

On certain multivalent functions.

By

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Let $A(p)$ be the class of functions of the form

$$f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n \quad (p \in \mathbb{N} = \{1, 2, 3, \dots\})$$

which are analytic in $D = \{z \mid |z| < 1\}$.

A function $f(z)$ in $A(p)$ is said to be p -valently starlike iff

$$\operatorname{Re} \frac{zf'(z)}{f(z)} > 0 \quad \text{in } D.$$

We denote by $S(p)$ the subclass of $A(p)$ consisting of functions which are p -valently starlike in D .

A function $f(z)$ in $A(p)$ is said to be p -valently close-to-convex iff there exists a function $g(z) \in S(p)$ such that

$$\operatorname{Re} \frac{zf'(z)}{g(z)} > 0 \quad \text{in } D.$$

We denote by $K(p)$ the subclass of $A(p)$ consisting of functions which are p -valently close-to-convex in D .

Livingston [2] introduced this class $K(p)$.

Ozaki [5, Theorem 3] proved that if $f(z) \in A(p)$ and

$$1 + \operatorname{Re} \frac{zf''(z)}{f'(z)} < \frac{k+p+1}{2} \quad \text{in } D,$$

then $f(z)$ is at most k -valent in D .

Moreover, by using Umezawa's result [7, Theorem 6] we can prove that if $f(z) \in A(p)$ and

$$1 + \operatorname{Re} \frac{zf''(z)}{f'(z)} < \frac{k+p+1}{2} \quad \text{in } D,$$

then $f(z)$ is convex of order at most p in one direction in D , and at most k -valent in D .

Nunokawa and Owa [4, Theorem 2] proved the following result.

LEMMA 1. Let $f(z) \in A(p)$ and

$$1 + \operatorname{Re} \frac{zf''(z)}{f'(z)} < \beta \quad \text{in } D$$

$$\text{where } p < \beta \leq p + \frac{1}{2}$$

Then we have

$$\left| \arg \frac{f'(z)}{z^{p-1}} \right| \leq 2(\beta - p) \sin^{-1} |z| \quad \text{in } D$$

and therefore

$$\operatorname{Re} \frac{f'(z)}{z^{p-1}} > 0 \quad \text{in } D.$$

This shows that $f(z)$ is p -valently close-to-convex or $f(z) \in K(p)$.

On the other hand, Nunokawa [3] proved the following result.

LEMMA 2. Let $f(z) \in A(p)$ and assume that

$$\operatorname{Re} \frac{f'(z)}{z^{p-1}} > 0 \quad \text{in } D$$

and

$$\left(\operatorname{Im} \frac{f'(z)}{z^{p-1}} \right) \left(\operatorname{Im} e^{-i\alpha} z \right) \neq 0, \quad z \in D(\alpha)$$

for some α ($0 \leq \alpha < \pi$), where

$$D(\alpha) = \left\{ z \mid 0 < |z| < 1 \text{ and } (\arg z - \alpha)(\arg z - \pi - \alpha) \neq 0 \right\}.$$

Then $f(z)$ is p -valently starlike in D , $f(z) \in S(p)$ or

$$\operatorname{Re} \frac{zf'(z)}{f(z)} > 0 \quad \text{in } D.$$

Applying LEMMA 1 and 2, we can prove the following theorem.

THEOREM 1. Let $f(z) \in A(p)$ and assume that

$$1 + \operatorname{Re} \frac{zf''(z)}{f'(z)} < p + \frac{1}{2} \quad \text{in } D$$

and

$$\left(\operatorname{Im} \frac{f'(z)}{z^{p-1}} \right) \left(\operatorname{Im} e^{-i\alpha} z \right) \neq 0 \quad z \in D(\alpha)$$

for some α ($0 \leq \alpha < \pi$) where

$$D(\alpha) = \left\{ z \mid 0 < |z| < 1 \text{ and } (\arg z - \alpha)(\arg z - \pi - \alpha) \neq 0 \right\}.$$

Then $f(z)$ is p -valently starlike in D , $f(z) \in S(p)$ or

$$\operatorname{Re} \frac{zf'(z)}{f(z)} > 0 \quad \text{in } D.$$

PROOF. Applying LEMMA 1 to $f(z)$, we have

$$\operatorname{Re} \frac{f(z)}{z^{p-1}} > 0 \quad \text{in } D.$$

From the assumption of THEOREM 1 and LEMMA 2, it follows that

$$\operatorname{Re} \frac{zf'(z)}{f(z)} > 0 \quad \text{in } D.$$

This completes our proof.

From THEOREM 1, we easily have the following corollary.

COROLLARY 1. Let $f(z) \in A(p)$ and assume that

$$1 + \operatorname{Re} \frac{zf''(z)}{f'(z)} < p + 1 \quad \text{in } D$$

and $f'(z)/z^{p-1}$ is typically real in D .

Then $f(z)$ is p -valently starlike in D or $f(z) \in S(p)$.

THEOREM 2. Let $f(z) \in A(p)$ and assume that

$$(1) \quad p - \frac{2p(\beta - p)}{k - p} < 1 + \operatorname{Re} \frac{zf''(z)}{f'(z)} < \beta \quad \text{in } D$$

where $1 + p \leq k$ and $(p + k - 1)/2 < \beta \leq (p + k + 1)/2$.

Then $f(z)$ is p -valently close-to-convex in D or $f(z) \in K(p)$.

PROOF. Let us put

$$(2) \quad H(z) = \frac{1}{\beta - p} \left\{ \beta - 1 - \frac{zf''(z)}{f'(z)} \right\} = \frac{zg'(z)}{g(z)}.$$

Then we have $H(0) = 1$, $\operatorname{Re} H(z) > 0$ in D and therefore it follows that $g(z) \in S(1)$.

A simple calculation of (2) gives

$$\frac{f'(z)}{pz^{p-1}} = \left(\frac{g(z)}{z} \right)^{p-\beta}$$

and then we have

$$\frac{zf'(z)}{z^{\frac{p+k}{2}} g(z)^{\frac{p-k}{2}}} = p \left(\frac{z}{g(z)} \right)^{\beta - \frac{p+k}{2}}$$

From the result by Robinson [6] and Komatu [1], we have

$$(3) \quad \frac{zf'(z)}{z^{\frac{p+k}{2}} g(z)^{\frac{p-k}{2}}} = p \left(\frac{g(z)}{z} \right)^{\frac{p+k}{2} - \beta} \rightarrow p \left(\frac{1}{1-z} \right)^{p+k-2\beta}$$

where the symbol $\rightarrow$ denotes the subordination.

From the assumption of THEOREM 2 and (3), we have

$$(4) \quad \operatorname{Re} \frac{zf'(z)}{z^{\frac{p+k}{2}} g(z)^{\frac{p-k}{2}}} > 0 \quad \text{in } D.$$

On the other hand, let us put

$$G(z) = z^{\frac{p+k}{2}} g(z)^{\frac{p-k}{2}}$$

Then we have

$$\begin{aligned} \operatorname{Re} \frac{zG'(z)}{G(z)} &= \frac{p+k}{2} + \left(\frac{p-k}{2} \right) \operatorname{Re} \frac{zg'(z)}{g(z)} \\ &= \frac{p+k}{2} + \frac{p-k}{2(\beta-p)} \left\{ \beta - 1 - \operatorname{Re} \frac{zf''(z)}{f'(z)} \right\}. \end{aligned}$$

From the assumption (1), we have

$$\begin{aligned} \operatorname{Re} \frac{zG'(z)}{G(z)} &> \frac{p+k}{2} + \frac{(p-k)}{2(\beta-p)} + \frac{(k-p)}{2(\beta-p)} \left\{ \beta - (\beta-p) \frac{(k+p)}{(k-p)} \right\} \\ &= 0 \quad \text{in } D. \end{aligned}$$

On the other hand, $G(z)$ has a zero of order p at $z=0$ and therefore we have

$$(5) \quad G(z) \in S(p).$$

From (4) and (5), we have

$$f(z) \in K(p).$$

This completes our proof.

COROLLARY 2. Let $f(z) \in A(p)$ and assume that $f(z)$ satisfies one of the following conditions:

- (6) $-p < 1 + \operatorname{Re} \frac{zf''(z)}{f'(z)} < p+1$ in D
- (7) $-\frac{1}{2}p < 1 + \operatorname{Re} \frac{zf''(z)}{f'(z)} < p + \frac{3}{2}$ in D .

Then $f(z)$ is p -valently close-to-convex in D or $f(z) \in K(p)$.

PROOF. Letting $k=\beta=p+1$ in THEOREM 2, we have (6) and letting $k=p+2$ and $\beta = p+3/2$ in THEOREM 2, we have (7).

Hence our conclusion follows in every case by THEOREM 2.

References

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