the usual metric, where  $\mathbb{Q}$  is the set of rationals and  $\mathbb{P}$  is the one of irrationals. The space  $AM(X)$  on the topological sum  $X = P \oplus (0, 1]$  is not completely metrizable. Indeed, define  $i : Q \rightarrow AM(X)$  by for all  $q \in Q$ ,

$$i(q)(x, y) = \begin{cases} |x - y| & \text{if } (x, y) \in P^2 \text{ or } (0, 1]^2, \\ |x - q| + y & \text{if } x \in P \text{ and } y \in (0, 1], \\ x + |y - q| & \text{if } x \in (0, 1] \text{ and } y \in P. \end{cases}$$

Then it is a closed embedding from  $Q \notin \mathfrak{M}_1$ . Consequently,  $AM(X) \notin \mathfrak{M}_1$ .

Combining the efforts of C. Bessaga [3], T. Banach [1], O. Pikhurko [14] and M. Zarichnyi [17] (see also [4, Theorem 2] and [2, Theorem 1.2], and moreover [9] in the unbounded case), we can obtain the following lemma, which is the another key ingredient in the proof of Theorem.

**Lemma.** *For a metrizable space  $Y$  and a closed subset  $A \subset Y$ , there exists a continuous function  $e : AM(A) \rightarrow AM(Y)$  such that  $e(d)|_{A^2} = d$  for any  $d \in AM(A)$ .*

Using this lemma, we can prove the following proposition, which is a generalization of Example.

**Proposition.** *If a metrizable space  $X$  contains a closed topological copy of  $\mathbb{P}$ , then  $AM(X)$  is not completely metrizable.*

*Sketch of Proof.* Take any closed embedding  $h : \mathbb{P} \rightarrow X$ , and put  $A_1 = h([0, 1/3] \cap \mathbb{P})$  and  $A_2 = h([2/3, 1] \cap \mathbb{P})$  with the natural metric induced by  $\mathbb{P}$ . Let  $Q' = [2/3, 1] \cap \mathbb{Q}$  and define an embedding  $i : Q' \rightarrow AM(A_1 \oplus A_2)$  by the same way as Example. Due to the above lemma, we can find an extensor  $e : AM(A_1 \oplus A_2) \rightarrow AM(X)$ . Then the composition  $e \circ i$  is a closed embedding from  $Q' \notin \mathfrak{M}_1$ . Therefore  $AM(X) \notin \mathfrak{M}_1$ .  $\square$

Theorem will be shown as follows:

*Sketch of Proof.* We only discuss the case that  $AM(X) \in \mathfrak{M}_n$ . Supposing that  $X \notin \mathfrak{A}_n$ , we can find a bounded complete metric space  $Y$  so that  $X$  is not an  $\mathfrak{A}_n$  subset of  $Y$ . Set  $B = X \setminus \text{int}_Y X$  and  $Z = \text{cl}_Y B \setminus X$ , where “ $\text{int}_Y$ ” is the interior operator and “ $\text{cl}_Y$ ” is the closure operator on  $Y$  respectively, so  $B$  is not an  $\mathfrak{A}_n$  set in  $Y$  and  $Z \notin \mathfrak{M}_n$ . Take two open subsets  $U$  and  $U'$  of  $\text{cl}_Y B$  so that  $Z \cap \text{cl}_Y U \notin \mathfrak{M}_n$ ,  $Z \cap U' \neq \emptyset$  and  $\text{cl}_Y U \cap \text{cl}_Y U' = \emptyset$ . Fix any  $a \in Z \cap U'$  and choose  $\{a_n\} \subset X \cap U'$  converging to  $a$ . Let  $A = \{a_n\}$ ,  $B' = B \cap \text{cl}_Y U$  and  $Z' = Z \cap \text{cl}_Y U$ . Define an embedding  $i : Z' \rightarrow AM(A \oplus B')$  by the same technique as Example and take an extending map  $e : AM(A \oplus B') \rightarrow AM(X)$  as in Lemma, so  $AM(X)$  admits a closed embedding  $e \circ i$  from  $Z' \notin \mathfrak{M}_n$ . It follows that  $AM(X) \notin \mathfrak{M}_n$ .  $\square$

**Remark 1.** We can consider that there exists certain “complementary” relation between  $X$  and  $AM(X)$ . In Example, the subspaces  $Q \notin \mathfrak{M}_1$  and  $P \notin \mathfrak{A}_1$  are complementary to each other in  $[0, 1]$ . The embedding  $i$  constructs a metric on  $X = P \oplus (0, 1]$  by connecting  $q \in [0, 1] \setminus P$  and  $0 \in [0, 1] \setminus (0, 1]$ . Similarly, we can define an embedding  $j : P \rightarrow PM(X) \setminus AM(X)$  by for any  $p \in P$ ,

$$i(p)(x, y) = \begin{cases} |x - y| & \text{if } (x, y) \in P^2 \text{ or } (0, 1]^2, \\ |x - p| + y & \text{if } x \in P \text{ and } y \in (0, 1], \\ x + |y - p| & \text{if } x \in (0, 1] \text{ and } y \in P. \end{cases}$$

Then  $i(Q)$  and  $j(P)$  are complementary to each other in some kind of “boundary” between  $AM(X)$  and  $PM(X) \setminus AM(X)$ . We can observe the similar relations between  $Q' \notin \mathfrak{M}_1$  and  $A_2 \notin \mathfrak{A}_1$  in Proposition, and between  $Z' \notin \mathfrak{M}_n$  and  $B' \notin \mathfrak{A}_n$  in Theorem, and they induces certain complementary relation between  $X$  and  $AM(X)$  in the Borel hierarchy.

**Remark 2.** In [12], the space  $AM(X)$  is homeomorphic to Hilbert space  $\ell_2$  if  $X$  is compact, and homeomorphic to the Banach space  $\ell_\infty$  of bounded real-valued functions on natural numbers with the sup-norm if  $X$  is  $\sigma$ -compact but not compact. It is known that  $\ell_\infty$  is isometrically universal for separable metric spaces (the Fréchet embedding theorem, cf. [6, p. 101]), see also [10, Lemma 2.8] in the non-separable case. Observe that the embeddings  $i$  in Example, Proposition, and Theorem are in fact isometric. Remark that the extension  $e$  in Lemma can be chosen as an isometric operator. Hence the composition  $e \circ i$  is also isometric, and more  $AM(X)$  admits an isometric embedding from some metric spaces. Using these techniques, Y. Ishiki and the author [10] have investigated the isometric universality of spaces of metrics.

## Acknowledgements

The research [13] was motivated by some works of Yoshito Ishiki, and the author would like to thank him for his valuable suggestions. This article was partially supported by the Research Institute for Mathematical Sciences, an International Joint Usage/Research Center located in Kyoto University.

## References

- [1] T. Banach, *AE(0)-spaces and regular operators extending (averaging) pseudometrics*, Bull. Pol. Acad. Sci. Math. **42** (1994), 197–206.
- [2] T. Banach, N. Brodskiy, I. Stasyuk and E.D. Tymchatyn, *On continuous extension of uniformly continuous functions and metrics*, Colloq. Math. **116** (2009), no. 2, 191–202.
- [3] C. Bessaga, *On linear operators and functors extending pseudometrics*, Fund. Math. **142** (1993), no. 2, 101–122.

- [4] C. Bessaga and T. Banach, *On linear operators extending [pseudo]metrics*, Bull. Pol. Acad. Sci. Math. **48** (2000), no. 1, 35–49.
- [5] R. Engelking, *General Topology*, Revised and Complete Edition, Sigma Ser. in Pure Math., **6**, Heldermann Verlag, Berlin, 1989.
- [6] J. Heinonen, P. Koskela, N. Shanmugalingam and J. Tyson, *Sobolev spaces on metric measure spaces, An approach based on upper gradients*, New Math. Monogr., **27** Cambridge University Press, Cambridge, 2015.
- [7] Y. Ishiki, *An interpolation of metrics and spaces of metrics*, arXiv: 2003.13227 [math.MG].
- [8] Y. Ishiki, *Spaces of metrics are Baire*, arXiv: 2402.04565 [math.MG].
- [9] Y. Ishiki, *An isometric extensor of metrics*, arXiv: 2407.03030 [math.MG].
- [10] Y. Ishiki and K. Koshino, *On isometric universality of spaces of metrics*, arXiv: 2409.17701 [math.MG].
- [11] A.S. Kechris, *Classical descriptive set theory*, Grad. Texts in Math. **156**, Springer-Verlag, New York, 1995.
- [12] K. Koshino, *Recognizing the topologies of spaces of metrics with the topology of uniform convergence*, Bull. Pol. Acad. Sci. Math. **70** (2022), 165–171.
- [13] K. Koshino, *On the Borel complexity and the complete metrizability of spaces of metrics*, Anal. Geom. Metr. Spaces, accepted.
- [14] O. Pikhurko, *Extending metrics in compact pairs*, Mat. Stud. **3** (1994), 103–106.
- [15] K. Sakai, *Topology of Infinite-Dimensional Manifolds*, Springer, SMM, Springer, Tokyo, 2020.
- [16] A.H. Stone, *Absolute  $F_\sigma$  spaces*, Proc. Amer. Math. Soc. **13** (1962), 495–499.
- [17] M. Zarichnyi, *Regular linear operators extending metrics: a short proof*, Bull. Polish Acad. Sci. Math. **44** (1996), no. 3, 267–269.

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